Indian J. Pure Appl. Math., **50**(2): 333-342, June 2019
© Indian National Science Academy
DOI: 10.1007/s13226-019-0328-5

MORE VARIANTS OF ERDŐS-SELFRIDGE SUPERELLIPTIC CURVES AND THEIR RATIONAL POINTS

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(*Received* 4 *October* 2017; *accepted* 30 *May* 2018)

Developing on the works of Bennett and Siksek and more recently of Das, Laishram and Saradha, we study rational points on several other variants of Erdős-Selfridge super elliptic curve.

Key words : Super elliptic curves; rational solutions; ternary forms; exponential Diophantine equations

1. Introduction

Throughout the paper, we use the following notation. Let $2 = p_1 < p_2 < \cdots$ denote the sequence of all primes. Let $x, y \in \mathbb{Q}$ with $y \neq 0, k \geq h \geq 0$ be integers and $i_1 < i_2 < \cdots < i_h$ be h integers in $[1, k]$. Take $i_0 = 0$ and $i_{h+1} = k + 1$. Further $\ell \geq 2$ always denotes a prime. For any integer $n \geq 1$, let $P(n)$ denote the greatest prime factor of n and take $P(1) = 1$. For any two integers $0 \leq i < j$ define

$$\Delta(i, j) = (x + i + 1) \cdots (x + j - 1).$$

Here and henceforth, an empty product is taken to be equal to 1. Put

$$\Delta_h = \Delta(i_0, i_1)\Delta(i_1, i_2) \cdots \Delta(i_h, i_{h+1}). \quad (1)$$

Thus Δ_h is product of terms $x + i, 1 \leq i \leq k$ with the h terms $x + i_1, \cdots, x + i_h$ omitted. When $h = 0$, we understand that no term is omitted and

$$\Delta_0 = (x + 1) \cdots (x + k).$$

Bennett and Siksek [2] considered rational solutions of

$$\Delta_0 = y^\ell \tag{2}$$

in x and $y \neq 0$. They showed that if (2) holds, then

$$\ell \leq e^{3^k}. \tag{3}$$

This can be viewed as a rational analogue to the Schinzel-Tijdeman theorem on integral solutions to the superelliptic equation $f(x) = y^\ell$ where $f(x)$ is a polynomial. In [4], the result of Bennett and Siksek was extended to the case $h = 1$ as follows. Take

$$\Delta_1 = y^\ell, \qquad 1 \leq i_1 \leq k. \tag{4}$$

Let

$$\theta = \begin{cases} \pi(\frac{k-1}{2}) + 1 & \text{if } k \text{ is odd} \\ \pi(\frac{k}{2}) & \text{if } k \text{ is even and } \frac{k}{2} \text{ is prime} \\ \pi(\frac{k}{2}) + 1 & \text{if } k \text{ is even and } \frac{k}{2} \text{ is not prime.} \end{cases} \tag{5}$$

Thus p_θ is the least prime $\geq k/2$. When $i_1 = 1$ or k, equation (4) is similar to the equation (2). Hence (3) holds. It was shown in [4] that if (4) holds with $2 \leq i_1 \leq k - p_\theta$ or $p_\theta < i_1 < k$, then (3) holds.

Remark : The condition on i_1 in the above result has been removed by Edis [13].

In this paper, we investigate the equation (1) for any $h \geq 1$ and the case $h = 2$ in particular.

***Theorem* 1.1** — *With the notation as introduced above, let $k \geq 4h + 3$. Suppose the equation*

$$\Delta_h = y^\ell \tag{6}$$

has a rational solution in $x, y \neq 0$ with ℓ prime. Assume that there exists some pair (i_g, i_{g+1}) with $0 \leq g \leq h$ such that $i_{g+1} - i_g - 1 \geq p_\theta$. Then (3) holds.

Remark : Suppose $h = 0$. Then $g = 0, (i_0, i_1) = (0, k + 1)$ and $i_1 - i_0 - 1 = k \geq p_\theta$ for $k \geq 3$. Then Theorem 1.1 gives the result of Bennett and Siksek. Suppose $h = 1$. Then $i_1 < k, (i_0, i_1) = (0, i_1), (i_1, i_2) = (i_1, k + 1)$. Thus if either $i_1 - 1 \geq p_\theta$ or $k - i_1 \geq p_\theta$, then by Theorem 1.1, we recover the result in [4] for $k \geq 7$. The cases $3 \leq k \leq 6$ required some well known results on Fermat-type equations.

We shall apply Theorem 1.1 along with combinatorial arguments to show the following result for the case $h = 2$.

***Theorem* 1.2** — *Let $k \geq 4$. Suppose the equation*

$$\Delta_2 = y^\ell \tag{7}$$

has a rational solution in $x, y \neq 0$ with ℓ prime. Assume that there exists some pair (i_g, i_{g+1}) with $0 \leq g \leq 2$ such that $i_{g+1} - i_g - 1 \geq p_\theta$. Then (3) holds.

2. Preliminaries

Write $x = \frac{n}{s}$ and $y = \frac{m}{t}, m \neq 0$ with s, t positive integers and gcd$(n, s) =$ gcd$(m, t) = 1$. Further we may also assume that

$$\ell > e^{3^k}, \tag{8}$$

otherwise, (3) is valid. Equation (7) can be re-written as

$$(n+s)\cdots(n+(i_1-1)s)(n+(i_1+1)s)\cdots(n+(i_2-1)s)\cdots(n+(i_h+1)s)\cdots(n+ks) = \frac{s^{k-h}m^\ell}{t^\ell}.$$

Since the left hand side is an integer and gcd$(n, s) = \gcd(m, t) = 1$, we get $s^{k-h} = t^\ell$. As ℓ is a prime $> k - h$, there is an integer $d \geq 1$ such that $s = d^\ell$ and $t = d^{k-h}$. Thus (7) gives rise to the equation

$$\Delta^{(h)} := (n+d^\ell)\cdots(n+(i_1-1)d^\ell)(n+(i_1+1)d^\ell)\cdots(n+(i_2-1)d^\ell)\cdots(n+(i_h+1)d^\ell)\cdots(n+kd^\ell) = m^\ell$$

with gcd$(n, d) = 1$. Note that $\Delta^{(h)}$ is a product of $k - h$ terms. We also denote this product as

$$\Delta^{(h)} = (n + j_1 d^\ell)\cdots(n + j_{k-h} d^\ell)$$

with $1 \leq j_1 < \cdots < j_{k-h} \leq k$. Thus we consider the equation

$$(n + j_1 d^\ell)\cdots(n + j_{k-h} d^\ell) = m^\ell. \tag{9}$$

Further we can write each term

$$n + j_t d^\ell = a_{j_t} x_{j_t}^\ell, \quad 1 \leq t \leq k - h,$$

with $P(a_{j_t}) < k$ and a_{j_t} is $\ell-$th power free. Equation (9), when d^ℓ is replaced by any common difference $D \geq 1$ has been studied in several papers. Let

$$(n + t_1 D)\cdots(n + t_{k-h} D) = m^\ell \tag{10}$$

with $n \geq 1, 0 \leq t_1 < t_2 < \cdots < t_{k-h} < k$ and $\gcd(n, D) = 1$. This is an equation dealing with perfect powers in a product with successive terms in arithmetic progression and h terms missing.

In 1955, Erdős [5] proved the following result when $D = 1$.

Suppose (10) with $D = 1, \ell \geq 3, P(m) > k$ hold and

$$h \leq (1-\epsilon)\frac{k \log\log k}{\log k}, \epsilon > 0.$$

Then k is bounded by an absolute constant.

This result was considerably sharpened by Shorey [14, 15] and later by Nesterenko and Shorey [7]. As a result of [7], one obtains that k is bounded by an absolute constant whenever

$$h < .5168k,\ \ell \geq 7.$$

The results for $\ell > 2$ depend on the theory of linear forms in logarithms, irrationality measures of Baker based on hyper geometric method and estimates of Halberstam and Roth on difference between consecutive k free integers.

When $\ell = 2$, in the above result of Erdős, h was taken to be $h \leq c_1 \frac{k}{\log k}$, where c_1 is an absolute constant. This was sharpened by Shorey [15] to $h \leq (1-\epsilon)k\frac{\log\log k}{\log k}$ and later relaxed further by Balasubramanian and Shorey [1]. Thus we see that when k is sufficiently large, the equation does not hold under the restriction on h and $P(m) > k$.

Let now $D > 1$. Shorey and Tijdeman combined several elementary arguments of Erdős with the application of box principle on numerous occasions in a beautiful paper (see [16, p. 343]). Their result for $\ell = 2$ was improved by Saradha and Shorey in [12] and for $\ell = 3$ by Das, Laishram and Saradha in [3]. We combine the results of these three papers and present it as below.

Let $\epsilon > 0$. Equation (10) with $D > 1$ and $P(m) > k$ implies that there exist a computable number $c(\epsilon)$ and an absolute constant c_2 such that either

$$k \leq c(\epsilon) \text{ } or \text{ } \ell^{\omega(D)} \leq c_2 k \frac{\log\log k}{\log k}$$

provided

$$h < (1-\epsilon)k\frac{\log\log k}{\log k} \tag{11}$$

where $\omega(D)$ denotes the the number of distinct prime divisors of D.

In other words, we see that under the hypothesis of the above result,

$$\ell \leq c_2 k \frac{\log\log k}{\log k}$$

provided k is sufficiently large and $\gg h\log h/\log\log h$. We may view Theorem 1.1 as being less restrictive on k at the cost of a worse bound for ℓ. The condition $P(m) > k$ in the above result would ensure that there exists a prime which divides only one term of the product and this is an easy case to form a required ternary equation, see Lemma 3.1 below. So we may think of the restriction $i_{g+1} - i_g - 1 \geq p_\theta$ in Theorem 1.1 as more relaxed than the condition $P(m) > k$.

For complete results on $h = 1$, we refer to [9] and [10]. Now we restrict to $h = 2$. Equation (10) was completely solved for $D = 1$ by Mukhopadhyaya and Shorey [6] for $\ell = 2$ and by Saradha and Shorey [11] for $\ell \geq 3$. From the results of these two papers we get

Let $k \geq 4$. The only solutions of (10) with $n \geq 1, D = 1$ and $h = 2$ are given by $(n,k) \in \{(1,4),(2,4),(1,5),(2,5),(1,6),(2,6),(3,6),(4,6),(5,6),(1,7),(3,7),(4,7),(3,8),$ $(14,8),(2,9),(1,10),(2,10),(5,10),(1,11),(4,11)\}$.

3. Lemmas

We shall begin with a lemma on forming a suitable ternary form from equation (7) which is similar to Lemma 2.1 of [2] and Lemma 3.1 of [4].

Lemma 3.1 — Let $k \geq h+2$. Suppose (7) has a rational point (x,y) with $y \neq 0$. Let $p > 2h+1$ be a prime either dividing d or dividing at least one term and at most two terms of $\Delta^{(h)}$. Then there are non-zero integers a, b, c, u, v, w satisfying

$$au^\ell + bv^\ell + cw^\ell = 0$$

such that

(1) a, b, c are ℓ-th power free integers

(2) $P(abc) < k$

(3) $p \nmid abc$

(4) p divides precisely one of u, v, w.

Proof : Suppose $p|d$. Then $p \nmid (n + j_t d^\ell)$ for $1 \leq t \leq k-h$, since gcd$(n,d) = 1$. Thus for $1 \leq s,t \leq k-h$, the identity

$$a_{j_t}x_{j_t}^\ell - a_{j_s}x_{j_s}^\ell = (n + j_t d^\ell) - (n + j_s d^\ell) = (j_t - j_s)d^\ell \tag{12}$$

gives a required ternary form provided $j_t - j_s$ is not divisible by p. If $j_t - j_s$ is divisible by p for all choices of $1 \leq s,t \leq k-h$ then

$$(k-h-1)p \leq j_{k-h} - j_1 \leq k-1$$

giving

$$p \le (k-1)/(k-h-1) \le 1+h$$

which contradicts the assumption on p. Hence we can choose a pair (j_s, j_t) such that $j_t - j_s$ is not divisible by p. For this choice we will get a ternary form as asserted in the lemma.

Let $p \nmid d$. Suppose p divides only one term of $\Delta^{(h)}$, say, $n + j_s d^\ell, 1 \le s \le k-h$. Then p occurs to an ℓ-th power in this term. Hence $p \nmid a_j$ and $p | x_{j_s}$. Forming an equation as in (12), we get a ternary form as required provided there exists some $t \ne s$ such that $j_s - j_t$ is not divisible by p. Suppose no such t exists. This means $j_s - j_t$ for $1 \le t < s$ and $s+1 \le t \le k-h$ are all divisible by p. Thus

$$(s-1)p \le j_s - j_1 \le k-1 \text{ and } (k-h-s)p \le j_{k-h} - j_s \le k-1.$$

which implies that

$$p \le (k-1)/(s-1) \text{ for } s \ne 1 \text{ and } p \le (k-1)/(k-h-s) \text{ for } s \ne k-h.$$

Hence

$$p \le (k-1)/(k-h-1)$$

which contradicts the assumption on p.

Suppose p divides exactly two terms, say $n + j_s d^\ell$ and $n + j_t d^\ell$ with $j_s < j_t$. Then $j_t = j_s + rp$ for $r \ge 1$. There are $[rp/2]$ pairs of integers $(j_s + u, j_s + rp - u)$ with $1 \le u \le [rp/2]$ and $\min(j_s - 1, k - j_s - rp)$ pairs of integers $(j_s - u, j_s + rp + u)$ with $1 \le u \le \min(j_s - 1, k - j_s - rp)$. Among these pairs there can be at most h pairs in which one of the element equals an index corresponding to an omitted term. Thus there are at least $[rp/2] + \min(j_s - 1, k - j_s - rp) - h$ pairs with both elements not equalling an omitted index. Thus if

$$[rp/2] + \min(j_s - 1, k - j_s - rp) - h \ge 1 \tag{13}$$

then there exist two terms of the form $n + (j_s + u)d^\ell, n + (j_s + rp - u)d^\ell, 1 \le u \le [rp/2]$ or $n + (j_s - u)d^\ell, n + (j_s + rp + u)d^\ell, 1 \le u \le \min(j_s - 1, k - j_s - rp)$ in $\Delta^{(h)}$. Then we form the identities

$$(n + j_s d^\ell)(n + (j_s + rp)d^\ell) - (n + (j_s + u)d^\ell)(n + (j_s + rp - u)d^\ell) = -u(rp - u)d^{2\ell}$$

with $1 \le u \le [rp/2]$ or

$$(n + j_s d^\ell)(n + (j_s + rp)d^\ell) - (n + (j_s - u)d^\ell)(n + (j_s + rp + u)d^\ell) = u(rp - u)d^{2\ell}$$

with $1 \le u \le \min(j_s - 1, k - j_s - rp)$. Since p divides only $n + j_s d^\ell$ and $n + (j_s + rp)d^\ell$ we see that p does not divide $u(rp - u)$ and the second term on the left hand side of the above two identities and it occurs to an ℓth power in the first term on the left hand side. Thus the above identities give rise to a required ternary form. Condition (13) is valid since $p > 2h + 1$. □

The lemma below is similar to Lemma 3.2 in [4].

Lemma 3.2 — Suppose (7) has a rational point (x, y) with $y \neq 0$ and $\ell \geq 7$. Let $2h + 1 < p$ be a prime either dividing d or dividing at least one term and at most two terms of $\Delta^{(h)}$. Further assume that $p \le e^k$. Then

$$\log \ell \le \frac{16(\prod_{q<k, q \neq p} q) + 1}{6} \log(\sqrt{p} + 1) \le 3^k.$$

PROOF : Since by (8), $\ell > e^k$, we see that the prime p as in the hypothesis of the lemma satisfies, $p \neq \ell$. We now argue as in [2] to get the inequality as asserted. □

4. PROOF OF THEOREM 1.1

By assumption, there exists some pair (i_g, i_{g+1}) with $0 \le g \le h$ such that $i_{g+1} - i_g - 1 \ge p_\theta$. Hence p_θ divides $\Delta^{(h)}$. Further $p_\theta \ge k/2$ implies $p_\theta > 2h + 1$ since $k \ge 4h + 3$ and p_θ divides at most two terms in $\Delta^{(h)}$. Hence by Lemma 3.2, taking $p = p_\theta$, inequality (3) is valid. □

5. PROOF OF THEOREM 1.2

The following lemma, proved by Saradha and Shorey in [8, Lemma 13] is useful to form ternary equations as required in Lemma 3.1.

Lemma 5.1 — Let $\ell \ge 5$. Let a, b, c be non-zero integers such that $P(abc) \le 3$. Then the equation

$$ax^\ell - by^\ell = cz^\ell$$

in non zero integers x, y, z with

$$\gcd(ax^\ell, by^\ell, cz^\ell) = 1, \quad \mathrm{ord}_2(by^\ell) \ge 4$$

has no solution.

From now on, we say that property $\mathcal{E}$ holds if we can form a ternary equation as in Lemma 5.1. We assume that equation (9) holds with $h = 2$. Suppose there exists a prime $p \ge \max(p_\theta, 7)$ dividing $\Delta^{(h)}$, then as in Theorem 1.1, (3) holds. Now $\max(p_\theta, 7) = p_\theta$ for $k \ge 11$. By hypothesis $\Delta^{(h)}$ is divisible by p_θ. Thus we need to consider only $4 \le k \le 10$. In these cases, $\max(p_\theta, 7) = 7$. Hence if $\Delta^{(h)}$ is divisible by a prime ≥ 7, then (3) holds. So we may assume that

$$P(\Delta^{(h)}) \le 5.$$

Let us assume that by^ℓ corresponds to the term in $\Delta^{(h)}$ in which 2 appears to the maximum power. By equation (9), the power of 2 in the product of the rest of the terms is at most $ord_2(k-1)! \le 7$ since $k \le 10$. Thus

$$ord_2(by^\ell) \ge \ell - 7. \tag{14}$$

Note that for any other term ax^ℓ of $\Delta^{(h)}$, we have

$$ax^\ell - by^\ell = rd^\ell \text{ with } |r| \le 9. \tag{15}$$

Thus if property $\mathcal{E}$ holds, then by Lemma 5.1, $\ell \le 13$, thus proving the theorem. If $P(\Delta^{(h)}) \le 3$ or $P(a_{j_1} \cdots a_{j_{k-h}}) \le 3$, such a ternary equation can easily be formed with $P(abr) \le 3$. In this case, then property $\mathcal{E}$ holds. Hence we may assume that $k \ge 6$ and

$$P(\Delta^{(h)}) = 5 \text{ and } P(a_{j_1} \cdots a_{j_{k-h}}) = 5. \tag{16}$$

Thus by equation (9), it follows that 5 divides exactly two of the a_i's. Further by the given hypothesis, there exists $1 \le g \le h$ with

$$i_{g+1} - i_g \ge p_\theta + 1. \tag{17}$$

Let $k = 6$. Then $p_\theta = 3$ and by (16) 5 must divide a_1 and a_6. Hence the indices of missing terms must belong to [2, 3, 4, 5] from which we see (17) cannot be satisfied. Thus $k \ne 6$.

Let $k = 7$. Then $p_\theta = 5$ and 5 divides a_1, a_6 or a_2, a_7. So the indices of the missing terms belong to either [2, 3, 4, 5, 7] or [1, 3, 4, 5, 6]. Thus (17) is not satisfied. Thus $k \ne 7$.

Let $8 \le k \le 10$. Then $p_\theta = 5$ and 5 divides exactly two terms and by (16), two other terms are omitted. Let us denote by R the product of the remaining $k-4$ terms in $\Delta^{(h)}$. Then $P(R) \le 3$. If by^ℓ, the term in which power of 2 appears to the maximum, coincides with one of the terms in R, then it is easy to see that property $\mathcal{E}$ holds. For instance, let $k = 9$ and 5 divide a_1 and a_6. By (17) the missing terms correspond to the indices $(2, 8), (2, 9), (3, 9)$. Let us consider the case of $(2, 8)$. Then $R = (n + 3d^\ell)(n + 4d^\ell)(n + 5d^\ell)(n + 7d^\ell)(n + 9d^\ell)$ with $P(R) \le 3$. Now it is clear that if by^ℓ coincides with a term in R then we can pick another term in R which differs by rd^ℓ with $|r| \le 4$. The resulting ternary equation as in (15) satisfies property $\mathcal{E}$.

Thus we may assume that by^ℓ corresponds to one of the terms which are divisible by 5. Thus every term in $\Delta^{(h)}$ has absolute value at least

$$2^{(\ell-7)} \times 5 - 10d^\ell. \tag{18}$$

On the other hand, the power of 2 in any term in R is at most 3 since $k \leq 10$. Further it can be easily checked that there are at least two terms in R, say, $a_\mu x_\mu^\ell$ and $a_\nu x_\nu^\ell$ which are composed of only the prime 2. Thus

$$d^\ell \leq |\mu - \nu| d^\ell \leq |a_\mu x_\mu^\ell - a_\nu x_\nu^\ell| \leq 2^3$$

which is a contradiction for $d > 1$. When $d = 1$, we use (18) to get

$$2^{(\ell-7)} \times 5 - 10 \leq |a_\mu x_\mu^\ell| \leq 2^3$$

which again gives a contradiction. □

Remark 1 : It may be possible to remove the condition $i_{g+1} - i_g - 1 \geq p_\theta$ in Theorem 1.2 following the ideas of [13].

Remark 2 : In [4] which deals with the case $h = 1$, it was assumed that $d > 1$ due to known results. We point out here that these known results, quoted therein are true for $n \geq 0$. Nevertheless, as seen in the proof of Theorem 1.2, one can avoid the assumption $d > 1$.

Acknowledgement

The author likes to thank Indian National Science Academy for awarding Senior Scientist Fellowship under which this work was done. She also thanks DAE-Center for Excellence in Basic Sciences, Mumbai University for providing facilities to carry out this work.

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