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ON THE GLOBAL ATTRACTOR OF MODIFIED SWIFT-HOHENBERG EQUATION IN 3D CASE

Ning Duan and Xianyun Xu

School of Science, Jiangnan University, Wuxi 214122, *P. R. China*
e-mails: dn@jiangnan.edu.cn; dachuangjndx@163.com

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In this paper, we study the global dynamic for the solution semiflow of the modified Swift-Hohenberg equation in 3D case. We show that the equation has a global attractor in $H^4(\Omega)$ when the initial value belongs to $H^1(\Omega)$.

Key words : Global attractor; modified Swift-Hohenberg equation; absorbing set.

1. Introduction

The dynamic properties of diffusion equation and diffusion system such as the global asymptotical behaviors of solutions and global attractors are important for the study of diffusion model, which ensure the stability of diffusion phenomena and provide the mathematical foundation for the study of diffusion dynamics. There are many studies on the existence of global attractors for diffusion equations. For the classical results we refer the reader to [1-4].

In this paper, we consider the modified Swift-Hohenberg equation

$$\frac{\partial u}{\partial t} + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = 0, (x,t) \in \Omega \times (0,T), \tag{1}$$

where $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3) \subset \mathbb{R}^3$, a and b are arbitrary constants. On the basis of physical considerations, as usual Eq.(1) is supplemented with the periodic boundary value conditions

$$\varphi|_{x_i=0} = \varphi|_{x_i=L_i}, \quad i = 1, 2, 3, \tag{2}$$

for u, ∇u, Δu and $\nabla\Delta u$, and the initial condition

$$u(x, 0) = u_0(x), \quad x \in \Omega. \tag{3}$$

The Swift-Hohenberg equation is one of the universal equations used in the description of pattern formation in spatially extended dissipative systems (see [5]), which arise in the study of viscous film folw and bifurcating solutions of the Navier-Stokes [6], plasma confinement in toroidal devices [7], convective hydrodynamics [8]. Noticing that, the usual Swift-Hohenberg equation [8] is recovered for $b = 0$. The additional term $b|\nabla u|^2$, reminiscent of the Kuramoto-Sivashinsky equation, which arises in the study of various pattern formation phenomena involving some kind of phase turbulence or phase transition (see [9-11]), breaks the symmetry $u \to -u$.

During the past years, many authors have paid much attention to the Swift-Hohenberg equation (see, e.g. [8, 12, 13])

$$\frac{\partial u}{\partial t} = \alpha u - (1 + \Delta)^2 u - u^3.$$

However, only a few people devoted to the modified Swift-Hohenberg equation. It was Doelman *et al*. [15] who first studied the modified Swift-Hohenberg equation for a pattern formation system with two unbounded spatial directions that is near the onset to instability. Polat [10] considered the modified Swift-Hohenberg equation in 2D case. In his paper, for the modified Swift-Hohenberg equation as (1)-(3), the existence of a global attractor is proved. Recently, Song *et al*. [5] studied the long time behavior for modified Swift-Hohenberg equation in H^k $(k \geq 0)$ space. By using an iteration procedure, regularity estimates for the linear semigroups and a classical existence theorem of global attractor, they proved that the 2D modified Swift-Hohenberg equation possesses a global attractor in Sobolev space H^k for all $k \geq 0$, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm. Duan and Gao [14] studied the optimal control problem for the 1D modified Swift-Hohenberg equation.

In this paper, we are interested in the existence of global attractors for the problem (1)-(3). We shall show that the equation has a global attractor in H^4 when the initial value belongs to H^1. The outline of this paper is as follows. In the next section, we give some preparations, and we also give the main results on the existence of global attractor for problem (1)-(3); In Section 3, the uniform estimates of solutions are established; In the last section, the main result on the existence of global attractor is proved.

2. Preliminaries

We denote by $H = L^2(\Omega)$, $(\cdot, \cdot)$ the H-inner product and by $\|\cdot\|$ the corresponding H-norm, denote $A = -\Delta$, where Δ is the Laplace operator. On the other hand, we assume that the initial function has zero mean, i.e. $\int_\Omega u_0(x)dx = 0$, then it follows that $\int_\Omega u(x,t)dx = 0$ for $t > 0$. Here, as [3], we set

$$\dot{H}^k_{per} = \{u|u \in H^k_{per}(\Omega), \int_\Omega u(x,t)dx = 0,\}, \;\; k = 1, 2, \cdots .$$

Similar to the proof in [16], we have the following results on global existence and uniqueness of solution to problem (1)-(3).

Lemma 2.1 — Assume that $u_0 \in H^1_{per}(\Omega)$, then problem (1)-(3) admits a unique solution u such that

$$u \in L^\infty([0,T]; H^1_{per}(\Omega)) \cap L^2([0,T]; H^3_{per}(\Omega)).$$

By Lemma 2.1, we can define the operator semigroup

$$S(t)u_0 : \dot{H}^1_{per}(\Omega) \times \mathbb{R}^+ \to \dot{H}^1_{per}(\Omega),$$

which is $(\dot{H}^1_{per}, \dot{H}^1_{per})$-continuous. In what follows, we always assume that $\{S(t)\}_{t\geq 0}$ is the semigroup generated by the weak solutions of problem (1)-(3). It is sufficient to see that the restriction of $\{S(t)\}$ on the affined space $\dot{H}^1_{per}(\Omega)$ is a well defined semigroup.

In order to prove the existence of global attractor, we give some definitions and results.

Definition 2.2 — (see [1, 17]). Suppose that E is a bounded subset of H^1 and B is a bounded subset of H^4. Then, B is said to be a bounded (H^1, H^4)-absorbing set for $\{S(t)\}_{t\geq 0}$, if there exists a time $T > 0$ depending on B such that

$$S(t)E \subseteq B, \;\; \forall t \geq T.$$

Definition 2.3 — (see [1, 17]). $\{S(t)\}_{t\geq 0}$ is said to be (H^1, H^4)-asymptotically compact, if for any bounded $\{u_{0,n}\}_{n=1}^\infty$ in H^1 and $t_n \to \infty$, $\{S(t_n)u_{0,n}\}_{n=1}^\infty$ has a convergent subsequence in $H^4(\Omega)$.

Proposition 2.4 — (see [1, 17]). Suppose that $\mathcal{A}$ is an (H^1, H^1)-global attractor for $\{S(t)\}_{t\geq 0}$. Suppose further that $\{S(t)\}_{t\geq 0}$ has a bounded (H^1, H^4)-absorbing set and $\{S(t)\}_{t\geq 0}$ is (H^1, H^4)-asymptotically compact. Then $\mathcal{A}$ is also an (H^1, H^4)-global attractor.

The main result of this article is given by the following theorem, which provides the existence of global attractors of problem (1)-(3).

***Theorem* 2.5** — *Suppose that $u_0 \in H^1(\Omega)$, a is sufficiently large, then the problem (1)-(3) has an (H^1, H^4)-global attractor for the solution $u(x,t)$, which is invariant and compact in $H^4(\Omega)$ and attracts every bounded subset of $H^1(\Omega)$ with respect to the norm topology of $H^4(\Omega)$.*

3. Uniform Estimates of Solutions

In this section, we establish the uniform estimates of solutions of problem (1)-(3) as $t \to \infty$.

Lemma 3.1 — Suppose that $u_0 \in L^2(\Omega)$ and a is sufficiently large, then for problem (1)-(3), we have

$$\|u(t)\| \leq M_0, \ \ \forall t \geq T_0,$$

and

$$\int_t^{t+1} \|Au(t)\|^2 d\tau \leq M_0, \ \ t \geq T_0.$$

Here, M_0 is a positive constant depending on a and b. T_0 depends on a, b and R where $\|u_0\|^2 \leq R^2$.

PROOF : Multiplying equation (1) by u and integrating the resulting relation over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + a\|u\|^2 + \|u\|_4^4 = -2(\Delta u, u) - b(|\nabla u|^2, u). \tag{4}$$

Note that

$$(|\nabla u|^2, u) = \int_\Omega |\nabla u|^2 u dx = -\int_\Omega u^2 \Delta u dx - \int_\Omega |\nabla u|^2 u dx,$$

that is

$$b(|\nabla u|^2, u) = -\frac{1}{2} b \int_\Omega u^2 \Delta u dx \leq \|u\|_4^4 + \frac{b^2}{16}\|\Delta u\|^2.$$

On the other hand,

$$-2(\Delta u, u) \leq \frac{1}{2}\|\Delta u\|^2 + 2\|u\|^2.$$

Therefore

$$\frac{d}{dt}\|u\|^2 + (1 - \frac{b^2}{8})\|\Delta u\|^2 \leq (4 - 2a)\|u\|^2. \tag{5}$$

Using Poincaré's inequality and Hölder's inequality, we get

$$\|u\|^2 \leq C_0\|\nabla u\|^2 = -C_0(u, \Delta u) \leq \frac{1}{2}\|u\|^2 + \frac{C_0^2}{2}\|\Delta u\|^2.$$

By (5) and the above inequality, we obtain

$$\frac{d}{dt}\|u\|^2 + \left[\frac{1}{C_0^2}(1 - \frac{b^2}{8}) - 2(2 - a)\right]\|u\|^2 \leq 0,$$

where a is sufficiently large, it satisfies $\frac{1}{C_0^2}(1-\frac{b^2}{8})-2(2-a)>0$. Using Gronwall's inequality, we deduce that

$$\|u\|^2 \le e^{-\left[\frac{1}{C_0^2}(1-\frac{b^2}{8})-2(2-a)\right]t}\|u_0\|^2 \le C_1, \tag{6}$$

for all $t \ge T^* = \frac{8C_0^2}{8-b^2-16C_0^2(2-a)} \ln \frac{R^2}{C_1}$. Integrating (5) over $(t, t+1)$ with $t \ge T^*$, we have

$$\int_t^{t+1} \|\Delta u(T)\|^2 d\tau \le C_2. \tag{7}$$

Using a mean value theorem for integrals we obtain the existence of a time $t_0' \in (T^*, T^*+1)$ such that the following estimate holds uniformly

$$\|\Delta u(t_0')\|^2 \le C_3.$$

Lemma 3.2 — Suppose that $u_0 \in \dot{H}^1_{per}(\Omega)$ and a is sufficiently large, then for problem (1)-(3), we have

$$\|\nabla u(t)\| \le M_1, \ \ \forall t \ge T_1,$$

and

$$\int_t^{t+1} \|\nabla \Delta u(t)\|^2 d\tau \le M_1, \ \ t \ge T_1.$$

Here, M_1 is a positive constant depending on a, b. T_1 depends on a, b and R where $\|u_0\|^2_{H^1_{per}} \le R^2$.

PROOF : Multiplying equation (1) by $-\Delta u$ and integrating the resulting relation over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\nabla \Delta u\|^2 + a\|\nabla u\|^2 = 2\|\Delta u\|^2 + b(|\nabla u|^2, \Delta u) + (u^3, \Delta u).$$

Note that

$$(u^3, \Delta u) = -(\nabla u^3, \nabla u) = -3\int_\Omega u^2 |\nabla u|^2 dx \le 0,$$

and

$$(|\nabla u|^2, \Delta u) = -2(\nabla u \Delta u, \nabla u) = 0.$$

On the other hand, by Nirenberg's inequality, we have

$$2\|\Delta u\|^2 \le 2(C_1'\|\nabla \Delta u\|^{\frac{2}{3}}\|u\|^{\frac{1}{3}} + C_2'\|u\|)^2 \le \frac{1}{2}\|\nabla \Delta u\|^2 + \frac{C_3}{2}.$$

Summing up, we get

$$\frac{d}{dt}\|\nabla u\|^2 + \|\nabla \Delta u\|^2 + 2a\|\nabla u\|^2 \le C_3, \tag{8}$$

where a is sufficiently large, it satisfies $a > 0$. Using Gronwall's inequality, we get

$$\|\nabla u\|^2 \leq e^{-2at}\|\nabla u_0\|^2 + \frac{C_3}{2a} \leq \frac{C_3}{a}, \tag{9}$$

for all $t \geq T' = \max\{T^*, \frac{1}{2a}\ln\frac{2aR^2}{C_3}\}$. Integrating (8) over $(t, t+1)$ with $t \geq T'$, we have

$$\int_t^{t+1} \|\nabla\Delta u\|^2 d\tau \leq C_4.$$

Using a mean value theorem for integrals we obtain the existence of a time $t_0 \in (T', T'+1)$ such that the following estimate holds uniformly

$$\|\nabla\Delta u(t_0)\|^2 \leq C_5.$$

Set $T_1 = T'$, we complete the proof. □

Lemma 3.3 — Suppose that $u_0 \in \dot{H}^1_{per}(\Omega)$ and a is sufficiently large, then for problem (1)-(3), we have

$$\|Au(t)\| \leq M_2, \ \ \forall t \geq T_2,$$

and

$$\int_t^{t+1} \|u_t\|^2 d\tau \leq M_2, \ \ t \geq T_2.$$

Here, M_2 is a positive constant depending on a, b. T_2 depends on a, b and R where $\|u_0\|^2_{H^1_{per}} \leq R^2$.

PROOF : Multiplying equation (1) by $\Delta^2 u$ and integrating the resulting relation over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 + a\|\Delta u\|^2 = -b(|\nabla u|^2, \Delta^2 u) - (u^3, \Delta^2 u) + 2\|\nabla\Delta u\|^2.$$

Using Nirenberg's inequality, we derive that

$$2b^2\|\nabla u\|_4^4 \leq 2b^2(C_1'\|\Delta^2 u\|^{\frac{1}{4}}\|\nabla u\|^{\frac{3}{4}} + C_2'\|\nabla u\|)^4 \leq \frac{1}{8}\|\Delta^2 u\|^2 + C_6,$$

and

$$2\|u\|_6^6 \leq 2(C_1'\|\Delta^2 u\|^{\frac{1}{4}}\|u\|^{\frac{3}{4}} + C_2'\|u\|)^6 \leq \frac{1}{8}\|\Delta^2 u\|^2 + C_7,$$

and

$$2\|\nabla\Delta u\|^2 \leq 2(C_1'\|\Delta^2 u\|^{\frac{3}{4}}\|u\|^{\frac{1}{4}} + C_2'\|u\|)^2 \leq \frac{1}{4}\|\Delta^2 u\|^2 + C_8.$$

Therefore, we have

$$-b(|\nabla u|^2, \Delta^2 u) \leq 2b^2\|\nabla u\|_4^4 + \frac{1}{8}\|\Delta^2 u\|^2 \leq \frac{1}{4}\|\Delta^2 u\|^2 + C_6,$$

and

$$-(u^3, \Delta^2 u) \leq \frac{1}{8}\|\Delta^2 u\|^2 + 2\|u\|_6^6 \leq \frac{1}{4}\|\Delta^2 u\|^2 + C_7.$$

Summing up, we get

$$\frac{d}{dt}\|\Delta u\|^2 + \frac{1}{2}\|\Delta^2 u\|^2 + 2a\|\Delta u\|^2 \leq 2(C_6 + C_7 + C_8), \tag{10}$$

where a is sufficiently large, it satisfies $a > 0$. Using Gronwall's inequality, we get

$$\|\Delta u\|^2 \leq e^{-2a(t-t_0')}\|\Delta u(t_0')\|^2 + \frac{C_6 + C_7 + C_8}{a} \leq \frac{2(C_6 + C_7 + C_8)}{a}, \tag{11}$$

for all $t \geq T_0' = \max\{T_0, t_0' + \frac{1}{2a}\ln\frac{aR^2}{2(C_6+C_7+C_8)}\}$. Setting $t \geq T_0''$, taking $s \in (t, t+1)$, integrating (l0) over $(s, t+1)$, we obtain

$$\|\Delta u(t+1)\|^2 \leq C + \|\Delta u(s)\|^2.$$

Integrating the above inequality with respect to s in $(t, t+1)$, using (7), we deduce that

$$\|\Delta u(t+1)\|^2 \leq C + \int_t^{t+1} \|\Delta u(s)\|^2 dx \leq C_9, \quad \forall t \geq T_0''. \tag{12}$$

By Sobolev's embedding theorem, Lemma 3.1, Lemma 3.2 and (11), we deduce that

$$\|u\|_\infty + \|\nabla u\|_6 \leq C_{10}. \tag{13}$$

Multiplying equation (1) by u_t, integrating the resulting relation over Ω, we derive that

$$\begin{aligned}
&\|u_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 - \frac{d}{dt}\|\nabla u\|^2 + \frac{a}{2}\frac{d}{dt}\|u\|^2 \\
=& - b(|\nabla u|^2, u_t) - (u^3, u_t) \\
\leq& |b|\|u_t\|\|\nabla u\|_4^2 + \|u\|_\infty^2\|u\|\|u_t\| \\
\leq& \frac{1}{2}\|u_t\|^2 + b^2\|\nabla u\|_4^4 + \|u\|_\infty^4\|u\|^2,
\end{aligned}$$

that is

$$\|u_t\|^2 + \frac{d}{dt}(\|\Delta u\|^2 + a\|u\|^2 - 2\|\nabla u\|^2) \leq C_{11}. \tag{14}$$

Integrating (14) over $(t+1, t+2)$, using Lemma 3.1, Lemma 3.2 and (12), we get

$$\int_{t+1}^{t+2} \|u_t\|^2 dx \leq C_{12}, \quad \forall t \geq T_0''.$$

Using a mean value theorem for integrals we obtain the existence of a time $t_1 \in (T_0''+1, T_0''+2)$ such that the following estimate holds uniformly

$$\|u_t(t_1)\|^2 \leq C_{13}.$$

Then, we complete the proof. □

Lemma 3.4 — Suppose that $u_0 \in \dot{H}^1_{per}(\Omega)$ and a is sufficiently large, then for problem (1)-(3), we have

$$\|\nabla\Delta u(t)\| \leq M_3, \ \ \forall t \geq T_3,$$

and

$$\int_t^{t+1} \|A^{\frac{1}{2}} u_t(t)\|^2 dt \leq M_3, \ \ \forall t \geq T_3.$$

Here, M_3 is a positive constant depending on a, b. T_3 depends on a, b and R where $\|u_0\|^2_{H^1} \leq R^2$.

PROOF : Acting the Laplace operator on (1), we obtain

$$\frac{\partial \Delta u}{\partial t} + \Delta^3 u + 2\Delta^2 u + a\Delta u + b\Delta|\nabla u|^2 + \Delta(u^3) = 0. \tag{15}$$

the equation (15) is supplemented with the boundary by the following boundary conditions

$$\varphi|_{x_i=0} = \varphi|_{x_i=L_i}, \quad i = 1, 2, 3, \tag{16}$$

for u and the derivatives of u at least of order≥ 2 and ≤ 5, By Nirenberg's inequality, we have

$$\|\Delta u\|_4 \leq C_1'\|\nabla\Delta^2 u\|^{\frac{1}{4}}\|\Delta u\|^{\frac{3}{4}} + C_2'\|\Delta u\|.$$

Multiplying (15) by $\Delta^2 u$ and integrating on Ω, using (13) and above inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta u\|^2 + \|\nabla\Delta^2 u\|^2 + a\|\nabla\Delta u\|^2 \\
=&2\|\Delta^2 u\|^2 + b(\Delta|\nabla u|^2, \Delta^2 u) + (\Delta u^3, \Delta^2 u) \\
=&2\|\Delta^2 u\|^2 - b(2\nabla u\Delta u, \nabla\Delta^2 u) - 3(u^2\nabla u, \nabla\Delta^2 u) \\
\leq&2\|\Delta^2 u\|^2 + 2|b|\|\nabla\Delta^2 u\|\|\nabla u\|_4\|\Delta u\|_4 + 3\|u\|^2_\infty\|\nabla u\|\|\nabla\Delta^2 u\| \\
\leq&\frac{1}{2}\|\nabla\Delta^2 u\|^2 + \frac{C_{14}}{2}.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\|\nabla\Delta u\|^2 + \|\nabla\Delta^2 u\|^2 + 2a\|\nabla\Delta u\|^2 \leq C_{14}. \tag{17}$$

By Gronwall's inequality, we immediately obtain

$$\|\nabla\Delta u(t)\|^2 \leq e^{-2a(t-t_0)}\|\nabla\Delta u(t_0)\|^2 + \frac{C_{14}}{2a} \leq \frac{C_{14}}{a}. \tag{18}$$

for all $t \geq T_1^* = \max\{T_1, t_0 + \frac{1}{2a}\ln\frac{C_{14}}{aC_{15}}\}$. Combining (6), (9), (11) and (18) together gives

$$\|\nabla u\|_\infty \leq C_{15}, \quad \|\Delta u\|_q \leq C_{16}, \quad \forall t \geq T_1^*, \tag{19}$$

Multiplying equation (15) by u_t, integrating the resulting relation over Ω, we obtain

$$\begin{aligned}
&\|\nabla u_t\|^2 + \frac{1}{2}\frac{d}{dt}(\|\nabla\Delta u\|^2 - 2\|\Delta u\|^2 + \|\nabla u\|^2) \\
\leq &|b||(\Delta|\nabla u|^2, u_t)| + |(\Delta u^3, u_t)| \\
\leq &2|b|\|\nabla u\|_\infty\|\Delta u\|\|\nabla u_t\| + 3\|u\|_\infty^2\|\nabla u\|\|\nabla u_t\| \\
\leq &\frac{1}{2}\|\nabla u_t\|^2 + \frac{C_{17}}{2},
\end{aligned}$$

that is

$$\|\nabla u_t\|^2 + \frac{d}{dt}(\|\nabla\Delta u\|^2 - 2\|\Delta u\|^2 + \|\nabla u\|^2) \leq C_{17}, \tag{20}$$

Letting $t \geq T_1^*$, taking $s \in (t, t+1)$, integrating the above inequality over $(s, t+1)$, we obtain

$$\|\nabla\Delta u(t+1)\|^2 \leq C_{17} + \|\nabla\Delta u(s)\|^2 + \|\Delta u(s)\|^2 + \|\nabla u(s)\|^2 + \|\Delta u(t+1)\|^2.$$

Integrating the above inequality with respect to s in $(t, t+1)$, we have

$$\|\nabla\Delta u(t+1)\|^2 \leq C_{17} + \int_t^{t+1}\|\nabla\Delta u(s)\|^2 ds + 2M_2 + M_1 \leq C_{18}, \forall t \geq T_1^*. \tag{21}$$

Integrating (20) over $(t+1, t+2)$, using (21), we get

$$\int_{t+1}^{t+2}\|A^{\frac{1}{2}}u_t\|^2 d\tau \leq C_{19}, \ \ \forall t \geq T_1^*.$$

Using a mean value theorem for integrals we obtain the existence of a time $t_2 \in (T_1^*+1, T_1^*+2)$ such that the following estimate holds uniformly

$$\|A^{\frac{1}{2}}u_t(t_2)\|^2 \leq C_0^{''}.$$

Then, we complete the proof. □

Lemma 3.5 — Suppose that $u_0 \in \dot{H}^1_{per}(\Omega)$ and a is sufficiently large, then for problem (1)-(3), we have

$$\|u_t\| \leq M_4, \ \ \forall t \geq T_4.$$

Here, M_4 is a positive constant depending on a, b. T_4 depends on a, b, and R where $\|u_0\|^2_{H^1_{per}} \leq R^2$.

PROOF : Setting $v = u_t$, differentiating (1) with respect to the time t, we deduce that

$$v_t + \Delta^2 v + 2\Delta v + av + b(|\nabla u|^2)_t + (u^3)_t = 0. \tag{22}$$

Multiplying (22) by v, integrating the resulting relation over Ω, we derive that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\Delta v\|^2 + a\|v\|^2 - 2\|\nabla v\|^2 + 2b(\nabla u \nabla v, v) + 3(u^2 v, v) = 0. \tag{23}$$

Using Sobolev's embedding theorem, noticing the boundary value conditions, we get

$$\begin{aligned} & 2\|\nabla v\|^2 - 2b(\nabla u \nabla v, v) - 3(u^2 v, v) \\ \leq & 2\|\nabla v\|^2 + 2|b|\|\nabla u\|_\infty \|\nabla v\|\|v\| + 3\|u\|^2_\infty \|v\|^2 \\ \leq & 3\|\nabla v\|^2 + b^2\|\nabla u\|^2_\infty \|v\|^2 + 3\|u\|^2_\infty \|v\|^2 \\ \leq & \frac{1}{2}\|\Delta v\|^2 + (\frac{9}{4} + b^2 C^2_{15} + 3C^2_{10})\|v\|^2. \end{aligned} \tag{24}$$

Adding (23) and (24) together gives

$$\frac{d}{dt}\|v\|^2 + \|\Delta v\|^2 + (2a - \frac{9}{2} - 2b^2 C^2_{15} - 6C^2_{10})\|v\|^2 \leq 0, \tag{25}$$

where a is sufficiently large, it satisfies $2a - \frac{9}{2} - 2b^2 C^2_{15} - 6C^2_{10} > 0$. Using Gronwall's inequality, we derive that

$$\|v\|^2 \leq e^{-\left(2a - \frac{9}{2} - 2b^2 C^2_{15} - 6C^2_{10}\right)(t - t_1)} \|v(t_1)\|^2 \leq C_{20}, \tag{26}$$

for all $t \geq t_1 + \frac{2}{4a - 9 - 4b^2 C^2_{15} - 12C^2_{10}} \ln \frac{C_{20}}{C_{13}}$. Then, the proof is completed. □

Lemma 3.6 — Suppose that $u_0 \in \dot{H}^1_{per}(\Omega)$ and a is sufficiently large, then for problem (1)-(3), we have

$$\|A^{\frac{1}{2}} v_t(t)\| \leq M_5, \ \ \forall t \geq T_5.$$

Here, M_5 is a positive constant depending on a, b. T_5 depends on a, b and R where $\|u_0\|^2_{H^1_{per}} \leq R^2$.

PROOF : Multiplying (22) by Av, integrating the resulting relation over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2+\|\nabla\Delta v\|^2+a\|\nabla v\|^2\\
\le&2\|\Delta v\|^2+|b(2\nabla u\nabla v,\Delta v)|+3|(u^2v,\Delta v)|\\
\le&2\|\Delta v\|^2+2|b|\|\nabla u\|_\infty\|\nabla v\|\|\Delta v\|+3\|u\|_\infty^2\|v\|\|\Delta v\|\\
\le&2\|\Delta v\|^2+|b|C_{15}(\|\nabla v\|^2+\|\Delta v\|^2)+\frac{3}{2}C_{10}^2(\|v\|^2+\|\Delta v\|^2) \qquad (27)\\
\le&C_{21}\|\Delta v\|^2+C_{22}\|v\|^2+|b|C_{15}\|\nabla v\|^2\\
\le&C_{21}\|\Delta v\|^2+C_{22}C_{20}+|b|C_{15}\|\nabla v\|^2\\
\le&\frac{1}{2}\|\nabla\Delta v\|^2+\frac{C_{21}^2}{2}\|\nabla v\|^2+C_{22}C_{20}+|b|C_{15}\|\nabla v\|^2.
\end{aligned}$$

Hence

$$\frac{d}{dt}\|\nabla v\|^2+\|\nabla\Delta v\|^2+(2a-C_{21}^2-2|b|C_{15})\|\nabla v\|^2\le 2C_{20}C_{22},$$

where a is sufficiently large, it satisfies $2a-C_{21}^2-2|b|C_{15}>0$. By Gronwall's ienquality, we can obtain

$$\begin{aligned}
\|\nabla v\|^2\le&e^{-[2a-C_{21}^2-2|b|C_{15}](t-t_2)}\|\nabla v(t_2)\|^2+\frac{2C_{20}C_{22}}{2a-C_{21}^2-2|b|C_{15}}\\
\le&C_0''e^{-[2a-C_{21}^2-2|b|C_{15}](t-t_2)}+\frac{2C_{20}C_{22}}{2a-C_{21}^2-2|b|C_{15}} \qquad (28)\\
\le&\frac{4C_{20}C_{22}}{2a-C_{21}^2-2|b|C_{15}},
\end{aligned}$$

for all $t\ge t_2+\frac{1}{2a-C_{21}^2-2|b|C_{15}}\ln\frac{C_0''[2a-C_{21}^2-2|b|C_{15}]}{2C_{20}C_{22}}$. Then, the proof is completed. □

Lemma 3.7 — Suppose that $u_0\in\dot{H}_{per}^1(\Omega)$ and a is sufficiently large, then for problem (1)-(3), we have

$$\|A^2u(t)\|\le M_6,\ \ \forall t\ge T_6.$$

Here, M_6 is a positive constant depending on γ. T_6 depends on γ and R where $\|u_0\|_{H_{per}^1}^2\le R^2$.

PROOF : For the equation (1), by Lemmas 3.1-3.6, we have

$$\|\Delta^2u\|\le(\|u_t\|+a\|u\|^2+2\|\nabla u\|^2+|b|\||\nabla u|^2\|+\|u^3\|)\le C_{22},\ \ \forall t\ge T.$$

On the other hand, by Sobolev's embedding theorem, we have

$$\|\Delta u\|_\infty\le C_{23}.$$

Then, the lemma is proved. □

4. PROOF OF THEOREM 2.5

Consider the problem (1)-(3), we first show that $\{S(t)\}_{t\geq 0}$ has an (H^1, H^1)-global attractor, and then prove this attractor is an (H^1, H^4)-attractor of problem (1)-(3).

We suppose that M_1 and M_6 are the constants in Lemma 3.2 and Lemma 3.7, respectively. Denote

$$B_1 = \{u \in \dot{H}^1_{per} : \|A^{\frac{1}{2}}u\| \leq M_1\}, \tag{29}$$

$$B_2 = \{u \in \dot{H}^4_{per} : \|A^2 u\| \leq M_6\}. \tag{30}$$

Using Lemmas 3.2 and 3.7, we can obtain B_1 is a bounded $(\dot{H}^1_{per}, \dot{H}^1_{per})$-absorbing set for $\{S(t)\}_{t\geq 0}$ and B_2 is a bounded $(\dot{H}^1_{per}, \dot{H}^4_{per})$-absorbing set for $\{S(t)\}_{t\geq 0}$. Note that the embedding $\dot{H}^4_{per} \hookrightarrow \dot{H}^1_{per}$ is compacted. Using Lemma 3.3, we obtain $\{S(t)\}_{t\geq 0}$ is $(\dot{H}^1_{per}, \dot{H}^1_{per})$-asymptotically compact. Hence, $\{S(t)\}_{t\geq 0}$ has an $(\dot{H}^1_{per}, \dot{H}^1_{per})$-global attractor $\mathcal{A}$. In the following, we show that $\mathcal{A}$ is actually an $(\dot{H}^1_{per}, \dot{H}^4_{per})$-global attractor for $\{S(t)\}_{t\geq 0}$.

Lemma 4.1 — Suppose that $u_0 \in \dot{H}^1_{per}(\Omega)$, a is sufficiently large, then for the solution $u(x,t)$ of the problem (1)-(3), the dynamical system $\{S(t)\}_{t\geq 0}$ is $(\dot{H}^1_{per}, \dot{H}^4_{per})$-asymptotically compact.

PROOF : By the equation (1), we can obtain

$$A^2 u = -u_t - 2\Delta u - au - b|\nabla u|^2 - u^3. \tag{31}$$

Assume $\{u_{0,n}\}_{n=1}^{\infty}$ is bounded in $\dot{H}^1_{per}(\Omega)$ and $t_n \to \infty$. In the following we prove that $\{S(t_n)u_{0,n}\}_{n=1}^{\infty}$ has a convergent subsequence in $\dot{H}^4_{per}(\Omega)$. Denote

$$u_n(t) = S(t)u_{0,n}, \text{ and } v_n(t_n) = \frac{du_n}{dt}|_{t=t_n}.$$

Note that $\{u_{0,n}\}_{n=1}^{\infty}$ is bounded in $\dot{H}^1_{per}$. Then, there exists $R > 0$ such that

$$\|u_{0,n} + A^{\frac{1}{2}}u_{0,n}\| \leq R, \ \forall n = 1, 2, \cdots.$$

By Lemmas 3.6 and 3.7, there exists $T > 0$ such that

$$\|v_n\|_{D(A^{\frac{1}{2}})} \leq M_5, \ \|u_n\|_{D(A^2)} \leq M_6, \ \forall t \geq T, \ n = 1, 2, \cdots. \tag{32}$$

Since $t_n \to \infty$, there exists $N > 0$ such that $t_n \geq T$ for all $n \geq N$. Therefore, by (32), we get

$$\|v_n(t_n)\|_{D(A^{\frac{1}{2}})} \leq M_5, \ \|u_n(t_n)\|_{D(A^2)} \leq M_6, \ \forall n \geq N. \tag{33}$$

Note that the embedding $D(A^{\frac{1}{2}}) \hookrightarrow H$ and $D(A^2) \hookrightarrow D(A)$ are compacted. Hence, by (32), there exist $v \in D(A^{\frac{1}{2}})$, $\Delta u \in D(A)$, $\nabla u \in \dot{H}^3_{per}$ and $u \in \dot{H}^4_{per}$ such that, up to a subsequence,

$$\begin{cases} v_n(t_n) \to v \text{ strongly in } H, \\ \Delta u_n(t_n) \to \Delta u \text{ strongly in } D(A^{\frac{1}{2}}), \\ \nabla u_n(t_n) \to \nabla u \text{ strongly in } D(A), \\ u_n(t_n) \to u \text{ strongly in } \dot{H}^3_{per}. \end{cases} \tag{34}$$

By (32) and Sobolev's embedding theorem, we obtain

$$\|u_n(t_n)\|_{W^{2,\infty}} \le C, \ \ \forall n \ge N.$$

It then follows from (32) and (34) that

$$\|u_n(t_n) - u\| \to 0, \ \ \|v_n(t_n) - v\|^2 \to 0, \ \ \|\Delta u_n(t_n) - \Delta u\|^2 \to 0,$$

and

$$\begin{aligned} &\||\nabla u_n(t_n)|^2 - |\nabla u|^2\| + \|(u_n(t_n))^3 - u^3\| \\ =&\|[\nabla u_n(t_n) + \nabla u][\nabla u_n(t_n) - \nabla u]\| + \|(u_n(t_n) - u)((u_n(t_n))^2 + u_n(t_n)u + u^2)\| \\ \le&\|\nabla u_n(t_n) + \nabla u\|_\infty \|\nabla u_n(t_n) - \nabla u\| + \|u_n(t_n) - u\|\|(u_n(t_n))^2 + u_n(t_n)u + u^2\|_\infty \\ \le&C(\|\nabla u_n(t_n) - \nabla u\| + \|u_n(t_n) - u\|) \to 0. \end{aligned}$$

Therefore

$$A^2 u_n(t_n) \to -v - 2\Delta u - au - b|\nabla u|^2 - u^3, \ \text{ strongly in } H,$$

that is $\{u_n(t_n)\}_{n=1}^\infty$ converges to $A^{-2}(-v-2\Delta u-au-b|\nabla u|^2-u^3)$ in $\dot{H}^4_{per}(\Omega)$. Then, we complete the proof. □

Now we give the proof of the main result.

PROOF OF THEOREM 2.5 : Note that $\{S(t)\}_{t\ge 0}$ has an $(\dot{H}^1_{per}, \dot{H}^1_{per})$-global attractor $\mathcal{A}$. By Lemma 3.7, B_2 is a bounded $(\dot{H}^1_{per}, \dot{H}^4_{per})$-absorbing set for $\{S(t)\}_{t\ge 0}$. On the other hand, by Lemma 4.1, we can obtain $\{S(t)\}_{t\ge 0}$ is $(\dot{H}^1_{per}, \dot{H}^4_{per})$-asymptotically compact. Then, by Proposition 2.4, $\mathcal{A}$ is actually an $(\dot{H}^1_{per}, \dot{H}^4_{per})$-global attractor for $\{S(t)\}_{t\ge 0}$. The proof of Theorem 2.5 is completed.

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