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CONTINUOUS DEPENDENCE OF THE VALUE FUNCTION ON A CLASS OF ONE-DIMENSIONAL AUTONOMOUS OPTIMAL CONTROL PROBLEMS AND EXISTENCE OF SOLUTIONS

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We give in this paper the proof of existence of solutions for a class of one-dimensional autonomous optimal control problems. The second result in the paper is about convergent subsequences of solutions to problems, we show that the value function depends continuously on the problems.

Key words : Optimal control problems; infinite horizon; value function; global attractor.

1. Introduction

Many problems in engineering, ecology and resources exploitation can be formulated as infinite horizon optimal control problems (P) of the form

$$\text{Maximize} \int_{t_0}^{\infty} e^{-\delta t} F(t, x(t), u(t)) dt$$

over measurable functions $u(.)$ and absolutely continuous functions $x(.)$ satisfying

$$\dot{x}(t) = f(t, x(t), u(t)) \text{ for a. e. } t \geq t_0, \tag{1.1}$$
$$u(t) \in U(t, x(t)) \text{ for a.e. } t \geq t_0,$$

$$x(t) \in \Omega(t) \text{ for } t \geq t_0,$$
$$x(t_0) = x_0, \tag{1.2}$$

where $t_0 \in \mathbb{R}$, $x = (x_1, ..., x_n)$ and $u = (u_1, ..., u_m)$ are respectively, the state and the control, $f : [t_0, +\infty[\times\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $F : [t_0, +\infty[\times\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are functions, for $t \geq t_0$, the sets $U = U(t, x(t)) \subseteq \mathbb{R}^m, \Omega(t) \subseteq \mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$. δ is a given positive discount rate. A measurable function $u(.) : [t_0, +\infty[\to \mathbb{R}^m$ satisfying $u(t) \in U(t, x(t))$ a.e. is called an admissible control (for (P)). An absolutely continuous function $x(.) : [t_0, +\infty[\to \mathbb{R}^n$ satisfying (1.1),(1.2) and $x(t) \in \Omega(t)$ for $t \geq t_0$ is called an admissible state trajectory (corresponding to $u(.)$). Suitable conditions on the data ensuring that, for a given control function $u(.)$ and initial data $(\tau, \xi) \in [t_0, +\infty[\times\mathbb{R}^n$, there is a unique absolutely continuous function $x(.) : [\tau, +\infty[\to \mathbb{R}^n$ satisfying $\dot{x}(t) = f(t, x(t), u(t))$ a.e. on $[\tau, +\infty[$, $x(\tau) = \xi$ and the previous integral makes sense. It is assumed in autonomous problems that $F(x, u)$ is measurable and locally bounded in x uniformly in u, thus, for any admissible pair (x, u) and $\delta > 0$, the integral $\int_0^\infty e^{-\delta t} F(x(t), u(t))dt$ makes sense.

The problem (P) is embedded in a family of optimal control problems $\{(A_{\tau,\xi}) : (\tau, \xi) \in [t_0, +\infty[\times\mathbb{R}^n\}$, parameterized by the initial time and state $(\tau, \xi) \in [t_0, +\infty[\times\mathbb{R}^n$:

$$(A_{\tau,\xi}) \text{ Maximize } \left\{ \begin{array}{c} \int_\tau^\infty e^{-\delta t} F(t, x(t), u(t))dt \; / \dot{x} = f(t, x, u), \\ u(t) \in U(t, x(t)) \text{ a.e. on } [\tau, +\infty[, \\ x(t) \in \Omega(t) \text{ for } t \geq \tau \text{ and } x(\tau) = \xi \} \end{array} \right\}.$$

In this case $(P) = (A_{t_0,x_0})$. With this family of problems, the value function is

$$V(\tau, \xi) = \sup\{(A_{\tau,\xi})\},$$

which describes the variation of the optimal rent functional for $(A_{\tau,\xi})$ when (τ, ξ) range over $[t_0, +\infty[\times\mathbb{R}^n$.

Dynamic programming approach leads to the Hamilton-Jacobi-Bellman (HJB) equation characterization of the value function. Because of the importance of this equation in optimal feedback control, it has been discussed in the literature for many years. Many of these discussions were devoted to show the existence of a unique non-smooth solution called a 'viscosity solution' see [7, 8, 11]. However, the (HJB) equation is, in general, not analytically solvable, there are in the literature only few numerical approximations and algorithm of resolution for the equation such as the one reported in [12]. In autonomous infinite horizon optimal control problem (P), the (HJB) equation can be considered as a first-order Hamilton-Jacobi equation,

$$H(x, \phi(x), \nabla\phi(x)) = 0 \text{ in } \Omega$$

where H is the hamiltonian and Ω an open set in $\mathbb{R}^n$.

For some classes of Hamilton-Jacobi equations, Trélat proves that, under suitable assumptions, the viscosity solution is sub-analytic see [15]. (HJB) operators have been the object of intensive study during the last years, for a general review of their theory and applications see [4-6, 10, 13, 14].

The value function may be characterized as the unique (generalized) solution of the Hamilton-Jacobi (HJ) equation. This important result in nonlinear deterministic control can take a number of guises depending on the class of optimal control problems considered and the notion of "generalized" solution adopted. As well known, (HJ) equations may fail to have C^1 solutions ϕ. It becomes necessary then to give meaning to functions ϕ drawn from a larger function class, which are solutions to (HJ) equations in some extended sense. Many concepts of extended solutions to (HJ) equations, each providing a characterization of V, have been proposed, like lower Dini solutions and others. For the most part, the various extended solution concepts (e.g. the various forms of viscosity solutions [7, 11]) coincide when the functions ϕ involved are locally Lipschitz continuous.

Thanks to [9], the solution $x(.)$ (optimal state trajectory) of the one state variable autonomous infinite horizon optimal control problem (P) must always be monotonic if the hamiltonian is strictly concave in the control u, this property concerning the state trajectory is useful in the rest of the paper and it has the following corollary,

Lemma 1.1 — Suppose the problem (P),

$$\begin{cases} \max \int\limits_0^\infty e^{-\delta t} F(x(t), u(t)) dt \\ \dot{x}(t) = u(t), \ t \geq 0 \\ x(0) = a \\ u : [0, +\infty) \to \mathbb{R}, \text{ measurable} \end{cases}$$

admits a solution $x(.)$. If F is strictly concave in u, then $x(.)$ is monotone.

PROOF : If F is strictly concave in u, then the hamiltonian is also strictly concave in u, from [9, p. 164] the solution $x(.)$ is monotone.

In this paper, we prove the existence of solutions for a class of optimal control problems we consider without the compactness condition on the set of controls. The second result in the paper is about convergent subsequences of solutions x_n to problems $P_n \in P$, we show that the value function V depends continuously on the problem P.

Specification of a problem set

From now on we shall consider but problems (P) of the following type:

$$\begin{cases} \max \int\limits_0^\infty e^{-\delta t} F(x(t), u(t)) dt \\ \dot{x}(t) = u(t),\ t \geq 0 \\ x(0) = a \\ u : [0, +\infty) \to \mathbb{R},\ \text{measurable} \\ u \in U \end{cases} .$$

The problem may have as possible interpretation the optimal exploitation stock, where $x(t)$ can be considered as a real valued capital stock at time t, $\dot{x}(t)$ the rate investment, U the set of the admissible controls; $F(x(t), u(t))$ the net income rate which result, δ is a fixed discount rate and the discounted integral of the net income, when maximized is the present value of the initial capital stock a.

We specify such a problem by a pair $(F, \delta) = P$ and an initial state a. If an initial state is fixed, the problem is denoted by $P(a)$. The family of problems $P(a)$ is denoted by $P = (F, \delta)$.

To further simplifying the analysis we consider only those $F(.,.)$ for which investment $\dot{x}(t) > 0$ for large $x(t)$ as well as disinvestment $\dot{x}(t) < 0$ for large $-x(t)$ is not profitable, and therefore the optimally steered stock $x(t)$ decreases for large initial stock $x(0)$ and increases for large $-x(0)$. We give the definition,

Definition 1.1 — An interval $[a, b]$ $(a < b)$ is called *(global) attractor* for F, if

1/ $F(x, 0) \geq F(y, u)$ whenever $b \leq x \leq y$, $0 \leq u$,

2/ $F(x, 0) \geq F(y, u)$ whenever $y \leq x \leq a$, $0 \geq u$.

Remark 1.1 : F admits a global attractor means that is not profitable to invest (take strategy $u \geq 0$) for large stock $x(t)$ ($x(t) > b$), because the net income F which result decreases, passing from $F(x, 0)$ to $F(y, u)$ once $b \leq x \leq y$ (see 1/ of definition 1.1). Therefore whenever we have at time t a large stock $x(t)$ ($x(t) > b$), we must disinvest (take strategy $u \leq 0$), this implies the optimally steered stock $x(t)$ decreases for large initial stock $x(0)$.

As well as, it is not profitable to disinvest (take strategy $u \leq 0$) for large $-x(t)$ ($x(t) < a$), because the net income F which result also decreases, passing from $F(x, 0)$ to $F(y, u)$ once $y \leq x \leq a$ (see 2/ of Definition1.1). Therefore whenever we have at time t a large stock$-$ $x(t)$ ($x(t) < a$), we must invest (take strategy $u \geq 0$), this implies the optimally steered stock $x(t)$ increases for large $-x(0)$.

In conclusion, on the right of the global attractor $[a, b]$ $(a < b)$ it's profitable to disinvest as well as to invest on his left.

Definition 1.2 — A set W of controls is called equi-integrable, if

1/ $\forall \epsilon > 0$ $\exists \delta > 0$ such that $\int\limits_A |w| < \epsilon$ $\forall A \subset \mathbb{R}$, measurable with $|A| < \delta$, $\forall w \in W$,

2/ $\forall \epsilon > 0$ $\exists B \subset \mathbb{R}$, measurable with $|B| < \infty$ such that $\int\limits_{\mathbb{R} \setminus B} |w| < \epsilon$ $\forall w \in W$.

Let $\mathcal{P}$ be the set of problems (F, δ) such that δ is a positive number, the set of the admissible controls U is equi-integrable and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is strictly concave in u and admits a global attractor.

Definition 1.3 — The function

$$V(a) = \sup \left\{ \begin{array}{c} \int\limits_0^\infty e^{-\delta t} F(x(t), u(t)) dt \; / u : [0, +\infty) \to \mathbb{R}, \text{ measurable,} \\ \dot{x} = u, u \in U \text{ and } x(0) = a \end{array} \right\},$$

is called the value of a (for the problem P) and $V = V_P$ is called the value function.

If $(x(.), u(.))$ is optimal, then

$$V(x(0)) = \int\limits_0^\infty e^{-\delta t} F(x(t), u(t)) dt.$$

2. Specification of a Topology on P and Preliminary Results

For open intervals I, J in $\mathbb{R}$ put

$$|F|_{I,J} = \sup_{x \in I, u \in J} |F(x, u)|.$$

For $(F, \delta) \in \mathcal{P}$, and open intervals I, J such that I contains an attractor for F, put

$$\mathcal{V}(F, \delta, I, J, \epsilon) = \left\{ \begin{array}{c} (F', \delta') \in \mathcal{P} / \left|\delta - \delta'\right| < \epsilon, I \text{ contains an attractor for } F' \\ \text{and } |F - F'|_{I,J} < \epsilon \end{array} \right\}.$$

Obviously the $\mathcal{V}(F, \delta, I, J, \epsilon)$ form a filter base of subsets $\mathcal{P}$ containing (F, δ).

Let τ be the topology generated by these sets i.e. the shortest topology such that $\mathcal{V}(F, \delta, I, J, \epsilon)$ form a base of the neighborhood filter.

τ is a Hausdorff topology admitting a countable base. It is this topology which will be considered for the rest of the article.

In the following lemma we show the existence of solution if both of control and trajectory state are bounded.

Lemma 2.1 — For every bounded intervals I_1, I_2 of the real line $\mathbb{R}$ with I_1 not separated from zero, the following problem $\bar{P}(a)$, $(a \in \mathbb{R})$ admits a solution

$$\begin{cases} \max \int\limits_0^\infty e^{-\delta t} F(x(t), \dot{x}(t)) dt \\ \dot{x}(t) = u(t),\ t \geq 0 \\ u(t) \in I_1,\ t \geq 0, \\ x(t) \in I_2,\ t \geq 0, \\ x(0) = a \end{cases}.$$

PROOF : Let $x(.)$ an absolutely continuous function with $x(0) = a$.

Put

$$\mathcal{V}(x(.)) = \int\limits_0^\infty e^{-\delta t} F(x(t), \dot{x}(t)) dt.$$

The value of a (for P) is

$$V(a) = \sup \left\{ \mathcal{V}(x(.)) / x(.) \text{ admissible}, x(0) = a \right\}.$$

There is a sequence of absolutely continuous $x_n(.)$ $(n \in \mathbb{N})$, with $x_n(0) = a$ and

$$V(a) = \lim_n \mathcal{V}(x_n(.)).$$

The sequence $x_n(.)$ is bounded in $L^\infty([0, +\infty))$. On the other part, the control u is supposed to be bounded. Therefore the sequence $x_n(.)$ is bounded in $W^{1,\infty}([0, +\infty))$, and hence, due to the theorem of Banach-Alaoglu-Bourbaki (see [2]), this sequence $x_n(.)$ converge for the topology weak-* to $x(.)$, up to a subsequence.

One deduce from the concavity of G that $x_n(.)$ converge uniformly to $x(.)$ which is solution of $\bar{P}(a)$ and $\lim_n \mathcal{V}(x_n(.)) = \mathcal{V}(x(.))$ Q.e.d.

Definition 2.1 — For $P(a), (P \in \mathcal{P}, a \in \mathbb{R})$, define $P^+(a)$ to be the problem

$$\begin{cases} \max \int_0^\infty e^{-\delta t} F(x(t), \dot{x}(t)) dt \\ \dot{x}(t) = u(t), \ t \geq 0 \\ u \in U \\ x : [0, +\infty[\to \mathbb{R} \text{ absolutely continuous} \\ \text{and either increasing or decreasing,} \\ x(0) = a \end{cases}.$$

Lemma 2.2 — $P(a)$ and $P^+(a)$ are equivalent i.e. have the same solutions.

PROOF : 1) Let $x^+(.) be$ solution of the problem $P^+(a)$, we show that $x^+(.)$ is also solution of the problem $P(a)$.

Let $x(.)$ be any absolutely continuous such that the function $t \to e^{-\delta t} F(x(t), \dot{x}(t))$ is integrable, $x(0) = a$, $\dot{x} \in U$ and

$$\mathcal{V}(x(.)) = \int_0^\infty e^{-\delta t} F(x(t), \dot{x}(t)) dt.$$

Using dominated convergence (theorem of Lebesgue) one notes that there is a sequence of absolutely continuous functions x_n such that

$$-n \leq x_n(t) \leq n, \ (t \geq 0),$$
$$-n \leq \dot{x_n}(t) \leq n, \ (t \geq 0) \text{ and}$$
$$\mathcal{V}(x(.)) = \lim_n \int_0^\infty e^{-\delta t} F(x_n(t), \dot{x}_n(t)) dt.$$

For each $n \in \mathbb{N}$, consider the problem $P^n(a)$:

$$\begin{cases} \max \int_0^\infty e^{-\delta t} F(x(t), \dot{x}(t)) dt \\ x : [0, +\infty[\to \mathbb{R} \text{ absolutely continuous,} \\ -n \leq x(t) \leq n, \ (t \geq 0), \\ -n \leq \dot{x}(t) \leq n, \ (t \geq 0), \\ x(0) = a \end{cases}.$$

From Lemma 2.1 we have each $P^n(a)$ admits a solution $y_n(.)$ and due to Lemma 1.1, $y_n(.)$ is either increasing or decreasing. Let

$$V_n(a) := \int_0^\infty e^{-\delta s} F(y_n(s), \dot{y}_n(s)) ds.$$

Hence V_n is the value function for P^n. Let

$$V(a) = \sup\{\mathcal{V}(x(.)) \;/\; x(.) \text{ admissible for } P^+(a)\}.$$

V is the value function for P^+. Obviously

$$V_n(a) \leq V_{n+1}(a) \;;\; \lim_n V_n(a) = V(a) \; (n \in \mathbb{N}).$$

Since V_n is the value function for P^n then we have for each $n \in \mathbb{N}$,

$$V_n(a) \geq \int_0^\infty e^{-\delta t} F(x_n(t), \dot{x}_n(t)) dt.$$

By passing to the limit, we deduce

$$V(a) \geq \mathcal{V}(x(.)).$$

But by assumption $x^+(.)$ is solution of the problem $P^+(a)$,

$$V(a) = \int_0^\infty e^{-\delta t} F(x^+(t), \dot{x^+}(t)) dt.$$

This proves that $x^+(.)$ is solution of the problem $P(a)$, since $x(.)$ is an arbitrary admissible for $P(a)$.

2) Conversely, let $x(.)$ be solution of the problem $P(a)$, from lemma1.1, we have $x(.)$ is either increasing or decreasing, then obviously $x^+(.)$ is solution to the problem $P^+(a)$

Therefore from 1/ and 2/, $P(a)$ and $P^+(a)$ are equivalent.

3. Main Results

***Theorem* 3.1** — *Every* $P(a), (P \in \mathcal{P}, a \in \mathbb{R})$ *admits a solution.*

PROOF : Using the above Lemma 2.2. it suffices to prove that $P^+(a)$ admits a solution.

We now prove a priory estimates for $P^+(a)$ using the assumption that there is an attractor $S = [s_1, s_2]$ for F.

For $y \in \mathbb{R}$ put

$$\underline{y} = \min(y, s_1), \overline{y} = \max(y, s_2), \overline{F_y} = \sup\{F(x, u) / \underline{y} \leq x \leq \overline{y}, u \in \mathbb{R}\}.$$

If x is increasing then $x(0) \leq x(t)$, $(t \geq 0)$. If however $x(0) \geq s_2$ then, the constant $x(0)$ satisfies

$$\frac{1}{\delta}F(x(0),0) = \int_0^\infty e^{-\delta t}F(x(0),u(0))dt$$
$$\geq \int_0^\infty e^{-\delta t}F(x(t),\dot{x}(t))dt.$$

Hence we may assume without loss of generality that $x(t) \leq \overline{(x(0))}$, $(t \geq 0)$, and by the same argument $\underline{x(0)} \leq x(t)$, $(t \geq 0)$.

That is, we do not alter the solution set of $P^+(a)$ if we add the constraint

$$\underline{a} = \underline{x(0)} \leq x(t) \leq \overline{x(0)} = \overline{a}, (t \geq 0). \tag{2.1}$$

If x is increasing we have

$$\int_0^t \left|\dot{x}(s)\right| ds = x(t) - x(0)$$
$$\leq s_1 - x(0), (t \geq 0).$$

If x is decreasing we have

$$\int_0^t \left|\dot{x}(s)\right| ds = -x(t) + x(0)$$
$$\leq x(0) - s_2, (t \geq 0).$$

Hence

$$\int_0^t \left|\dot{x}(s)\right| ds \leq \max(\left|s_1 - x(0)\right|, \left|x(0) - s_2\right|, (t \geq 0). \tag{2.2}$$

As for the value of $x(.)$ we always assume that

$$\frac{1}{\delta}F(x(0),0) \leq \int_0^\infty e^{-\delta t}F(x(t),\dot{x}(t))dt =: V(x)$$

$F(x(t),\dot{x}(t)) \leq \overline{F_{x(0)}}$ for all $t \geq 0$.

And therefore,

$$\begin{aligned}\int_0^\infty e^{-\delta t}\left|F(x(t),\dot{x}(t))\right|dt &\leq \int_0^\infty e^{-\delta t}\left|\overline{F_{x(0)}}-F(x(t),\dot{x}(t))\right|dt+\frac{1}{\delta}\left|\overline{F_{x(0)}}\right| \\ &= \frac{1}{\delta}\overline{F_{x(0)}}-V(x)+\frac{1}{\delta}\left|\overline{F_{x(0)}}\right| \\ &\leq \frac{2}{\delta}\left|\overline{F_{x(0)}}\right|-\frac{1}{\delta}F(x(0),0).\end{aligned}$$

Hence we have

$$\int_0^\infty e^{-\delta t}\left|F(x(t),\dot{x}(t))\right|dt \leq \frac{2}{\delta}\left|\overline{F_{x(0)}}\right|-\frac{1}{\delta}F(x(0),0). \tag{2.3}$$

We now prove existence of solutions to $P^+(a)$ using the estimates (2.1), (2.2), (2.3) and "the standard argument see [1]". Let

$$V(a)=\sup\left\{\begin{array}{c}\mathcal{V}(x(.))/x \text{ absolutely continuous and either increasing or decreasing,} \\ \dot{x}=u, u\in U \text{ and } x(0)=a\end{array}\right\}.$$

There is a sequence of absolutely continuous $x_n(.)$ $(n\in\mathbb{N})$, with $\dot{x}_n=u_n, u_n\in U$, $x_n(0)=a$ and

$$V(a)=\lim_n \mathcal{V}(x_n(.))$$

and such x_n is either increasing or decreasing. Passing to a subsequence we may without loss of generality assume that all x_n are say increasing. In any case we may assume that (2.1), (2.2), (2.3) hold for $x=x_n$ $(n\in\mathbb{N})$.

From (2.2) the sequence $\dot{x}_n$ is bounded in $L^1(\mathbb{R})$. On the other part, $\dot{x}_n=u_n, u_n\in U$ and the set of the admissible controls U is supposed to be equi-integrable, then $\left\{\dot{x}_n, n\in\mathbb{N}\right\}$ is equi-integrable. Therefore, due to the theorem of Dunford-Pettis (see [3]), we conclude that there is a subsequence of $\dot{x}_n$ which converges weakly in $L^1(\mathbb{R})$ to some function u which is Lebesgue integrable.

Hence

$$\begin{aligned}\lim_n x_n(t) &= a+\lim_n\int_0^t \dot{x_n}(s)ds \\ &= a+\lim_n\int_0^t u(s)ds =: x(t).\end{aligned}$$

Due to (2.3) we may also assume that the function

$$s \to e^{-\delta s} F(x_n(s), \dot{x_n}(s)) =: v_n(s)$$

converge to some Lebesgue integrable function $v(.)$. Put

$$y_n(t) = \int_0^t v_n(s)ds \;,\; y(t) = \int_0^t v(s)ds$$

$$H(t, \begin{pmatrix} y \\ x \end{pmatrix}) = \left\{ \begin{pmatrix} v \\ u \end{pmatrix} / u \in \mathbb{R}, v \in \mathbb{R}, v \leq e^{-\delta t} F(x,u) \right\}.$$

As $F(x,.)$ is concave, $H(t, \begin{pmatrix} y \\ x \end{pmatrix})$ is closed convex.

For every $n \in \mathbb{N}$:

$$\begin{pmatrix} \dot{y}_n(t) \\ \dot{x}_n(t) \end{pmatrix} \in H(t, \begin{pmatrix} y_n(t) \\ x_n(t) \end{pmatrix}) \text{ for almost all } t \geq 0.$$

The weak convergence of $\begin{pmatrix} \dot{y}_n(t) \\ \dot{x}_n(t) \end{pmatrix}$ implies that

$$\begin{pmatrix} \dot{y}(t) \\ \dot{x}(t) \end{pmatrix} \in H(t, \begin{pmatrix} y(t) \\ x(t) \end{pmatrix}) \text{ for almost all } t \geq 0.$$

Hence for all $t \geq 0$:

$$\dot{y}(t) \leq e^{-\delta t} F(x(t), \dot{x}(t))$$

and

$$\begin{aligned} V(a) &= \lim_n \int_0^\infty v_n(t)dt \\ &= \int_0^\infty v(t)dt \\ &\leq \int_0^\infty e^{-\delta t} F(x(t), \dot{x}(t))dt. \end{aligned}$$

Therefore,

$$V(a)=\int_0^\infty e^{-\delta t}F(x(t),\dot{x}(t))dt$$

and $x(.)$ is optimal for $P^+(a)$ and hence for $P(a)$ $\square$

The second result is about convergent subsequences of solutions x_n to problems $P_n \in P$. It show that the value function V depends continuously on the problem P.

***Theorem* 3.2** — *Let in $\mathcal{P}$ a sequence P_n and P such that $\lim_n P_n = P$, and let x_n optimal for P_n with $x_n(0) = a$ for all $n \in \mathbb{N}$.*

There is a subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} = x$ locally uniformly on $[0, +\infty)$ and x is optimal for P.

PROOF : Let x_n be optimal for $P_n(a), P_n = (F_n, \delta_n) \in \mathcal{P}$ and $P = \lim_n P_n = (F, \delta)$. There are bounded open intervals $I =]\sigma_1, \sigma_2[\subset J$ such that for large n every F_n admits an attractor $I_n \subset I, \lim_n \delta_n = \delta, \lim_n |F_n - F|_{I,J} = 0$. Since we may assume, e.g., that all x_n are increasing, this implies:

$$\min(a,\sigma_1)=\underline{a}\leq x_n(t)\leq\overline{a}=\max(a,\sigma_2),(t\geq 0).$$

From this, we deduce

$$\int_0^\infty \left|\dot{x}_n(s)\right|ds\leq\max(|s_1-\underline{a}|\,,|s_2-\bar{a}|),(n\in\mathbb{N})$$

with $[s_1, s_2]$ an attractor for F.

The proof then proceed's as does the proof of Theorem 3.1.

Corollary 3.1 — The value function V depends continuously on the problem $P, P \in \mathcal{P}$.

PROOF : If V_n resp. V is the value function for P_n resp. P, then using the above Theorem 3.2. we can deduce that $\lim_n V_n(b) = V(b)$ for all b, $b \in \mathbb{R}$.

CONCLUSION

It was shown in this paper that a problem $P(a)$, where P belongs to the family $\mathcal{P}$ we considered always admits a solution, and the value function V depends continuously on the problem P. Most problems of optimal exploitation or capital stocks or resource stocks admit global attractor for the net income function. There are, on the other hand, interesting problems without such an attractor, which is to say, that an extension of the above results to include these problems seems to be an important task.

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