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LOGARITHMIC STABILITY OF THE REFRACTIVE INDEX FOR THE ACOUSTIC EQUATION FROM BOUNDARY MEASUREMENTS

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We prove a stability estimate of logarithmic type for the inverse problem consisting in the determination of the refractive index of an obstacle from boundary measurements. We present a simple and direct proof, which is essentially based on a global Carleman inequality and the complex geometrical optics solutions.

Key words : Inverse problems; stability estimate of logarithmic type; refractive index; Carleman inequality; complex geometrical optics solutions.

1. INTRODUCTION AND MAIN RESULT

The paper deals with an inverse problem for the acoustic equation with fixed frequency. The object is to identify the associated refractive index from boundary measurements. We prove the logarithmic stability estimate. The proofs involve the use of the complex geometrical optics solutions and a global Carleman inequality.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain with C^2 boundary such that $\Omega \subset B(0, R)$, ($B(0, R)$ is the ball with radius $R > 0$). Let $\Gamma = \partial\Omega$ be a boundary of Ω. We denote $n(x) \in L^\infty(\Omega)$ the refractive index. There exists a positive constant M such that we have:

$$\|n\|_{L^\infty(\Omega)} \leq M. \tag{1.1}$$

We assume that $n(x)$ is known on a neighborhood of the boundary $\partial\Omega$ and $n = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. Consider the following problem, at fixed frequency ω, associated to the acoustic equation:

$$\begin{cases} P_n u = \left(\Delta + \omega^2 n\right) u = 0, \text{ in } \Omega, \\ u_{|\partial\Omega} = f \in H^{1/2}(\Gamma). \end{cases} \tag{1.2}$$

Define the local Dirichlet to Neumann map associated to n by:

$$\begin{array}{rcl}\Lambda_n : H^{1/2}(\Gamma) & \to & H^{-1/2}(\Gamma)\\ f & \mapsto & \dfrac{\partial u}{\partial \nu}|_{\Gamma},\end{array}$$

where ν is the outward unit vector normal to $\Gamma = \partial\Omega$.

The inverse problem under consideration is to recover information about the refractive index from boundary measurements. This is the typical problem where one wishes to know the interior properties of a medium by making measurements at the boundary.

It has shown that in dimension $d \geq 3$, an L^∞ potential n is uniquely determined by the D-to-N map [11]. The L^∞ hypothesis on n can even be relaxed to L^s with $s > d = 2$ see [6], and the smoothness assumption on $\partial\Omega$ can be relaxed to $C^{1,1}$ (see [10]). It has also been shown that in dimension $d = 2$, a $W^{1,\infty}$ potential n is uniquely determined by D-to-N map provided n is close to zero [13] and close to ”most potentials” [15]. Recently, Kenig-Sjöstrand-Uhlmann in [9] showed that the same result holds even if the measurement is taken on possibly a very small subset of the boundary. Bukhgeim and Uhlmann in [4] proved the uniqueness result for the general case $q \in L^\infty(\Omega)$ from knowledge of the Cauchy data measured on particular subsets of the boundary depending on the geometry of the domain. In case $d = 2$, Bukhgeim [3] has recently proved that any potential n is determined by D-to-N map.

The issue of stability was first addressed by Alessandrini in [1] for the full data problem who established a log-type stability estimate for the conductivity and later by Heck-Wang [12] who proved a log-log-type stability estimate when the data is measured on a subset that is slightly large than half of the boundary. In this paper, we proved a simple logarithmic stability, using a global Carleman estimate, when the data is measured on all of the boundary.

In the case where an arbitrary part of $\partial\Omega$ Ammari and Uhlmann [2] proved that the knowledge of the partial Cauchy data determines uniquely the potential n provided that n is known in a neighborhood of $\partial\Omega$. Under the same condition and when $n \in H^s$, $s > d/2$, Fathallah in [8] proved the stability estimate from infinity measurement on any open subset of $\partial\Omega$. Bukhgeim and Uhlmann [4] showed by using a local Carleman estimates in order to construct complex geometrical optics solutions vanishing on the complementary of $\Gamma_1 \subset \partial\Omega$, that the knowledge of the partial Cauchy data for the Schrödinger equation on any open subset of the boundary determines uniquely the potential n provided that $n \in L^\infty(\Omega)$ is known in a neighborhood of the boundary. In [7] it is proved a formula for calculating the refractive index n for the acoustic equation from the partial Dirichlet to Neumann map (DN) associated to n. These results are applied to identify locations and values of small volume perturbations of this refractive index at fixed frequency.

Before citing the main result of this paper, we introduce the following subsets of Ω:

$$\Omega_1 \subset \Omega_2 \subset B(0, \frac{3R}{8}) \subset \Omega \subset B(0, R).$$

And let $x_0 \in \Omega_1$, we note that

$$\Omega_2 \setminus \Omega_1 = \{x \in \Omega, \frac{R}{8} < |x - x_0| \leq \frac{R}{4}\},$$

and

$$\Omega \setminus B(0, \frac{3R}{8}) = \{x \in \Omega, \frac{3R}{8} < |x - x_0| \leq \frac{R}{2}\}.$$

Now, we cite the main result of this paper in the following theorem :

***Theorem* 1.1** — *Let u_1 and u_2 two solutions to (1.2) corresponding respectively to n_1 and n_2. There exist two positives constants $\beta \in (0,1)$ and $C = C(\Omega, \beta)$ such that the following estimate holds*

$$\|n_1 - n_2\|_{H^{-1}(\Omega)} \leq \frac{C}{\left|\ln\left(\|\Lambda_{n_1} - \Lambda_{n_2}\|_{L^2(\Gamma)}\right)\right|^{\beta}}.$$

This paper is organized as following. In Section 2, using a global carleman estimate, we prove an important estimate of the Fourier transform of the refractive index $n = n_1 - n_2$. Section 3 is devoted to the proof of Theorem 1.1.

2. Estimate of the Fourier Transform

In this section, we will get an estimate of the Fourier transform of $n = n_1 - n_2$. We begin by citing the following lemma:

Lemma 2.1 — Let $w = u_1 - u_2$ be the solution of the following boundary value problem:

$$\begin{cases} P_{n_1} w(x) = \omega^2 n(x) u_2(x) = F(x), \text{ in } \Omega, \\ w = 0, \text{ on } \Gamma, \end{cases}$$

where u_1 and u_2 two solutions to (1.2) corresponding respectively to the refractive index n_1 and n_2 and $F \in L^2(\Omega)$. Then, there exist five positives constants τ_0, σ, $C = C(\Omega_2, \Omega_1, \tau_0)$, $\alpha_1 = \alpha_1(\Omega_2, \Omega_1, \tau_0)$ and $\alpha_2 = \alpha_2(\Omega_2, \Omega_1, \tau_0)$ such that for any $\tau \geq \tau_0$ the following estimate holds

$$\|w\|_{H^1(\Omega_2 \setminus \Omega_1)} \leq C \left[e^{-\tau\alpha_1} \|w\|_{H^1(\Omega)} + e^{\tau\alpha_2} \left(\|\partial_\nu w\|_{L^2(\Gamma)} + \|F\|_{L^2(\Omega)}\right)\right],$$

where C, α_1, α_2 independent on n_1, F, w and τ and the subset $\Omega_2 \setminus \Omega_1$ is defined in the introduction of this paper.

PROOF : Let

$$0 \leq \psi(x) = \ln\left(\frac{R^2}{|x-x_0|^2}\right) \in C^2(\overline{\Omega}).$$

Then

$$|\nabla \psi(x)| = \frac{2}{|x-x_0|} \geq \frac{2}{R} = m', \ \ x \in \Omega.$$

We set $m = \min(1, m')$ and

$$M = \max\left(1, \sum_{|\alpha| \leq 2} \|\partial^\alpha \psi\|_{C(\overline{\Omega})}\right).$$

For any given parameter σ, we set

$$\varphi(x) = e^{\sigma\psi(x)}, \ x \in \Omega.$$

It is necessary to introduce a cut-off function satisfying $0 \leq \chi_1 \leq 1$, $\chi_1 \in C^\infty(\mathbb{R}^d)$ ($d = 2$ or 3) and

$$\chi_1 = \begin{cases} 1, \text{ in }, B(0, \frac{3R}{8}), \\ 0, \text{ in } \Omega_3 \subset \Omega \setminus B(0, \frac{3R}{8}), \end{cases}$$

Where $\Omega \setminus B(0, \frac{3R}{8})$ is defined in the introduction of this paper. We set $w_1 = \chi_1 w$, we obtain the following system:

$$\begin{cases} P_{n_1} w_1(x) = \chi_1(x)F(x) + Q_1(x,D)w, \text{ in } \Omega, \\ w = 0, \text{ on } \Gamma, \end{cases}$$

where $Q_1(x,D)$ is a first order operator supported in $\Omega \setminus B(0, \frac{3R}{8})$ such that

$$Q_1 w = 2\nabla\chi.\nabla + \Delta\chi w.$$

We apply Carleman estimate by Corollary 2.1 in [5] to w_1, we have for $\tau \geq \tau_0 = 88M^6/m^4$ and $w = 0$ on Γ:

$$\begin{aligned} C\int_{B(0,\frac{3R}{8})} e^{2\tau\varphi}(|w|^2 + |\nabla w|^2)dx &\leq \int_{B(0,\frac{3R}{8})} e^{2\tau\varphi}\,|Q_1 w|^2\,dx + \int_{B(0,\frac{3R}{8})} e^{2\tau\varphi}\,|F(x)|^2\,dx \\ &+ \int_\Gamma e^{2\tau\varphi}|\nabla w|^2 d\sigma_x. \end{aligned} \tag{2.1}$$

Using the properties of χ_1, we easily prove

$$\int_{B(0,\frac{3R}{8})} e^{2\tau\varphi} |Q_1 w|^2 \, dx \leq \frac{C}{R^2} \int_{\Omega\setminus B(0,\frac{3R}{8})} e^{2\tau\varphi} \left(|w|^2 + |\nabla w|^2\right) dx. \tag{2.2}$$

Using (2.2) in (2.1), we obtain

$$\begin{aligned} CR^2 \int_{\Omega_2\setminus\Omega_1} e^{2\tau\varphi}(|w|^2 + |\nabla w|^2)dx \quad &\leq \quad \int_{\Omega\setminus B(0,\frac{3R}{8})} e^{2\tau\varphi} \left(|w|^2 + |\nabla w|^2\right) dx + \int_\Omega e^{2\tau\varphi} |F(x)|^2 \, dx \\ &+ \quad \int_\Gamma e^{2\tau\varphi} |\nabla w|^2 d\sigma_x. \end{aligned} \tag{2.3}$$

Then, we have

$$\varphi = e^{\sigma \ln\left(R^2/|x-x_0|^2\right)} = \frac{R^{2\sigma}}{|x-x_0|^{2\sigma}}.$$

Consequently, we obtain

$$\begin{aligned} CR^2 e^{2\tau\varphi_0} \int_{\Omega_2\setminus\Omega_1} (|w|^2 + |\nabla w|^2)dx \quad &\leq \quad e^{2\tau\varphi_1} \int_{\Omega\setminus B(0,\frac{3R}{8})} \left(|w|^2 + |\nabla w|^2\right) dx \\ &+ \quad e^{2\tau\varphi_2} \left(\int_\Omega |F(x)|^2 \, dx + \int_\Gamma |\nabla w|^2 d\sigma_x\right), \end{aligned} \tag{2.4}$$

where

$$\varphi_0 = \frac{R^{2\sigma}}{\left(\frac{R}{4}\right)^{2\sigma}} = 4^{2\sigma}, \qquad \varphi_1 = \frac{R^{2\sigma}}{\left(\frac{3R}{8}\right)^{2\sigma}} = \left(\frac{8}{3}\right)^{2\sigma}, \qquad \varphi_2 = \|\varphi\|_\infty .$$

We note $\varphi_0 - \varphi_1 = \alpha_1 > 0$ and $\varphi_2 - \varphi_0 = \alpha_2 > 0.$
Then, by (2.4) we get

$$\begin{aligned} CR^2 \int_{\Omega_2\setminus\Omega_1} (|w|^2 + |\nabla w|^2)dx \quad &\leq \quad e^{-2\tau\alpha_1} \int_{\Omega\setminus B(0,\frac{3R}{8})} \left(|w|^2 + |\nabla w|^2\right) dx \\ &+ \quad e^{2\tau\alpha_2} \left(\int_\Omega |F(x)|^2 \, dx + \int_\Gamma |\nabla w|^2 d\sigma_x\right). \end{aligned} \tag{2.5}$$

Finally, by (2.5) we obtain the following estimate:

$$\|w\|_{H^1(\Omega_2\setminus\Omega_1)} \leq C \left[e^{-\tau\alpha_1} \|w\|_{H^1(\Omega)} + e^{\tau\alpha_2} \left(\|\partial_\nu w\|_{L^2(\Gamma)} + \|F\|_{L^2(\Omega)}\right)\right] .\Box$$

Let $-1 < \delta < 0$ and introduce the weighted L^2-space

$$L^2_\delta(\mathbb{R}^3) = \{f \in L^2_{loc}(\mathbb{R}^3), \int_{\mathbb{R}^3} (1 + |x|^2)^\delta |f(x)|^2 dx < +\infty\}.$$

From [14], we know that the solutions of $\left(\frac{1}{\omega^2}\Delta + n_i\right) u_i = 0$ on $\mathbb{R}^3$ can be given by:

$$u_i = e^{x.\rho_i}(1 + \psi_{n_i}(x, \rho_i)), i = 1, 2. \tag{2.6}$$

for $|\rho_i|$ sufficiently large with $\psi_{n_i}(., \rho_i) \in L^2_\delta(\mathbb{R}^3)$. Moreover, we have

$$\|\psi_{n_i}(., \rho_i)\|_{L^2_\delta(\mathbb{R}^3)} \leq \frac{C}{|\rho_i|}. \tag{2.7}$$

In the later of this paper, we note $s = \min(|\rho_1|, |\rho_2|)$.
Now, we cite an estimate of the Fourier transform of n in the following theorem:

***Theorem* 2.1** — *For any $s \geq s_0$ and $\xi \in \mathbb{R}^d$ such that $|\xi| \leq s$, there exist a positive constant C such that the following estimate:*

$$|\hat{n}(\xi)| \leq C\left(e^{Ns} \|w\|_{H^1(\Omega_2 \setminus \Omega_1)} + s^{-1}\right),$$

where $N = |x|$ and C independent of s and ξ.

PROOF : From (2.6), we have

$$\begin{aligned}
u_2\overline{u}_1 &= e^{x.(\rho_2+\tilde{\rho_1})}(1 + \psi_{n_1}(x, \rho_1))(1 + \psi_{n_2}(x, \rho_2)), \\
&= e^{x.(\rho_2+\tilde{\rho_1})}(1 + \psi_{n_2}(x, \rho_2) + \psi_{n_1}(x, \rho_1) + \psi_{n_1}(x, \rho_1)\psi_{n_2}(x, \rho_2)), \\
&= e^{ix\xi} + e^{ix\xi}(\psi_{n_2}(x, \rho_2) + \psi_{n_1}(x, \rho_1) + \psi_{n_1}(x, \rho_1)\psi_{n_2}(x, \rho_2)).
\end{aligned}$$

Then, we get:

$$\int_\Omega n(x) u_2\overline{u}_1 dx = \int_\Omega n(x) e^{ix\xi} dx + g_1(\xi, s), \tag{2.8}$$

where

$$g_1(\xi, s) = e^{ix\xi}(\psi_{n_2}(x, \rho_2) + \psi_{n_1}(x, \rho_1) + \psi_{n_1}(x, \rho_1)\psi_{n_2}(x, \rho_2)).$$

And from (2.7), we get

$$|g_1(\xi, s)| \leq \frac{C}{s}, \tag{2.9}$$

where C independent of s and ξ.

Now, we introduce a cut-off function satisfying $0 \leq \chi_2 \leq 1$, $\chi_2 \in C^\infty(\mathbb{R}^d)$ and

$$\chi_2 = \begin{cases} 1, \text{ in } \Omega \setminus \Omega_2, \\ 0, \text{ in } \Omega_1, \end{cases}$$

where Ω_2 and Ω_1 are defined in the introduction of this paper.

We set $\tilde{w}(x) = \chi_2 w(x)$, we obtain the following system

$$\begin{cases} P_{n_1}\tilde{w}(x) = \chi_2(x)\omega^2 n(x)u_2(x) + Q_2(x,D)w, \text{ in } \Omega, \\ \tilde{w} = 0, \text{ on } \Gamma, \end{cases} \tag{2.10}$$

where $Q_2(x, D)$ is a first order operator supported in $\Omega_2 \setminus \Omega_1$ such that

$$Q_2 w = 2\nabla\chi.\nabla + \Delta\chi w.$$

From the problem (2.10), we use the integration by parts to prove the Green's formula given by the following identity

$$\begin{aligned} \int_\Omega P_{n_1}\tilde{w}\overline{u}_1 dx &= \int_\Omega (\chi_2(x)\omega^2 n(x)u_2(x) + Q_2(x,D)w)\overline{u}_1 dx, \\ &= \int_\Omega \tilde{w}\overline{P_{n_1}u_1} dx, \\ &= 0. \end{aligned}$$

From (2.8), we have

$$\int_\Omega \chi_2(x)\omega^2 n(x)e^{ix\xi}dx + g_1(\xi,s) + \int_\Omega Q_2(x,D)w\overline{u}_1 dx = 0. \tag{2.11}$$

On the other hand, we have

$$\int_\Omega Q_2(x,D)w\overline{u}_1 dx \le C\,\|w\|_{H^1(\Omega_2\setminus\Omega_1)}\,\|u_1\|_{L^2(\Omega)}\,.$$

Or, from (2.6) and (2.7), we have

$$\|u_i\|_{L^2(\Omega)} \le Ce^{Ns},$$

where $N = |x|$.

Consequently, we have

$$\int_\Omega Q_2(x,D)w\overline{u}_1 dx \le Ce^{Ns}\,\|w\|_{H^1(\Omega_2\setminus\Omega_1)}\,. \tag{2.12}$$

From (2.12), (2.11) and (2.9), we get

$$\int_\Omega \chi_2(x)\omega^2 n(x)e^{ix\xi}dx \le C\left(e^{Ns}\,\|w\|_{H^1(\Omega_2\setminus\Omega_1)} + s^{-1}\right).$$

Then, we have

$$|\hat{n}(\xi)| \leq C\left(e^{Ns}\|w\|_{H^1(\Omega_2\setminus\Omega_1)} + s^{-1}\right),$$

where C independent of s and ξ. □

3. PROOF OF THE STABILITY ESTIMATE

3.1 PROOF OF THEOREM 1.1

PROOF: From Theorem 2.1 and Lemma 2.1, we obtain

$$|\hat{n}(\xi)| \leq C\left[e^{Ns}\left(e^{-\tau\alpha_1}\|w\|_{H^1(\Omega)} + e^{\tau\alpha_2}\left(\|\partial_\nu w\|_{L^2(\Gamma)} + \|F\|_{L^2(\Omega)}\right)\right) + s^{-1}\right]. \tag{3.1}$$

We have,

$$\|F\|_{L^2(\Omega)} \leq Ce^{Ns},$$

and

$$\|w\|_{H^1(\Omega)} \leq Cse^{Ns}.$$

Consequently, (3.1) becomes:

$$|\hat{n}(\xi)| \leq C\left[se^{2Ns-\tau\alpha_1} + e^{2Ns+\tau\alpha_2}\|\partial_\nu w\|_{L^2(\Gamma)} + e^{2Ns+\tau\alpha_2} + s^{-1}\right]. \tag{3.2}$$

For $\tau = \gamma s$ with γ large, we have

$$se^{2Ns-\tau\alpha_1} \leq se^{s(2N-\gamma\alpha_1)} \leq e^{-\alpha_3 s} \leq \frac{1}{s},$$

where $\alpha_3 > 0$ independent of s.
And

$$e^{2Ns+\tau\alpha_2} \leq e^{\alpha_4 s},$$

where $\alpha_4 > 0$ independent of s.
Then, (3.2) becomes

$$|\hat{n}(\xi)| \leq C\left[e^{\alpha_4 s}\|\partial_\nu w\|_{L^2(\Gamma)} + e^{\alpha_4 s} + s^{-1}\right]. \tag{3.3}$$

On the other hand, for $0 < R < s$ we have

$$\begin{aligned}
\|n\|^2_{H^{-1}(\mathbb{R}^d)} &= \int_{|\xi|\leq R} |\hat{n}(\xi)|^2 < \xi >^{-2} d\xi + \int_{|\xi|>R} |\hat{n}(\xi)|^2 < \xi >^{-2} d\xi,\\
&\leq C\left(R^d sup\left|< \xi >^{-1}\hat{n}(\xi)\right|^2 + \frac{1}{R^2}\|n\|^2_{L^2(\Omega)}\right),\\
&\leq C\left(R^d\left\|< \xi >^{-1}\hat{n}(\xi)\right\|^2_{L^\infty(B(0,R))} + \frac{1}{R^2}\right).
\end{aligned}$$

From (3.3) and (1.1), we have

$$\|n\|_{H^{-1}(\Omega)} \le C\left(R^d\left(e^{\alpha_4 s}\|\partial_\nu w\|_{L^2(\Gamma)} + e^{\alpha_4 s} + s^{-1}\right) + \frac{1}{R^2}\right). \tag{3.4}$$

By (3.4) and choosing $R = s^{\frac{2}{(d+4)}}$, there exist two positive constants $\beta \in (0,1)$ independent of s and α_5 such that we have

$$\|n\|_{H^{-1}(\Omega)} \le C\left[e^{\alpha_5 s}\|\partial_\nu w\|_{L^2(\Gamma)} + e^{\alpha_5 s} + s^{-\beta}\right]. \tag{3.5}$$

We use the temporary notation $D = \|\partial_\nu w\|_{L^2(\Gamma)}$. The function $s \mapsto \frac{1}{s^\beta} + De^{\alpha_5 s} + e^{\alpha_5 s}$ attains its minimum at $\tilde{s}$ satisfying

$$-\frac{\beta}{\tilde{s}^{\beta+1}} + \alpha_5 De^{\alpha_5 \tilde{s}} + \alpha_5 e^{\alpha_5 \tilde{s}} = 0.$$

Using the elementary inequality $s^\varrho \le e^{\varrho s}$, $s \ge 1$, $\varrho > 0$, we obtain

$$\frac{1}{D} = \alpha_5 \frac{\tilde{s}^{\beta+1}e^{\alpha_5 \tilde{s}}}{\beta - \alpha_5 \tilde{s}^{\beta+1}e^{\alpha_5 \tilde{s}}} \le e^{\alpha_5(\beta+2)\tilde{s}}, \ \text{ if } \ \tilde{s} \ge 1.$$

That is

$$\frac{1}{\alpha_5(\beta+2)} \ln \frac{1}{D} \le \tilde{s}, \ \text{ if } \ \tilde{s} \ge 1. \tag{3.6}$$

We can take $s = \tilde{s}$ in (3.6) and taking into account (3.5), we easily obtain

$$\|n\|_{H^{-1}(\Omega)} \le \frac{C}{\left|\ln\left(\|\partial_\nu w\|_{L^2(\Gamma)}\right)\right|^\beta},$$

for another $\beta \in (0,1)$ independent of s. □

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