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ON SOME EQUALITIES AND INEQUALITIES FOR K-FRAMES

Fahimeh Arabyani Neyshaburi*, Ghadir Mohajeri Minaei** and Ehsan Anjidani***

Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

***Technical and Vocational University, Neyshabur Branch, Iran*

****Department of Mathematics, University of Neyshabur, P.O. Box* 91136-899, *Neyshabur, Iran*

e-mails: arabianif@yahoo.com; gh-minaei1391@yahoo.com; anjidani@neyshabur.ac.ir

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K-frame theory was recently introduced to reconstruct elements from the range of a bounded linear operator K in a separable Hilbert space. This significant property is worthwhile especially in some problems arising in sampling theory. Some equalities and inequalities have been established for ordinary frames and their duals. In this paper, we continue and extend these results to obtain several important equalities and inequalities for K-frames. Moreover, by applying Jensen's operator inequality we obtain some new inequalities for K-frames.

Key words : Frames; K-frames; Parseval K-frames; K-duals, Jensen's operator inequality.

1. Introduction and Preliminaries

Frames are redundant systems in separable Hilbert spaces, which provide non-unique representations of vectors and are applied in a wide range of applications [4-7]. Working on efficient algorithms for computing reconstruction of signals without noisy phase, Balan *et al*. [3] discover some new identities for Parseval frames. Then, these identities have been generalized by some researchers for the alternate dual frames and other types of frames [2, 11, 18, 19].

Recently, K-frames were introduced by Găvruţa [13] to study atomic systems with respect to a bounded operator $K \in B(\mathcal{H})$. Recall that atomic decomposition for a closed subspace $\mathcal{H}_0$ of a Hilbert space $\mathcal{H}$ introduced by Feichtinger *et al*. [10] with frame-like properties. Although, the sequences in atomic decompositions do not necessarily belong to $\mathcal{H}_0$. This interesting property, which was the main motivation of introducing atomic decomposition and K-frame theory, comes from some problems in sampling theory [15, 16]. In this work, we extend and improve these results to obtain some equalities and inequalities for K-frames. First, we are going to state some preliminaries of K-frames and their duals, which are used in our main results. Let $\mathcal{H}$ be a separable Hilbert space and I a countable index set and $K \in B(\mathcal{H})$. A sequence $F := \{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1.1}$$

Obviously, F is an ordinary frame if $K = I_{\mathcal{H}}$ and so K-frames are a generalization of the ordinary frames [7, 8, 13]. The constants A and B in (1.1) are called the lower and the upper bounds of F, respectively. A K-frame F is called a Parseval K-frame, whenever $\sum_{i\in I} |\langle f, f_i\rangle|^2 = \|K^*f\|^2$. Since F is a Bessel sequence, similar to ordinary frames the synthesis operator can be defined as $T_F : l^2 \to \mathcal{H}$; $T_F(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$. The operator T_F is bounded and its adjoint which is called the analysis operator is given by $T_F^*(f) = \{\langle f, f_i\rangle\}_{i\in I}$, and the frame operator is given by $S_F : \mathcal{H} \to \mathcal{H}$; $S_F f = T_F T_F^* f = \sum_{i\in I} \langle f, f_i\rangle f_i$. Unlike ordinary frames, the frame operator of a K-frame is not invertible in general. However, if K has closed range, then S_F from $R(K)$ onto $S_F(R(K))$ is an invertible operator [17].

In [1], the notion of duality for K-frames is introduced and several approaches for construction and characterization of K-frames and their dual are presented. Indeed, a Bessel sequence $\{g_i\}_{i\in I} \subseteq \mathcal{H}$ is called a K-*dual* of $\{f_i\}_{i\in I}$ if

$$Kf = \sum_{i\in I} \langle f, g_i\rangle f_i, \quad (f \in \mathcal{H}). \tag{1.2}$$

***Theorem* 1.1** — (Douglas [9]). *Let $L_1 \in B(\mathcal{H}_1, \mathcal{H})$ and $L_2 \in B(\mathcal{H}_2, \mathcal{H})$ be bounded linear mappings on given Hilbert spaces. Then the following assertions are equivalent:*

(i) $R(L_1) \subseteq R(L_2)$;

(ii) $L_1 L_1^* \le \lambda^2 L_2 L_2^*$, *for some* $\lambda > 0$;

(iii) There exists a bounded linear mapping $X \in B(\mathcal{H}_1, \mathcal{H}_2)$, such that $L_1 = L_2 X$.

Moreover, if (i), (ii) and (iii) are valid, then there exists a unique operator X so that $L_1 = L_2 X$ and

(a) $\|X\|^2 = \inf\{\alpha > 0, L_1 L_1^* \le \alpha L_2 L_2^*\}$;

(b) $N(L_1) = N(X)$;

(c) $R(X) \subset \overline{R(L_2^*)}$.

Remark 1.2 : Suppose $F = \{f_i\}_{i \in I}$ is a K-frame for $\mathcal{H}$ with the optimal bounds A and B, respectively. Clearly $B = \|S_F\|$. Also, the lower inequality (1.1) implies that

$$AKK^* \leq S_F = T_F T_F^*.$$

Hence, by taking $L_1 = K$ and $L_2 = T_F$ in Douglas' theorem, it follows that there exists an operator $X \in B(\mathcal{H}, l^2)$ such that

$$T_F X = K, \tag{1.3}$$

and $\{X^* \delta_i\}_{i \in I}$ is a K-dual of F, where $\{\delta_i\}_{i \in I}$ is the standard orthonormal basis of l^2, see [13]. Moreover, by Douglas' theorem the equation (1.3) has a unique solution as X_F such that

$$\|X_F\|^2 = \inf\{\alpha > 0, KK^* \leq \alpha T_F T_F^*\}. \tag{1.4}$$

Now, let A be the optimal lower bound of F. Then, we obtain

$$\begin{aligned} A &= \sup\{\alpha > 0 : \quad \alpha KK^* f \leq T_F T_F^*\} \\ &= (\inf\{\beta > 0 : \quad KK^* \leq \beta T_F T_F^*\})^{-1} \\ &= \|X_F\|^{-2}, \end{aligned}$$

which coincides with ordinary frames. Indeed, if $K = I_{\mathcal{H}}$, we easily obtain $X_F = T_F^* S_F^{-1}$ and so $\|X_F\|^{-2} = \|T_F^* S_F\|^{-2} = \|S_F^{-1}\|^{-1}$.

For further information in K-frame theory we refer the reader to [1, 10, 13, 17]. Throughout this paper, we suppose $\mathcal{H}$ is a separable Hilbert space, $K^\dagger$ the pseudo inverse of operator K, I a countable index set and for every $J \subset I$, we denote the complement of J by J^c. Also $I_{\mathcal{H}}$ denotes the identity operator on $\mathcal{H}$. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$, and we abbreviate $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. Also we denote the range of $K \in B(\mathcal{H})$ by $R(K)$, and the orthogonal projection of $\mathcal{H}$ onto a closed subspace V of $\mathcal{H}$ by π_V. Finally, we use of T_J and S_J to denote the synthesis operator and frame operator, whenever the index set is limited to $J \subset I$.

2. MAIN RESULTS

In this section we obtain several equalities and inequalities for K-frames, K-duals and Parseval K-frames. These results extend and improve some important results of [2, 11, 19].

***Theorem* 2.1** — *Let $F=\{f_i\}_{i\in I}$ be a K-frame for $\mathcal{H}$ and $\{g_i\}_{i\in I}$ be a K-dual of F. Then for every $J\subseteq I$ and $f\in\mathcal{H}$,*

$$\left(\sum_{i\in J}\langle f,g_i\rangle\overline{\langle Kf,f_i\rangle}\right)-\left\|\sum_{i\in J}\langle f,g_i\rangle f_i\right\|^2=\left(\sum_{i\in J^c}\overline{\langle f,g_i\rangle}\langle Kf,f_i\rangle\right)-\left\|\sum_{i\in J^c}\langle f,g_i\rangle f_i\right\|^2.$$

PROOF : Suppose that $J\subseteq I$, and consider the operator

$$M_Jf=\sum_{i\in J}\langle f,g_i\rangle f_i,\quad (f\in\mathcal{H}).$$

One can easily show that M_J is a well defined and bounded operator on $\mathcal{H}$. Moreover, we have $M_J+M_{J^c}=K$. Hence,

$$\begin{aligned}&\left(\sum_{i\in J}\langle f,g_i\rangle\overline{\langle Kf,f_i\rangle}\right)-\left\|\sum_{i\in J}\langle f,g_i\rangle f_i\right\|^2\\
&=\langle K^*M_Jf,f\rangle-\langle M_J^*M_Jf,f\rangle\\
&=\langle (K^*-M_J^*)M_Jf,f\rangle\\
&=\langle M_{J^c}^*(K-M_{J^c})f,f\rangle\\
&=\langle M_{J^c}^*Kf,f\rangle-\langle M_{J^c}^*M_{J^c}f,f\rangle\\
&=\langle f,K^*M_{J^c}f\rangle-\|M_{J^c}f\|^2\\
&=\left(\sum_{i\in J^c}\overline{\langle f,g_i\rangle}\langle Kf,f_i\rangle\right)-\left\|\sum_{i\in J^c}\langle f,g_i\rangle f_i\right\|^2.\square\end{aligned}$$

Using Theorem 2.1 we immediately obtain the following corollary.

Corollary 2.2 — Let $F=\{f_i\}_{i\in I}$ be a K-frame with a K-dual $\{g_i\}_{i\in I}$. Then for every $J\subseteq I$ and $f\in\mathcal{H}$,

$$\begin{aligned}&\langle T_J\{(X_Ff)_i\}_{i\in J},Kf\rangle-\|T_J\{(X_Ff)_i\}_{i\in J}\|^2\\
&=\overline{\langle T_{J^c}\{(X_Ff)_i\}_{i\in J^c},Kf\rangle}-\|T_{J^c}\{(X_Ff)_i\}_{i\in J^c}\|^2,\end{aligned}$$

where X_F is as in (1.4).

***Theorem* 2.3** — *Let $F=\{f_i\}_{i\in I}$ be a K-frame for $\mathcal{H}$ and $\{g_i\}_{i\in I}$ be a K-dual of F. Then for every bounded sequence $\{\alpha_i\}_{i\in I}$ and $f\in\mathcal{H}$,*

$$\begin{aligned}&\left(\sum_{i\in I}\alpha_i\langle f,g_i\rangle\overline{\langle Kf,f_i\rangle}\right)-\left\|\sum_{i\in I}\alpha_i\langle f,g_i\rangle f_i\right\|^2\\
&=\left(\sum_{i\in I}(1-\overline{\alpha_i})\overline{\langle f,g_i\rangle}\langle Kf,f_i\rangle\right)-\left\|\sum_{i\in I}(1-\alpha_i)\langle f,g_i\rangle f_i\right\|^2.\end{aligned}$$

In the sequel, we survey some inequalities on Parseval K-frames. These results also extend some important inequalities for frames in [2, 11].

***Theorem* 2.4** — *Assume that $F=\{f_i\}_{i\in I}$ is a Parseval K-frame for $\mathcal{H}$. For every $f\in\mathcal{H}$, $J\subseteq I$ and $E\subseteq J^c$ we have*

$$\begin{aligned}&\left\|\sum_{i\in J\cup E}\langle f,f_i\rangle f_i\right\|^2-\left\|\sum_{i\in J^c\setminus E}\langle f,f_i\rangle f_i\right\|^2\\ =\;&\left\|\sum_{i\in J}\langle f,f_i\rangle f_i\right\|^2-\left\|\sum_{i\in J^c}\langle f,f_i\rangle f_i\right\|^2+2Re\sum_{i\in E}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}.\end{aligned}$$

PROOF : For the subset J of I, we denote by S_J the operator defined by $S_Jf=\sum_{i\in J}\langle f,f_i\rangle f_i$. Hence we have $S_J+S_{J^c}=KK^*$. Hence $S_J^2-S_{J^c}^2=KK^*S_J-S_{J^c}KK^*$. In fact

$$\begin{aligned}S_J^2-S_{J^c}^2&=S_J^2-(KK^*-S_J)^2\\&=KK^*S_J+S_JKK^*-(KK^*)^2\\&=KK^*S_J-(KK^*-S_J)KK^*\\&=KK^*S_J-S_{J^c}KK^*.\end{aligned}$$

Hence, for every $f\in\mathcal{H}$ we obtain

$$\|S_Jf\|^2-\|S_{J^c}f\|^2=\langle KK^*S_Jf,f\rangle-\langle S_{J^c}KK^*f,f\rangle,$$

and consequently

$$\begin{aligned}&\left\|\sum_{i\in J\cup E}\langle f,f_i\rangle f_i\right\|^2-\left\|\sum_{i\in J^c\setminus E}\langle f,f_i\rangle f_i\right\|^2\\ =\;&\sum_{i\in J\cup E}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}-\overline{\sum_{i\in J^c\setminus E}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}}\\ =\;&\sum_{i\in J}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}-\overline{\sum_{i\in J^c}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}}+2Re\sum_{i\in E}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}\\ =\;&\left\|\sum_{i\in J}\langle f,f_i\rangle f_i\right\|^2-\left\|\sum_{i\in J^c}\langle f,f_i\rangle f_i\right\|^2+2Re\sum_{i\in E}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}.\square\end{aligned}$$

***Theorem* 2.5** — *Let $F = \{f_i\}_{i\in I}$ be a Parseval K-frame for $\mathcal{H}$. Then for every $J \subseteq I$ and $f \in \mathcal{H}$,*

$$\begin{aligned} & Re\left(\sum_{i\in J^c}\langle f, f_i\rangle\overline{\langle KK^*f, f_i\rangle}\right) + \left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2 \\ = \; & Re\left(\sum_{i\in J}\langle f, f_i\rangle\overline{\langle KK^*f, f_i\rangle}\right) + \left\|\sum_{i\in J^c}\langle f, f_i\rangle f_i\right\|^2 \geq \frac{3}{4}\|KK^*f\|^2. \end{aligned}$$

PROOF : Since $S_J^2 - S_{J^c}^2 = KK^*S_J - S_{J^c}KK^*$, we can write

$$S_J^2 + S_{J^c}^2 = 2\left(\frac{KK^*}{2} - S_J\right)^2 + \frac{(KK^*)^2}{2} \geq \frac{(KK^*)^2}{2},$$

and consequently

$$\begin{aligned} & KK^*S_J + S_{J^c}^2 + (KK^*S_J + S_{J^c}^2)^* \\ = \; & KK^*S_J + S_{J^c}^2 + S_JKK^* + S_{J^c}^2 \\ = \; & KK^*(S_J + S_{J^c}) + S_J^2 + S_{J^c}^2 \\ = \; & (S_J + S_{J^c})KK^* + S_J^2 + S_{J^c}^2 \geq \frac{3}{2}(KK^*)^2. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} & Re\left(\sum_{i\in J^c}\langle f, f_i\rangle\overline{\langle KK^*f, f_i\rangle}\right) + \left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2 \\ = \; & Re\left(\sum_{i\in J}\langle f, f_i\rangle\overline{\langle KK^*f, f_i\rangle}\right) + \left\|\sum_{i\in J^c}\langle f, f_i\rangle f_i\right\|^2 \\ = \; & \frac{1}{2}\left(\langle KK^*S_Jf, f\rangle + \langle S_{J^c}^2f, f\rangle + \langle f, KK^*S_Jf\rangle + \langle f, S_{J^c}^2f\rangle\right) \\ \geq \; & \frac{3}{4}\|KK^*f\|^2. \end{aligned}$$

This completes the proof. □

Notice that, in Theorem 2.5 if we take $K = I_{\mathcal{H}}$ it reduce to Theorem 2.2 [19]. Also, Theorem 2.5 leads to the following concept, which is a generalization of [11] for Parseval frames. Let $F = \{f_i\}_{i\in I}$ be a Parseval K-frame. Then we consider

$$v_+(F, K, J) = \sup_{f\neq 0}\frac{Re\left(\sum_{i\in J^c}\overline{\langle f, f_i\rangle}\langle KK^*f, f_i\rangle\right) + \left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2}{\|KK^*f\|^2},$$

$$v_-(F, K, J) = \inf_{f\neq 0}\frac{Re\left(\sum_{i\in J^c}\overline{\langle f, f_i\rangle}\langle KK^*f, f_i\rangle\right) + \left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2}{\|KK^*f\|^2}.$$

Now, we are going to present some properties of these notations. For this we need the next lemma.

Lemma 2.6 — Let K be a closed range operator and $F = \{f_i\}_{i\in I}$ be a K-frame for $\mathcal{H}$. Then

(i) $\left\|\sum_{i\in I}\langle f, f_i\rangle f_i\right\|^2 \leq \|S_F\| \sum_{i\in I} |\langle f, f_i\rangle|^2, \quad (f \in \mathcal{H})$;

(ii) $\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq \|K^\dagger\|^2\|X_F\|^2 \left\|\sum_{i\in I}\langle f, f_i\rangle f_i\right\|^2, \quad (f \in R(K))$, where X_F is as in equation (1.1).

PROOF : The first part is a known result for every Bessel sequence [8] and so for K-frames. Hence, we only need to show (ii). Let $f \in R(K)$. Then we have

$$
\begin{aligned}
\left(\sum_{i\in I} |\langle f, f_i\rangle|^2\right)^2 &= |\langle S_F f, f\rangle|^2 \\
&\leq \|S_F f\|^2\|f\|^2 \\
&= \|S_F f\|^2\|(K^\dagger)^* K^* f\|^2 \\
&\leq \|S_F f\|^2\|(K^\dagger)\|^2\|K^* f\|^2 \\
&\leq \|S_F f\|^2\|(K^\dagger)\|^2\|X_F\|^2 \sum_{i\in I} |\langle f, f_i\rangle|^2. \square
\end{aligned}
$$

***Theorem* 2.7** — *Suppose $F = \{f_i\}_{i\in I}$ is a Parseval K-frame for $\mathcal{H}$. The following assertion hold:*

(i) $\frac{3}{4} \leq v_-(F, K, J) \leq v_+(F, K, J) \leq \|K\|\|K^\dagger\|(1 + \|K\|)$

(ii) $v_+(F, K, J) = v_+(F, K, J^c)$, *and* $v_-(F, K, J) = v_-(F, K, J^c)$

(iii) $v_+(F, K, I) = v_-(F, K, I) = 1$, *and* $v_+(F, K, \emptyset) = v_-(F, K, \emptyset) = 1$.

PROOF : For (i), it is sufficient to show the upper inequality. Since F is a Bessel sequence, so by applying Lemma 2.6 (i) we obtain

$$
\begin{aligned}
\left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2 &\leq \|S_J\| \sum_{i\in J} |\langle f, f_i\rangle|^2 \\
&\leq \|S_J\| \sum_{i\in I} |\langle f, f_i\rangle|^2 \\
&\leq \|K\|^2\|K^* f\|^2 \\
&= \|K\|^2\|K^\dagger K K^* f\|^2 \\
&\leq \|K\|^2\|K^\dagger\|^2\|K K^* f\|^2.
\end{aligned}
$$

Moreover,

$$\begin{aligned} Re\left(\sum_{i\in J^c}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}\right) &\leq \left(\sum_{i\in I}|\langle f,f_i\rangle|^2\right)^{1/2}\left(\sum_{i\in I}|\langle KK^*f,f_i\rangle|^2\right)^{1/2} \\ &= \|K^*f\|\|K^*KK^*f\| \\ &= \|K^\dagger KK^*f\|\|K^*KK^*f\| \\ &\leq \|K\|\|K^\dagger\|\|KK^*f\|^2. \end{aligned}$$

Hence,

$$v_-(F,K,J)\leq v_+(F,K,J)\leq \|K\|\|K^\dagger\|(1+\|K\|\|K^\dagger\|).$$

On the other hand, in the proof of Theorem 2.4 we observed that

$$KK^*S_J - S_{J^c}KK^* = S_J^2 - S_{J^c}^2.$$

and consequently we have

$$\langle S_J^2 f,f\rangle + \langle S_{J^c}KK^*f,f\rangle = \langle S_{J^c}^2 f,f\rangle + \langle KK^*S_J f,f\rangle,$$

for every $f\in\mathcal{H}$. Thus

$$\begin{aligned} &\left\|\sum_{i\in J}\langle f,f_i\rangle f_i\right\|^2 + \overline{\sum_{i\in J^c}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}} \\ =\ &\left\|\sum_{i\in J^c}\langle f,f_i\rangle f_i\right\|^2 + \sum_{i\in J}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}. \end{aligned}$$

This shows (ii). Finally (iii) is easy to check. □

Using the result above-mentioned, we establish some equivalent results for Parseval K-frames.

Corollary 2.8 — Let $\{f_i\}_{i\in I}$ be a Parseval K-frame for $\mathcal{H}$. Then for every $J\subset I$ and $f\in\mathcal{H}$ the following conditions are equivalent.

(i) $v_+(F,K,J) = v_-(F,K,J) = 1$.

(ii) $\left\|\sum_{i\in J}\langle f,f_i\rangle f_i\right\|^2 = Re\sum_{i\in J}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}$.

(iii) $\left\|\sum_{i\in J^c}\langle f,f_i\rangle f_i\right\|^2 = Re\sum_{i\in J^c}\langle f,f_i\rangle\overline{\langle KK^*f,f_i\rangle}$.

PROOF : $(ii) \Leftrightarrow (iii)$ is clear. Also, $(i) \Rightarrow (ii)$ holds by a direct computation. Now, let (ii) holds, then

$$\begin{aligned}\sum_{i\in J^c}\overline{\langle f, f_i\rangle}\langle KK^*f, f_i\rangle + \left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2 &= \sum_{i\in I}\overline{\langle f, f_i\rangle}\langle KK^*f, f_i\rangle \\ &= \langle KK^*f, S_F f\rangle \\ &= \|KK^*f\|^2 .\end{aligned}$$

i.e., (i) holds. Hence $(i) \Leftrightarrow (iii)$ and similarly $(i) \Leftrightarrow (iii)$. □

Finally, we obtain the following more general result.

Corollary 2.9 — Let $\{f_i\}_{i\in I}$ be a Parseval K-frame for $\mathcal{H}$. Then for every $J \subset I$ and $f \in \mathcal{H}$ the following conditions are equivalent.

(i) $\left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2 = \sum_{i\in J}\langle f, f_i\rangle\overline{\langle KK^*f, f_i\rangle}$.

(ii) $\left\|\sum_{i\in J^c}\langle f, f_i\rangle f_i\right\|^2 = \sum_{i\in J^c}\langle f, f_i\rangle\overline{\langle KK^*f, f_i\rangle}$.

(iii) $S_J f \perp S_{J^c} f$.

(iv) $f \perp S_{J^c} S_J f$.

PROOF : Since

$$\|S_J f\|^2 - \langle KK^*S_J f, f\rangle = \|S_{J^c} f\|^2 - \langle KK^*S_{J^c} f, f\rangle,$$

we have $(i) \Leftrightarrow (ii)$. Moreover, S_J and S_{J^c} are positive, so we have

$$\langle S_{J^c} f, S_J f\rangle = \langle f, S_{J^c} S_J f\rangle = \langle (S_J KK^* - S_J^2) f, f\rangle, \quad (f \in \mathcal{H}).$$

This follows that, $(iii) \Leftrightarrow (iv)$ and $(i) \Leftrightarrow (iv)$. □

3. Some New Inequalities

In this section, applying Jensen's operator inequality we obtain some new inequalities for K-frames. First, recall that a continuous function h defined on an interval L is said to be operator convex if

$$h(\lambda A + (1-\lambda)B) \le \lambda h(A) + (1-\lambda)h(B),$$

for all $\lambda \in [0, 1]$ and all self-adjoint operators A,B with spectra in L. A general formulation of Jensen's operator inequality is given as follows (see [12]).

***Theorem* 3.1** — *Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces and $h : L \to \mathbb{R}$ be an operator convex function. For each $n \in \mathbb{N}$, the inequality*

$$h(\sum_{i=1}^{n} \Phi_i(A_i)) \leq \sum_{i=1}^{n} \Phi_i(h(A_i)), \tag{3.1}$$

holds for every n-tuple $(A_1, ..., A_n)$ of self-adjoint operators in $B(\mathcal{H})$ with spectra in L and every n-tuple $(\Phi_1, ..., \Phi_n)$ of positive linear mappings $\Phi_i : B(\mathcal{H}) \to B(\mathcal{K})$ such that $\sum_{i=1}^{n} \Phi_i(I_{\mathcal{H}}) = I_{\mathcal{K}}$.

Also, a variant of Jensen's operator inequality (3.1) for convex functions is presented as follows (see [14]).

***Theorem* 3.2** — *Let $A_1, ..., A_n$ be self-adjoin operators with spectra in $[m, M]$ for some scalars $m < M$ and $\Phi_1, ..., \Phi_n$ be positive linear mappings from $B(\mathcal{H})$ into $B(\mathcal{K})$ with $\sum_{i=1}^{n} \Phi_i(I_{\mathcal{H}}) = I_{\mathcal{K}}$, where $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ are the identity mappings on $\mathcal{H}$ and $\mathcal{K}$, respectively. If $h : [m, M] \to \mathbb{R}$ is a continuous convex function, then*

$$h\left(mI_{\mathcal{K}} + MI_{\mathcal{K}} - \sum_{i=1}^{n} \Phi_i(A_i)\right) \leq h(m)I_{\mathcal{K}} + h(M)I_{\mathcal{K}} - \sum_{i=1}^{n} \Phi_i(h(A_i)). \tag{3.2}$$

Now, we apply inequalities (3.1) and (3.2) to obtain some inequalities for K-frames.

***Theorem* 3.3** — *Suppose $F = \{f_i\}_{i\in I}$ is a Parseval K-frame for $\mathcal{H}$. Then for every subset J of I,*

(i) *if $h : [0, \|K\|^2] \to \mathbb{R}$ is an operator convex function, then*

$$h\left(\frac{KK^*}{2}\right) \leq \frac{h(S_J) + h(S_{J^c})}{2}; \tag{3.3}$$

(ii) *if $h : [0, \|K\|^2] \to \mathbb{R}$ is a convex function, then*

$$h\left(\|K\|^2 I_{\mathcal{H}} - \frac{KK^*}{2}\right) \leq h(\|K\|^2 I_{\mathcal{H}}) - \frac{h(S_J) + h(S_{J^c})}{2}. \tag{3.4}$$

PROOF : Let $A_1 = S_J$ and $A_2 = S_{J^c}$. Since S_J and S_{J^c} are positive operators with $S_J + S_{J^c} = KK^*$, the spectra of A_1 and A_2 are in $[0, \|K\|^2]$. Now, define $\Phi_1 = \frac{1}{2}I_{B(\mathcal{H})}$ and $\Phi_2 = \frac{1}{2}I_{B(\mathcal{H})}$, where $I_{B(\mathcal{H})}$ is the identity mapping on $B(\mathcal{H})$, and put $m = 0$ and $M = \|K\|^2$. Then, assertions (i) and (ii) follow from inequalities (3.1) and (3.2), respectively.

Corollary 3.4 — Suppose $F = \{f_i\}_{i\in I}$ is a Parseval K-frame for $\mathcal{H}$. Then for every subset J of I and $f \in \mathcal{H}$, we obtain

$$\begin{aligned}\frac{1}{2}\|KK^*f\|^2 &\leq \left\|\sum_{i\in J}\langle f, f_i\rangle f_i\right\|^2 + \left\|\sum_{i\in J^c}\langle f, f_i\rangle f_i\right\|^2 \\ &\leq 2\|K\|^2\|K^*f\|^2 - \frac{1}{2}\|KK^*f\|^2. \end{aligned} \tag{3.5}$$

PROOF : Since the function $h(x) = x^2$ is convex on $[0, \infty)$, by Theorem 3.3 we have

$$\frac{(KK^*)^2}{4} \leq \frac{S_J^2 + S_{J^c}^2}{2}, \tag{3.6}$$

and

$$\left(\|K\|^2 I_{\mathcal{H}} - \frac{KK^*}{2}\right)^2 \leq \|K\|^4 I_{\mathcal{H}} - \frac{S_J^2 + S_{J^c}^2}{2},$$

and so

$$\frac{S_J^2 + S_{J^c}^2}{2} \leq \|K\|^2 KK^* - \frac{(KK^*)^2}{4}. \tag{3.7}$$

Therefore, it follows from (3.6) and (3.7) that

$$\frac{(KK^*)^2}{2} \leq S_J^2 + S_{J^c}^2 \leq 2\|K\|^2 KK^* - \frac{(KK^*)^2}{2}.$$

Hence for every $f \in \mathcal{H}$, we obtain inequality (3.5). □

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