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INEQUALITIES CONCERNING RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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In this paper, we establish some inequalities for rational functions with prescribed poles which generalize and refine the results of Li, Mohapatra and Rodriguez, Aziz and Shah, Qasim and Liman and others. A result of O'Hara and Rodriguez for self-inversive polynomials is also extended to rational functions by one of our result as a special case.

Key words : Rational function; polynomial; poles; zeros; Bernstein's inequality.

1. INTRODUCTION

Let $\mathbb{P}_n$ denote the set of all complex polynomials $P(z) = \sum_{\nu=0}^{n} b_\nu z^\nu$ of degree at most n and $P'(z)$ its derivative. Let $\mathbb{D}_{k-}$ denote the region inside the circle $\mathbb{T}_k = \{z;\ |z| = k > 0\}$ and $\mathbb{D}_{k+}$ the region outside $\mathbb{T}_k$. For f defined on $\mathbb{T}_k$, in the complex plane $\mathbb{C}$, we set

$$M(f,k) := \max_{z\in\mathbb{T}_k} |f(z)| \ \text{ and } \ m^*(f,k) := \min_{z\in\mathbb{T}_k} |f(z)|.$$

If $P \in \mathbb{P}_n$, then concerning the estimate of $M(P',1)$ on $\mathbb{T}_1$, we have the following famous Bernstein's inequality (for reference see [8]),

$$M(P',1) \le nM(P,1). \tag{1.1}$$

Equality holds in (1.1) only for $P(z) = \lambda z^n, \lambda \neq 0$. Noting that these extremal polynomials have all zeros at the origin, so it is natural to seek improvements under appropriate condition on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials having no zeros in the unit circle $\mathbb{T}_1$, then (1.1) can be improved. In fact, Erdös conjectured and later Lax [5] verified that if $P(z) \neq 0$ in $\mathbb{D}_{1-}$, then (1.1) can be replaced by

$$M(P',1) \le \frac{n}{2}M(P,1). \tag{1.2}$$

For the class of polynomials $P \in \mathbb{P}_n$, not vanishing in $\mathbb{D}_{1+}$, we have

$$M(P', 1) \geq \frac{n}{2} M(P, 1). \tag{1.3}$$

The above inequality is due to Turán [9].

For $a_\nu \in \mathbb{C},\ \nu = 1, 2, ..., n$, let

$$W(z) := \prod_{\nu=1}^{n} (z - a_\nu)$$

and

$$B(z) := \prod_{\nu=1}^{n} \Big(\frac{1 - \overline{a}_\nu z}{z - a_\nu}\Big),$$

the Blaschke product.

Let

$$\mathbb{R}_n := \mathbb{R}_n(a_1, a_2, ..., a_n) = \left\{ \frac{P(z)}{W(z)} :\ P \in \mathbb{P}_n \right\}.$$

Then $\mathbb{R}_n$ is the set of rational functions with possible poles $a_1, a_2, ..., a_n$ and having a finite limit at ∞. Note that $B(z) \in \mathbb{R}_n$ and $|B(z)| = 1$ for $z \in \mathbb{T}_1$.

In the past few years, several papers pertaining to Bernstein type inequalities for rational functions have appeared in the study of rational approximation problems. In fact in 1995, Li *et al*. [6] proved some inequalities similar to (1.1), (1.2) and (1.3) for rational functions with poles outside the unit circle. Among other things, they proved the following generalization of (1.1).

***Theorem* A** — *Suppose $\lambda \in \mathbb{T}_1$. Let $t_1, t_2, ..., t_n$ be the n zeros of $B(z) + \lambda$ and $s_1, s_2, ..., s_n$ be the n zeros of $B(z) - \lambda$. If $r \in \mathbb{R}_n$ and $z \in \mathbb{T}_1$, then*

$$|r'(z)| \leq \frac{|B'(z)|}{2} \left\{ \max_{1 \leq k \leq n} |r(t_k)| + \max_{1 \leq k \leq n} |r(s_k)| \right\}. \tag{1.4}$$

Equality holds in (1.4) for $r(z) = \mu B(z)$ with $\mu \in \mathbb{T}_1$.

In the same paper, Li *et al*. [6] extended (1.3) to a class of rational functions by proving the following result.

***Theorem* B** — *Suppose $r \in \mathbb{R}_n$ has exactly n zeros and all lie in $\mathbb{T}_1 \cup \mathbb{D}_{1-}$. Then for $z \in \mathbb{T}_1$,*

$$|r'(z)| \geq \frac{|B'(z)|}{2} |r(z)|. \tag{1.5}$$

Equality holds in (1.5) for $r(z) = \alpha B(z) + \beta$ *with* $\alpha,\ \beta \in \mathbb{T}_1$.

They also extended (1.2) to rational functions by proving the following result.

***Theorem* C** — *Suppose* $r \in \mathbb{R}_n$ *has all its zeros in* $\mathbb{T}_1 \cup \mathbb{D}_{1+}$. *Then for* $z \in \mathbb{T}_1$, *we have*

$$|r'(z)| \leq \frac{|B'(z)|}{2} M(r, 1). \tag{1.6}$$

Equality holds in (1.6) for $r(z) = \alpha B(z) + \beta$ *with* $\alpha,\ \beta \in \mathbb{T}_1$.

Further, Aziz and Shah [3] proved the following result similar to Theorem A for rational functions not vanishing in $\mathbb{D}_{1-}$ which also generalizes a polynomial inequality due to Aziz ([1], Theorem 4).

***Theorem* D** — *Let* $r \in \mathbb{R}_n$ *and all the zeros of* r *lie in* $\mathbb{T}_1 \cup \mathbb{D}_{1+}$. *If* $\mu_1, \mu_2, ..., \mu_n$ *are the zeros of* $B(z) + \lambda$ *and* $t_1, t_2, ..., t_n$ *are the zeros of* $B(z) - \lambda$, *where* $\lambda \in \mathbb{T}_1$, *then for* $z \in \mathbb{T}_1$,

$$|r'(z)| \leq \frac{|B'(z)|}{2} \left\{ \left(\max_{1 \leq k \leq n} |r(\mu_k)| \right)^2 + \left(\max_{1 \leq k \leq n} |r(t_k)| \right)^2 \right\}^{\frac{1}{2}}. \tag{1.7}$$

The result is best possible and equality holds in (1.7) for $r(z) = B(z) + \lambda$ *with* $\lambda \in \mathbb{T}_1$.

Recently, Qasim and Liman [4] considered a specialized class of rational functions $r(s(z))$, defined by

$$(r \circ s)(z) = r(s(z)) := \frac{P(s(z))}{W(s(z))},$$

where $s(z)$ is a polynomial of degree m and $r \in \mathbb{R}_n$, so that $r(s(z)) \in \mathbb{R}_{mn}$, and

$$W(s(z)) = \prod_{j=1}^{mn} (z - a_j).$$

Also the Blaschke product is given by

$$B(z) = \frac{W^*(s(z))}{W(s(z))} = z^{mn} \frac{\overline{W(s(\frac{1}{\overline{z}}))}}{W(s(z))} = \prod_{j=1}^{mn} \big(\frac{1 - \overline{a}_j z}{z - a_j}\big).$$

They proved several sharp results concerning Bernstein type inequalities for rational functions.

The main aim of this paper is to prove some compact generalizations of Theorems A, B, C and D. We shall also generalize and improve some results of Qasim and Liman [4]. As we will see a result of O'Hara and Rodriguez for self-inversive polynomials is also extended to rational functions by one of our result as a special case.

2. Main Results

From now on, we shall always assume that all poles $a_1, a_2, ..., a_{mn}$ of $r(s(z))$ lie in $\mathbb{D}_{1+}$. For the case when all poles are in $\mathbb{D}_{1-}$, we can obtain analogous results with suitable modification.

***Theorem* 1** — *Let $r(s(z)) \in \mathbb{R}_{mn}$ and $r(s(z)) \neq 0$ in $\mathbb{D}_{k+}$, where $k \leq 1$. If $\max_{z\in\mathbb{T}_1} |s(z)| = M'$, then for $z \in \mathbb{T}_1$, we have*

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{(1+k)}\right\}|r(s(z))|. \tag{2.1}$$

The result is best possible for $k = 1$ and equality holds in (2.1) for rational functions of the form $r(s(z)) = \alpha B(z) + \beta$ where $\alpha,\ \beta \in \mathbb{T}_1$ and $s(z) = z^m$.

Remark 1 : The special case of the above theorem with $k = 1$ was proved by Qasim and Liman ([4], Corollary 3).

For $k = 1$, the above theorem simplifies to inequality (1.5) where $s(z) = z$.

***Theorem* 2** — *Let $r(s(z)) \in \mathbb{R}_{mn}$ and $r(s(z)) \neq 0$ in $\mathbb{D}_{k+}$, where $k \leq 1$. If $\max_{z\in\mathbb{T}_1} |s(z)| = M'$, then for $z \in \mathbb{T}_1$, we have*

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{(1+k)}\right\}\Big(|r(s(z))| + m^*(r\circ s, k)\Big). \tag{2.2}$$

The result is best possible for $k = 1$ and equality holds in (2.2) for rational functions of the form $r(s(z)) = B(z) + \lambda$ where $\lambda \in \mathbb{T}_1$ and $s(z) = z^m$.

For $k = 1$, the above Theorem 2 reduces to the following result which generalizes and refines inequality (1.5).

Corollary 1 — Let $r(s(z)) \in \mathbb{R}_{mn}$ and $r(s(z)) \neq 0$ in $\mathbb{D}_{1+}$. If $\max_{z\in\mathbb{T}_1} |s(z)| = M'$, then for $z \in \mathbb{T}_1$, we have

$$|r'(s(z))| \geq \frac{|B'(z)|}{2mM'}\Big\{|r(s(z))| + m^*(r\circ s, 1)\Big\}. \tag{2.3}$$

The result is best possible and equality in (2.3) holds for $r(s(z)) = B(z) + \lambda$ where $\lambda \in \mathbb{T}_1$ and $s(z) = z^m$.

Remark 2 : For $s(z) = z$, the above Corollary 1 simplifies to a result of Aziz and Shah ([3], Theorem 3).

Next, we prove the following result for rational functions not vanishing in the unit circle, and having prescribed poles, which, in particular, provides a refinement and generalization of (1.7).

***Theorem* 3** — *Let $r(s(z)) \in \mathbb{R}_{mn}$ be such that $r(s(z)) \neq 0$ in $\mathbb{D}_{1-}$ and all the zeros of $s(z)$ lie in $\mathbb{D}_{1-}$. If $\mu_1, \mu_2, ..., \mu_{mn}$ are the zeros of $B(z) + \lambda$ and $t_1, t_2, ..., t_{mn}$ are the zeros of $B(z) - \lambda$, where $\lambda \in \mathbb{T}_1$, then for $z \in \mathbb{T}_1$,*

$$|r'(s(z))| \leq \frac{|B'(z)|}{2mm'}\left\{\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2 + \left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2 - 2m^{*2}\right\}^{\frac{1}{2}}, \tag{2.4}$$

where $m^* = \min_{z\in\mathbb{T}_1}|r(s(z))|$ and $m' = \min_{z\in\mathbb{T}_1}|s(z)|$.

Remark 3 : If we let $s(z) = z$ in (2.4), we will improve (1.7).

We now prove the following result which should be compared with the polynomial inequality proved by Malik [7]. More precisely, we have

***Theorem* 4** — *Let $r(s(z)) \in \mathbb{R}_{mn}$ and $r^*(s(z)) = B(z)r(\overline{s(\frac{1}{\bar{z}})})$, where $s(z)$ has all zeros in $\mathbb{D}_{1-}$. Then, we have*

$$\begin{aligned}\frac{\max_{z\in\mathbb{T}_1}|r(s(z))|}{\max_{z\in\mathbb{T}_1}|s(z)|} &\leq m\max_{z\in\mathbb{T}_1}\left\{\left|\frac{r'(s(z))}{B'(z)}\right| + \left|\frac{r^{*'}(s(z))}{B'(z)}\right|\right\} \\ &\leq \frac{\max_{z\in\mathbb{T}_1}|r(s(z))|}{\min_{z\in\mathbb{T}_1}|s(z)|}.\end{aligned} \tag{2.5}$$

If we take $s(z) = z$ in Theorem 4, we get the following generalization of a result of Malik [7] where the polynomial $P(z)$ is replaced by a rational function $r(z)$ with prescribed poles and the factor z^n by a Blaschke product $B(z)$.

Corollary 3 — Let $r \in \mathbb{R}_n$ and $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$, then we have

$$\max_{z\in\mathbb{T}_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| + \left|\frac{r^{*'}(z)}{B'(z)}\right|\right\} = \max_{z\in\mathbb{T}_1}|r(z)|. \tag{2.6}$$

A rational function $r \in \mathbb{R}_n$ is said to be self-inversive if $r^*(z) = \zeta r(z)$ for some $\zeta \in \mathbb{T}_1$. If we assume $r \in \mathbb{R}_n$ to be self-inversive, we get the following result from Corollary 3, which extends a result of O' Hara and Rodriguez for self-inversive polynomials.

Corollary 4 — Let $r \in \mathbb{R}_n$ be a self-inversive rational function, then

$$\max_{z\in\mathbb{T}_1}\left|\frac{r'(z)}{B'(z)}\right| = \frac{1}{2}\max_{z\in\mathbb{T}_1}|r(z)|. \tag{2.7}$$

Finally, we establish the following result which, as a special case, provides a generalization of polynomial inequality due to Aziz and Dawood ([2], Remark 1).

***Theorem* 5** — *Let $r(s(z)) \in \mathbb{R}_{mn}$ and all its zeros lie in $\mathbb{T}_1 \cup \mathbb{D}_{1-}$. If $r^*(s(z)) = B(z)\overline{r(s(\frac{1}{\bar{z}}))}$, then for $z \in \mathbb{T}_1$, we have*

$$\frac{\min_{z\in\mathbb{T}_1}|r(s(z))|}{\max_{z\in\mathbb{T}_1}|s(z)|} \leq m \min_{z\in\mathbb{T}_1}\left\{\left|\frac{r'(s(z))}{B'(z)}\right| - \left|\frac{{r^*}'(s(z))}{B'(z)}\right|\right\}$$
$$\leq \frac{\min_{z\in\mathbb{T}_1}|r(s(z))|}{\min_{z\in\mathbb{T}_1}|s(z)|}. \tag{2.8}$$

If we take $s(z) = z$ in (2.8), we get the following Corollary, which is a generalization of the corresponding result for polynomials ([2], Remark 1).

Corollary 5 — Let $r \in \mathbb{R}_n$ and all its zeros lie in $\mathbb{T}_1 \cup \mathbb{D}_{1-}$. If $r^*(s(z)) = B(z)\overline{r(s(\frac{1}{\bar{z}}))}$, then for $z \in \mathbb{T}_1$, we have

$$\min_{z\in\mathbb{T}_1}|r(z)| = \min_{z\in\mathbb{T}_1}\left\{\left|\frac{r'(z)}{B'(z)}\right| - \left|\frac{{r^*}'(z)}{B'(z)}\right|\right\}. \tag{2.9}$$

3. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 1 — Let $r(s(z)) \in \mathbb{R}_{mn}$. If all the zeros of $r(s(z))$ lie in $\mathbb{T}_1 \cup \mathbb{D}_{1+}$, then for $z \in \mathbb{T}_1$ and $r(s(z)) \neq 0$,

$$Re\left(\frac{z(r(s(z)))'}{r(s(z))}\right) \leq \frac{1}{2}|B'(z)|.$$

The above lemma is due to Qasim and Liman [4].

Lemma 2 — If $z \in \mathbb{T}_1$ then

$$Re\left(\frac{z(W(s(z)))'}{W(s(z))}\right) = \frac{mn - |B'(z)|}{2}.$$

Proof of Lemma 2 : Let $W^*(s(z)) = z^{mn}\overline{W(s(\frac{1}{\bar{z}}))}$, so that

$$z(W^*(s(z)))' = mnz^{mn}\overline{W(s(\frac{1}{\bar{z}}))} - z^{mn-1}\overline{W'(s(\frac{1}{\bar{z}}))s'(\frac{1}{\bar{z}})},$$

which implies for $z \in \mathbb{T}_1$,

$$z(W^*(s(z)))' = mnW^*(s(z)) - z^{mn}\,\overline{z(W(s(z)))'}.$$

Since $W^*(s(z)) \neq 0$ for $z \in \mathbb{T}_1$, the above inequality is equivalent to

$$\frac{z(W^*(s(z)))'}{W^*(s(z))} = mn - \left(\overline{\frac{z(W(s(z)))'}{W(s(z))}}\right),$$

which gives for $z \in \mathbb{T}_1$,

$$Re\left(\frac{z(W^*(s(z)))'}{W^*(s(z))}\right) + Re\left(\frac{z(W(s(z)))'}{W(s(z))}\right) = mn. \tag{3.1}$$

Recall that

$$B(z) = \frac{W^*(s(z))}{W(s(z))} = \prod_{j=1}^{mn} \left(\frac{1 - \overline{a}_j z}{z - a_j}\right).$$

This gives

$$\frac{z\ B'(z)}{B(z)} = \sum_{j=1}^{mn} \left(\frac{-z\overline{a}_j}{1 - \overline{a}_j z} - \frac{z}{z - a_j}\right).$$

Hence for $z \in \mathbb{T}_1$, we have

$$\frac{z\ B'(z)}{B(z)} = \sum_{j=1}^{mn} \left(\frac{|\overline{a}_j|^2 - 1}{|z - a_j|^2}\right).$$

Since $|a_j| > 1$ for all $1 \leq j \leq mn$, it follows from above that $\frac{z\ B'(z)}{B(z)}$ is real and positive. Also $|B(z)| = 1$ for $z \in \mathbb{T}_1$, therefore, we have

$$\frac{z\ B'(z)}{B(z)} = \left|\frac{z\ B'(z)}{B(z)}\right| = |B'(z)|, \quad z \in \mathbb{T}_1. \tag{3.2}$$

Again

$$B(z) = \frac{W^*(s(z))}{W(s(z))},$$

so that

$$B'(z) = \frac{W(s(z))(W^*(s(z)))' - W^*(s(z))(W(s(z)))'}{(W(s(z)))^2}.$$

This implies

$$\frac{z\ B'(z)}{B(z)} = \frac{z\ (W^*(s(z)))'}{W^*(s(z))} - \frac{z\ (W(s(z)))'}{W(s(z))}. \tag{3.3}$$

From (3.2) and (3.3), it follows that for $z \in \mathbb{T}_1$,

$$|B'(z)| = \frac{z\,(W^*(s(z)))'}{W^*(s(z))} - \frac{z\,(W(s(z)))'}{W(s(z))},$$

which implies for $z \in \mathbb{T}_1$,

$$Re\bigg(\frac{z(W^*(s(z)))'}{W^*(s(z))}\bigg) - Re\bigg(\frac{z(W(s(z)))'}{W(s(z))}\bigg) = |B'(z)|. \tag{3.4}$$

On subtracting (3.1) and (3.4), we get

$$Re\bigg(\frac{z(W(s(z)))'}{W(s(z))}\bigg) = \frac{mn - |B'(z)|}{2},$$

and this completes the proof of Lemma 2.

Lemma 3 — Suppose $\lambda \in \mathbb{T}_1$, then the following hold. The equation $B(z) = \lambda$ has exactly mn simple roots, say $t_1, t_2, ..., t_{mn}$, and all lie on the unit circle $\mathbb{T}_1$; and if $r(s(z)) \in \mathbb{R}_{mn}$ and $z \in \mathbb{T}_1$, then

$$B'(z)r(s(z)) - s'(z)r'(s(z))[B(z) - \lambda] = \frac{B(z)}{z}\sum_{k=1}^{mn} C_k r(s(t_k))\bigg|\frac{B(z)-\lambda}{z-t_k}\bigg|^2, \tag{3.5}$$

where $C_k = C_k(\lambda)$ is defined for $k = 1, 2, ..., mn$ by

$$C_k^{-1} = \sum_{j=1}^{mn} \frac{|a_j|^2 - 1}{|t_k - a_j|^2}. \tag{3.6}$$

Furthermore, for $z \in \mathbb{T}_1$, we have

$$\frac{zB'(z)}{B(z)} = \sum_{k=1}^{mn} C_k\bigg|\frac{B(z)-\lambda}{z-t_k}\bigg|^2. \tag{3.7}$$

The above lemma is due to Qasim and Liman [4].

Lemma 4 — Let $r(s(z)) \in \mathbb{R}_{mn}$. If all the zeros of $r(s(z))$ lie in $\mathbb{T}_1 \cup \mathbb{D}_{1+}$, then for $z \in \mathbb{T}_1$,

$$|(r(s(z)))'| \leq |(r^*(s(z)))'| - |B'(z)| \min_{z \in \mathbb{T}_1} |r(s(z))|,$$

where $r^*(s(z)) = B(z)\overline{r(s(\frac{1}{\bar{z}}))}$.

PROOF OF LEMMA 4 : By hypothesis, $r(s(z)) \neq 0$ in $\mathbb{D}_{1-}$. Suppose first that $r(s(z)) \neq 0$ on $\mathbb{T}_1$, so that all the zeros of $r(s(z))$ lie in $\mathbb{D}_{1+}$, and therefore, $m^* = m^*(r \circ s, 1) > 0$. Let α be a complex number with $|\alpha| < 1$, then for $z \in \mathbb{T}_1$,

$$|\alpha m^* B(z)| < |r(s(z))|.$$

Therefore, it follows by Rouche's theorem that

$$R(z) = r(s(z)) + \alpha m^* B(z)$$

has the same number of zeros in $\mathbb{D}_{1+}$ as $r(s(z))$. Thus $R(z)$ has mn zeros in $\mathbb{D}_{1+}$. This is true even if $m^* = 0$ as well. Hence in any case all the zeros of $R(z)$ lie in $\mathbb{T}_1 \cup \mathbb{D}_{1+}$. Therefore, by Lemma 1, by virtue of (3.2), $|B'(z)| > 0$, we have for $z \in \mathbb{T}_1$ such that $R(z) \neq 0$,

$$Re\left(\frac{zR'(z)}{|B'(z)|R(z)}\right) \leq \frac{1}{2}.$$

This implies for $z \in \mathbb{T}_1$ (which are not the zeros of $R(z)$), that

$$\left|\frac{zR'(z)}{|B'(z)|R(z)}\right| \leq \left|\frac{zR'(z)}{|B'(z)|R(z)} - 1\right|.$$

Equivalently

$$|zR'(z)| \leq \left|zR'(z) - |B'(z)|R(z)\right|, \tag{3.8}$$

for the points z on $\mathbb{T}_1$ which are not the zeros of $R(z)$. But (3.8) is trivially true for points z on $\mathbb{T}_1$ which are the zeros of $R(z)$ as well, it follows that

$$|R'(z)| \leq \left|zR'(z) - |B'(z)|R(z)\right| \text{ for all } z \in \mathbb{T}_1. \tag{3.9}$$

Since

$$R^*(z) = B(z)\overline{R(\frac{1}{\overline{z}})},$$

we have

$$z(R^*(z))' = zB'(z)\overline{R(\frac{1}{\overline{z}})} - \frac{B(z)}{z}\overline{R'(\frac{1}{\overline{z}})},$$

and therefore for $z \in \mathbb{T}_1$, we get

$$\begin{aligned} |(R^*(z))'| &= |zB'(z)\overline{R(z)} - B(z)\overline{zR'(z)}| \\ &= |B(z)|\left|\frac{zB'(z)}{B(z)}\overline{R(z)} - \overline{zR'(z)}\right|. \end{aligned} \tag{3.10}$$

Now, by (3.2), we have $\frac{zB'(z)}{B(z)} = |B'(z)|$, therefore from (3.9) and (3.10), it follows by using $|B(z)| = 1$ for $z \in \mathbb{T}_1$,

$$|R'(z)| \leq |(R^*(z))'| \text{ for } z \in \mathbb{T}_1. \tag{3.11}$$

Since $R(z)=r(s(z))+\alpha m^*B(z)$, we have

$$\begin{aligned}R^*(z)=&B(z)\overline{R(\frac{1}{\overline{z}})}\\&=B(z)\overline{\{r(s(\frac{1}{\overline{z}}))+\alpha m^*B(\frac{1}{\overline{z}})\}}\\&=r^*(s(z))+\overline{\alpha}m^*,\end{aligned}$$

for $z\in\mathbb{T}_1$, and hence from (3.11), we get

$$|(r(s(z)))'+\alpha m^*B'(z)|\leq|(r^*(s(z)))'|, \tag{3.12}$$

for every α with $|\alpha|<1$. We can choose the argument of α suitably, such that

$$|(r(s(z)))'+\alpha m^*B'(z)|=|(r(s(z)))'|+|\alpha|m^*|B'(z)|,$$

it follows from (3.12) that

$$|(r(s(z)))'|+|\alpha|m^*|B'(z)|\leq|(r^*(s(z)))'| \text{ for } z\in\mathbb{T}_1. \tag{3.13}$$

Finally letting $|\alpha|\to 1-$ in (3.13), we get the desired result. This completes the proof of Lemma 4.

Lemma 5 — Suppose that $\lambda\in\mathbb{T}_1$. Let $t_1,t_2,...,t_{mn}$ are the zeros of $B(z)-\lambda$ and $\mu_1,\mu_2,...,\mu_{mn}$ are the zeros of $B(z)+\lambda$. If $r(s(z))\in\mathbb{R}_{mn}$, then for $z\in\mathbb{T}_1$, we have

$$|(r(s(z)))'|^2+|(r^*(s(z)))'|^2\leq\frac{|B'(z)|^2}{2}\left\{\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2+\left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2\right\}.$$

Equality holds for $r(s(z))=\mu B(z)$, where $\mu\in\mathbb{T}_1$.

PROOF OF LEMMA 5 : Since $t_1,t_2,...,t_{mn}$ are the zeros of $B(z)-\lambda$, $\lambda\in\mathbb{T}_1$, therefore by Lemma 3, we obtain for $z\in\mathbb{T}_1$,

$$\begin{aligned}|B'(z)r(s(z))-&s'(z)r'(s(z))B(z)+\lambda s'(z)r'(s(z))|\\&=\left|\frac{B(z)}{z}\sum_{k=1}^{mn}C_kr(s(t_k))\left|\frac{B(z)-\lambda}{z-t_k}\right|^2\right|\\&\leq\left|\frac{B(z)}{z}\right|\sum_{k=1}^{mn}|C_k||r(s(t_k))|\left|\frac{B(z)-\lambda}{z-t_k}\right|^2\\&\leq\sum_{k=1}^{mn}|C_k|\left|\frac{B(z)-\lambda}{z-t_k}\right|^2\max_{1\leq k\leq mn}|r(s(t_k))|.\end{aligned} \tag{3.14}$$

Since for $z \in \mathbb{T}_1$, $|B(z)| = 1$ and $C_k \geq 0$, $k = 1, 2, ..., mn$, it follows from (3.14) with the help of (3.6) and (3.7), that

$$\begin{aligned} |B'(z)r(s(z)) - s'(z)r'(s(z))B(z) + \lambda s'(z)r'(s(z))| \\ \leq \sum_{k=1}^{mn} C_k \left| \frac{B(z)-\lambda}{z-t_k} \right|^2 \max_{1\leq k\leq mn} |r(s(t_k))| \\ = |B'(z)| \max_{1\leq k\leq mn} |r(s(t_k))|. \end{aligned} \tag{3.15}$$

Replacing λ by $-\lambda$ in (3.15) and noting that $\mu_1, \mu_2, ..., \mu_{mn}$ are the zeros of $B(z) + \lambda$, we get

$$\begin{aligned} |B'(z)r(s(z)) - s'(z)r'(s(z))B(z) - \lambda s'(z)r'(s(z))| \\ \leq |B'(z)| \max_{1\leq k\leq mn} |r(s(\mu_k))|. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), it follows that

$$\begin{aligned} |B'(z)r(s(z)) - s'(z)r'(s(z))B(z) + \lambda s'(z)r'(s(z))|^2 \\ + |B'(z)r(s(z)) - s'(z)r'(s(z))B(z) - \lambda s'(z)r'(s(z))|^2 \\ \leq |B'(z)|^2 \left\{ \left(\max_{1\leq k\leq mn} |r(s(t_k))| \right)^2 + \left(\max_{1\leq k\leq mn} |r(s(\mu_k))| \right)^2 \right\}. \end{aligned} \tag{3.17}$$

Using the identity

$$|\alpha + \beta|^2 + |\alpha - \beta|^2 = 2|\alpha|^2 + 2|\beta|^2$$

in (3.17), with $\alpha = B'(z)r(s(z)) - s'(z)r'(s(z))B(z)$ and $\beta = \lambda s'(z)r'(s(z))$, we get for $z,\ \lambda \in \mathbb{T}_1$,

$$\begin{aligned} 2\left\{ |B'(z)r(s(z)) - s'(z)r'(s(z))B(z)|^2 + |s'(z)r'(s(z))|^2 \right\} \\ \leq |B'(z)|^2 \left\{ \left(\max_{1\leq k\leq mn} |r(s(t_k))| \right)^2 + \left(\max_{1\leq k\leq mn} |r(s(\mu_k))| \right)^2 \right\}. \end{aligned} \tag{3.18}$$

Now $r^*(s(z)) = B(z)\overline{r(s(\frac{1}{\bar{z}}))}$, therefore, proceeding similarly as in the proof of Lemma 4, we have for $z \in \mathbb{T}_1$, that

$$|(r^*(s(z)))'| = |B(z)| \left| \frac{zB'(z)}{B(z)} \overline{r(s(z))} - \overline{zr'(s(z))s'(z)} \right|.$$

Using the fact that $|B(z)| = 1$ and $\frac{zB'(z)}{B(z)}$ is real for $z \in \mathbb{T}_1$, we get

$$\begin{aligned} |(r^*(s(z)))'| = |B(z)| \left| \frac{zB'(z)}{B(z)} r(s(z)) - zr'(s(z))s'(z) \right| \\ = |B'(z)r(s(z)) - r'(s(z))s'(z)B(z)|. \end{aligned} \tag{3.19}$$

Using (3.19) in (3.18), we get

$$|(r(s(z)))'|^2 + |(r^*(s(z)))'|^2 \leq \frac{|B'(z)|^2}{2}\left\{\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2 + \left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2\right\}.$$

For sharpness, let $r(s(z)) = \mu B(z), \mu \in \mathbb{T}_1$. It is easy to see for $z \in \mathbb{T}_1$, that

$$|(r(s(z)))'|^2 + |(r^*(s(z)))'|^2 = |B'(z)|^2.$$

Also, since $t_1, t_2, ..., t_{mn}$ are the zeros of $B(z) - \lambda$ and $\mu_1, \mu_2, ..., \mu_{mn}$ are the zeros of $B(z) + \lambda$, which all lie on the unit circle $\mathbb{T}_1$, where $\lambda \in \mathbb{T}_1$. It follows that for $z \in \mathbb{T}_1$, we have

$$\max_{1\leq k\leq mn}|r(s(t_k))| = \max_{1\leq k\leq mn}|r(s(\mu_k))| = 1,$$

and hence

$$\frac{|B'(z)|^2}{2}\left\{\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2 + \left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2\right\} = |B'(z)|^2,$$

thus establishing the sharpness in the Lemma 5.

Lemma 6 — If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $\mathbb{T}_1 \cup \mathbb{D}_{1-}$, then

$$\min_{z\in\mathbb{T}_1}|P'(z)| \geq n \min_{z\in\mathbb{T}_1}|P(z)|. \tag{3.20}$$

The result is best possible and equality in (3.20) holds for polynomials having all their zeros at the origin.

The above lemma is due to Aziz and Dawood [2].

4. Proofs of the Theorems

Proof of Theorem 1 : Let $r(s(z)) = \frac{P(s(z))}{W(s(z))} \in \mathbb{R}_{mn}$, where $s \in \mathbb{P}_m$. Since $r(s(z))$ has all its zeros in $\mathbb{T}_k \cup \mathbb{D}_{k-}$, $k \leq 1$, it follows that $P(s(z))$ has all its mn zeros in $\mathbb{T}_k \cup \mathbb{D}_{k-}$, $k \leq 1$. By direct calculation, we obtain

$$\frac{z(r(s(z)))'}{r(s(z))} = \frac{z(P(s(z)))'}{P(s(z))} - \frac{z(W(s(z)))'}{W(s(z))}. \tag{4.1}$$

If $z_1, z_2, ..., z_{mn}$ are the zeros of $P(s(z))$, then $|z_j| \leq k \leq 1$, $1 \leq j \leq mn$ and we have from

(4.1) for $0 \leq \theta < 2\pi$,

$$\begin{aligned}
Re\left(\frac{z(r(s(z)))'}{r(s(z))}\right)\Big|_{z=e^{i\theta}} &= Re\left(\frac{z(P(s(z)))'}{P(s(z))}\right)\Big|_{z=e^{i\theta}} - Re\left(\frac{z(W(s(z)))'}{W(s(z))}\right)\Big|_{z=e^{i\theta}} \\
&= \sum_{j=1}^{mn} Re\left(\frac{e^{i\theta}}{e^{i\theta}-z_j}\right) - Re\left(\frac{z(W(s(z)))'}{W(s(z))}\right)\Big|_{z=e^{i\theta}} \\
&\geq \sum_{j=1}^{mn}\left(\frac{1}{1+|z_j|}\right) - \left(\frac{mn-|B'(e^{i\theta})|}{2}\right) \text{ (by Lemma 2)} \\
&\geq \frac{mn}{1+k} - \frac{mn}{2} + \frac{|B'(e^{i\theta})|}{2} \\
&= \frac{1}{2}\left\{|B'(e^{i\theta})| + \frac{mn(1-k)}{1+k}\right\},
\end{aligned}$$

for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, other than the zeros of $r(s(z))$. Hence, we have

$$|(r(s(e^{i\theta})))'| \geq \frac{1}{2}\left\{|B'(e^{i\theta})| + \frac{mn(1-k)}{1+k}\right\}|r(s(e^{i\theta}))|, \tag{4.2}$$

for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, other than the zeros of $r(s(z))$. Since (4.2) is true for the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are the zeros of $r(s(z))$ also, it follows that

$$|r'(s(z))s'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{mn(1-k)}{1+k}\right\}|r(s(z))| \text{ for } z \in \mathbb{T}_1. \tag{4.3}$$

Now using inequality (1.1) in (4.3), we get (2.1) and this completes the proof of Theorem 1.

PROOF OF THEOREM 2 : Since $r(s(z))$ has all its zeros in $\mathbb{T}_k \cup \mathbb{D}_{k-}, k \leq 1$. Also $m^*(r \circ s, k) = \min_{z \in \mathbb{T}_k} |r(s(z))|$. Therefore, $m^*(r \circ s, k) \leq |r(s(z))|$, for $z \in \mathbb{T}_k$. First, we suppose that $r(s(z))$ has no zeros on $|z| = k$, then $m^*(r \circ s, k) > 0$ and for every complex number α with $|\alpha| < 1$, we have

$$|\alpha m^*(r \circ s, k)| < |r(s(z))| \text{ for } z \in \mathbb{T}_k.$$

Therefore, it follows by Rouche's theorem that the rational function

$$R(z) = r(s(z)) + \alpha m^*(r \circ s, k)$$

has exactly mn zeros in $\mathbb{D}_{k-}$. In case $r(s(z))$ has some zeros on $\mathbb{T}_k$, then $m^*(r \circ s, k) = 0$ and in this case $R(z) = r(s(z))$ has all its zeros in $\mathbb{T}_k \cup \mathbb{D}_{k-}, k \leq 1$. Applying Theorem 1 to $R(z)$, we get for $z \in \mathbb{T}_1$,

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{(1+k)}\right\}|r(s(z)) + \alpha m^*(r \circ s, k)|,$$

which on choosing the argument of α suitably in the right hand side so that for $z \in \mathbb{T}_1$,

$$|r'(s(z))| \geq \frac{1}{2mM'}\left\{|B'(z)| + \frac{mn(1-k)}{(1+k)}\right\}\Big(|r(s(z))| + |\alpha| m^*(r \circ s, k)\Big). \tag{4.4}$$

Finally letting $|\alpha| \to 1-$ in (4.4), we get (2.2) and this completes the proof of Theorem 2.

PROOF OF THEOREM 3 : By hypothesis $r(s(z))$ does not vanish in $\mathbb{D}_{1-}$, therefore, by Lemma 4, we have

$$\big(|(r(s(z)))'| + m^*|B'(z)|\big)^2 \leq |(r^*(s(z)))'|^2 \text{ for } z \in \mathbb{T}_1.$$

Adding both sides $|(r(s(z)))'|^2$ and using Lemma 5, we get for $z \in \mathbb{T}_1$,

$$\begin{aligned}\big(|(r(s(z)))'| &+ m^*|B'(z)|\big)^2 + |(r(s(z)))'|^2 \\ &\leq \frac{|B'(z)|^2}{2}\left\{\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2 + \left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2\right\}.\end{aligned} \tag{4.5}$$

Since

$$\begin{aligned}\big(|(r(s(z)))'| &+ m^*|B'(z)|\big)^2 \\ &= |(r(s(z)))'|^2 + m^{*2}|B'(z)|^2 + 2m^*|B'(z)||(r(s(z)))'| \\ &\geq |(r(s(z)))'|^2 + m^{*2}|B'(z)|^2,\end{aligned}$$

it follows from (4.5) for $z \in \mathbb{T}_1$ that

$$2|(r(s(z)))'|^2 \leq \frac{|B'(z)|^2}{2}\left\{\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2 + \left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2 - 2m^{*2}\right\}.$$

This gives for $z \in \mathbb{T}_1$,

$$|r'(s(z))s'(z)| \leq \frac{|B'(z)|}{2}\left\{\left(\max_{1\leq k\leq mn}|r(s(t_k))|\right)^2 + \left(\max_{1\leq k\leq mn}|r(s(\mu_k))|\right)^2 - 2m^{*2}\right\}^{\frac{1}{2}},$$

which when combined with Lemma 6 proves Theorem 3 completely.

PROOF OF THEOREM 4 : We have from inequality (3.15) for $\lambda,\ z \in \mathbb{T}_1$, that

$$\begin{aligned}|B'(z)r(s(z)) &- s'(z)r'(s(z))B(z) + \lambda s'(z)r'(s(z))| \\ &\leq |B'(z)|\max_{1\leq k\leq mn}|r(s(t_k))| \\ &\leq |B'(z)|\max_{z\in\mathbb{T}_1}|r(s(z))|.\end{aligned} \tag{4.6}$$

Since the right hand side of (4.6) is independent of λ, therefore, we can suitably choose λ such that

$$\begin{aligned}|B'(z)r(s(z)) - s'(z)r'(s(z))B(z)| + |s'(z)r'(s(z))| \\ \leq |B'(z)| \max_{z\in\mathbb{T}_1} |r(s(z))| \text{ for } z \in \mathbb{T}_1.\end{aligned}$$

This implies with the help of (3.19) that for $z \in \mathbb{T}_1$,

$$|(r(s(z)))'| + |(r^*(s(z)))'| \leq |B'(z)| \max_{z\in\mathbb{T}_1} |r(s(z))|. \tag{4.7}$$

Since $|B'(z)| > 0$, we get from (4.7), Lemma 6 and by Gauss Lucas-Theorem that

$$\left|\frac{r'(s(z))}{B'(z)}\right| + \left|\frac{{r^*}'(s(z))}{B'(z)}\right| \leq \frac{\max_{z\in\mathbb{T}_1} |r(s(z))|}{m \min_{z\in\mathbb{T}_1} |s(z)|} \text{ for } z \in \mathbb{T}_1,$$

or

$$\max_{z\in\mathbb{T}_1} \left\{\left|\frac{r'(s(z))}{B'(z)}\right| + \left|\frac{{r^*}'(s(z))}{B'(z)}\right|\right\} \leq \frac{\max_{z\in\mathbb{T}_1} |r(s(z))|}{m \min_{z\in\mathbb{T}_1} |s(z)|}. \tag{4.8}$$

Also it is easy to verify that

$$\frac{z(r^*(s(z)))'}{r^*(s(z))} = \frac{zB'(z)}{B(z)} - \frac{\overline{(r(s(z)))'}}{z\,\overline{r(s(\frac{1}{\bar{z}}))}},$$

which implies from (3.2) for $z \in \mathbb{T}_1$ that

$$\frac{z(r^*(s(z)))'}{r^*(s(z))} = |B'(z)| - \left(\overline{\frac{z(r(s(z)))'}{r(s(z))}}\right).$$

This gives for $z \in \mathbb{T}_1$,

$$\begin{aligned}\left|\frac{z(r^*(s(z)))'}{r^*(s(z))}\right| + \left|\left(\overline{\frac{z(r(s(z)))'}{r(s(z))}}\right)\right| \\ &\geq \left|\frac{z(r^*(s(z)))'}{r^*(s(z))} + \left(\overline{\frac{z(r(s(z)))'}{r(s(z))}}\right)\right| \\ &= |B'(z)|. \end{aligned} \tag{4.9}$$

Since $|r(s(z))| = |r^*(s(z))|$ for $z \in \mathbb{T}_1$, it follows that

$$|r'(s(z))s'(z)| + |{r^*}'(s(z))s'(z)| \geq |B'(z)||r(s(z))|,$$

which on using inequality (1.1) to $s(z)$ that

$$\max_{z\in\mathbb{T}_1}\left\{\left|\frac{r'(s(z))}{B'(z)}\right|+\left|\frac{r^{*'}(s(z))}{B'(z)}\right|\right\}\geq\frac{\max_{z\in\mathbb{T}_1}|r(s(z))|}{m\max_{z\in\mathbb{T}_1}|s(z)|}. \tag{4.10}$$

Combining (4.8) and (4.10), we get (2.5) and this completes the proof of Theorem 4.

PROOF OF THEOREM 5 : Since all the zeros of $r(s(z))$ lie in $\mathbb{T}_1\cup\mathbb{D}_{1-}$, it follow that all the zeros of $r^*(s(z))=B(z)\overline{r(s(\frac{1}{\bar{z}}))}$ lie in $\mathbb{T}_1\cup\mathbb{D}_{1+}$ and hence on applying Lemma 4 to $r^*(s(z))$, we get for $z\in\mathbb{T}_1$, that

$$|(r^*(s(z)))'|\leq|(r(s(z)))'|-|B'(z)|\min_{z\in\mathbb{T}_1}|r^*(s(z))|,$$

which implies by using the fact

$$\min_{z\in\mathbb{T}_1}|r(s(z))|=\min_{z\in\mathbb{T}_1}|r^*(s(z))|$$

that for $z\in\mathbb{T}_1$,

$$|r'(s(z))s'(z)|-|r^{*'}(s(z))s'(z)|\geq|B'(z)|\min_{z\in\mathbb{T}_1}|r(s(z))|.$$

This gives on using inequality (1.1) to $s(z)$ for $z\in\mathbb{T}_1$, that

$$\left|\frac{r'(s(z))}{B'(z)}\right|-\left|\frac{r^{*'}(s(z))}{B'(z)}\right|\geq\frac{\min_{z\in\mathbb{T}_1}|r(s(z))|}{m\max_{z\in\mathbb{T}_1}|s(z)|},$$

which implies

$$\min_{z\in\mathbb{T}_1}\left\{\left|\frac{r'(s(z))}{B'(z)}\right|-\left|\frac{r^{*'}(s(z))}{B'(z)}\right|\right\}\geq\frac{\min_{z\in\mathbb{T}_1}|r(s(z))|}{m\max_{z\in\mathbb{T}_1}|s(z)|}. \tag{4.11}$$

Again since $r^*(s(z))=B(z)\overline{r(s(\frac{1}{\bar{z}}))}$, it follows from (3.19) for $z\in\mathbb{T}_1$, that

$$|(r^*(s(z)))'|=|B'(z)r(s(z))-r'(s(z))s'(z)B(z)|,$$

which implies by using the fact $|B(z)|=1$ for $z\in\mathbb{T}_1$, that

$$|(r^*(s(z)))'|\geq|r'(s(z))s'(z)|-|B'(z)||r(s(z))|.$$

This gives for $z\in\mathbb{T}_1$,

$$|B'(z)||r(s(z))|\geq|r^{'}(s(z))s'(z)|-|r^{*'}(s(z))s'(z)|,$$

which on using Lemma 6, gives that

$$\min_{z\in\mathbb{T}_1}\left\{\left|\frac{r'(s(z))}{B'(z)}\right|-\left|\frac{{r^*}'(s(z))}{B'(z)}\right|\right\}\leq\frac{\min_{z\in\mathbb{T}_1}|r(s(z))|}{m\min_{z\in\mathbb{T}_1}|s(z)|}. \tag{4.12}$$

Combining (4.11) and (4.12), we get (2.8) and this completes the proof of Theorem 5.

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