INTEGRAL POINTS ON THE ELLIPTIC CURVE

\[ E_{pq} : y^2 = x^3 + (pq - 12)x - 2(pq - 8)^1 \]

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Let \( p = 8k + 5, q = 8k + 3 \) be the twin prime pair for some nonnegative integer \( k \). Assume that \( \left( \frac{2}{p} \right) = -1 \) or \( \left( \frac{2}{q} \right) = -1 \). In this paper, we prove that the elliptic curve \( E_{pq} : y^2 = x^3 + (pq - 12)x - 2(pq - 8) \) has unique integral point \((2, 0)\).

**Key words**: Elliptic curve; integral point; Fibonacci (Lucas) sequence.

1. INTRODUCTION

Throughout this paper, let \( \mathbb{Q}, \mathbb{Z} \) and \( \mathbb{N} \) denote the field of rational numbers, the ring of rational integers and the set of nonnegative integers, respectively. Let \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \).

Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \). Siegel’s theorem states that for a fixed Weierstrass equation defining \( E \), the set of integral points on \( E \) is finite. However it is difficult to determine all integral points on elliptic curves. For example, the points \((x, y) = (28844402, \pm 154914585540)\) are the largest integral points on the elliptic curve

\[ y^2 = x^3 + 27x - 62, \quad (1) \]

proposed by Don Zagier [14], in 1987. Then the same problem of integral points on the elliptic curve (1) was dealt with by some authors and by using different methods, we refer the reader to ([2, 3, 12, 13, 15]). In the literature, there are many results and advanced methods have been developed in studying integral points on elliptic curves (see [1, 10, 11]). In [13], Yang and Fu proved that if \( n > 1 \) and both \( 6n^2 - 1 \) and \( 12n^2 + 1 \) are odd primes, then the elliptic curve \( y^2 = x^3 + (36n^2 - 9)x - 2(36n^2 - 5) \) has

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only the integral point \((x, y) = (2, 0)\). In this paper, using elementary number theory methods and some properties of generalized Fibonacci and Lucas sequences, we obtain the following results.

**Theorem 1.1** — Let \(p = 8k + 5, q = 8k + 3\) be the twin prime pair for some nonnegative integer \(k\). Assume that \(\left(\frac{5}{p}\right) = -1\) or \(\left(\frac{7}{q}\right) = -1\). Let \(E_{pq}\) be the elliptic curve given by the Weierstrass equation:

\[
E_{pq} : y^2 = x^3 + (pq - 12)x - 2(pq - 8).
\]

Then \(E_{pq}\) has only the integral point \((x, y) = (2, 0)\), where \((\cdot)\) denote Legendre symbol.

Let \(E\) be an elliptic curve defined over \(\mathbb{Z}\) given by a Weierstrass equation

\[
y^2 = x^3 + Ax + B, \ A, B \in \mathbb{Z}.
\]

By [9], VIII Corollary 7.2, we have \(E(\mathbb{Q})_{\text{tors}} \subseteq E(\mathbb{Z})\), where \(E(\mathbb{Q})_{\text{tors}}\) denotes the torsion subgroup of \(E(\mathbb{Q})\).

**Corollary 1.2** — Let \(p = 8k + 5, q = 8k + 3\) be the twin prime pair for some nonnegative integer \(k\). Let \(E_{pq}\) be the elliptic curve given by the Weierstrass equation:

\[
E_{pq} : y^2 = x^3 + (pq - 12)x - 2(pq - 8).
\]

Then we have

\[
E_{pq}(\mathbb{Q})_{\text{tors}} = \{O, (2, 0)\}.
\]

**Proof** : In particular, if we assume that \(\left(\frac{5}{p}\right) = -1\) or \(\left(\frac{7}{q}\right) = -1\), it is trivial from \(E_{pq}(\mathbb{Q})_{\text{tors}} \subseteq E_{pq}(\mathbb{Z})\) and Theorem 1.1. Next, we shall give another proof by reduction properties for arbitrary twin prime pair \(p = 8k + 5, q = 8k + 3\).

It is clear that \(\{O, (2, 0)\} \subseteq E_{pq}(\mathbb{Q})_{\text{tors}}\). If \(k = 0\), i.e., \(p = 5, q = 3\), we have

\[
E_{5,3} : y^2 = x^3 + 3x - 14
\]

It has discriminant \(\Delta = 2^73^35^2\), so \(\overline{E}_{5,3}\) is nonsingular modulo \(p\) for every prime \(p \geq 7\). Hence, we have

\[
\overline{E}_{5,3}(\mathbb{F}_7) = \{O, (0, 0), (1, 2), (1, 5), (2, 0), (3, 1), (3, 6), (5, 0)\}
\]

\[
\overline{E}_{5,3}(\mathbb{F}_{11}) = \{O, (1, 1), (1, 10), (2, 0), (3, 0), (5, 4), (5, 7), (6, 0),
\]

\[
(7, 3), (7, 8), (8, 4), (8, 7), (9, 4), (9, 7), (10, 2), (10, 9)\}
Since $E_{5,3}(\mathbb{Q})_{\text{tors}}$ injects into both of these groups, we see that $(2, 0)$ is the only nonzero torsion point in $E_{5,3}(\mathbb{Q})$. Hence, $E_{5,3}(\mathbb{Q})_{\text{tors}} = \{O, (2, 0)\}$.

Assume $k \geq 1$. Then the elliptic curve $E_{pq}$ has good reduction at 3 since the discriminant $\Delta = -64(pq)^3 + 576(pq)^2$ and $3 \nmid \Delta$. Since $p = 8k + 5$, $q = 8k + 3$ are the twin primes, we have $k = 3t + 1$ for some positive integer $t$. Reducing modulo 3, we have

$$\tilde{E}_{pq} : \quad y^2 = x^3 + 2x.$$

By a similar way as above, it's easy to calculate that $\sharp \tilde{E}_{pq}(\mathbb{F}_3) = 2$. By Proposition 2.3, we obtain $E_{pq}(\mathbb{Q})_{\text{tors}} = \{O, (2, 0)\}$. \hfill $\Box$

**Remark 1.3** : For the above elliptic curves $E_{pq}$, there exists $k$ such that $\text{rank}(E_{pq}) \geq 1$, e.g., $k = 7$, $p = 61$, $q = 59$, $\text{rank}(E_{3599}) = 1$.

This paper is organized as follows. In section 2, we will recall some concepts and fundamental results which will be used in the following sections. In section 3, we will prove our main results.

## 2. Preliminary

Let $\alpha$ and $\beta$ be non-zero integers with $\alpha^2 + 4\beta \neq 0$. The generalized Fibonacci sequence $(U_n(\alpha, \beta))$ and the Lucas sequence $(V_n(\alpha, \beta))$ are defined by the following recurrence relations:

$$U_0(\alpha, \beta) = 0, \quad U_1(\alpha, \beta) = 1, \quad U_{n+2}(\alpha, \beta) = \alpha U_{n+1}(\alpha, \beta) + \beta U_n(\alpha, \beta)$$

and

$$V_0(\alpha, \beta) = 2, \quad V_1(\alpha, \beta) = \alpha, \quad V_{n+2}(\alpha, \beta) = \alpha V_{n+1}(\alpha, \beta) + \beta V_n(\alpha, \beta)$$

for $n \geq 0$.

We have the well-known expressions named Binet’s formulas:

$$U_n(\alpha, \beta) = \frac{\eta^n - \theta^n}{\eta - \theta} \quad \text{and} \quad V_n(\alpha, \beta) = \eta^n + \theta^n$$

where $\eta = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2}$, and $\theta = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}$, $n \geq 1$.

If $\beta = -1$, we represent $(U_n)$ and $(V_n)$ by $(u_n)$ and $(v_n)$, respectively. Note that if $\alpha \geq 3$, then $u_n > 0$ for all $n \geq 1$.

**Lemma 2.1** — [8]. Let $t \geq 1$ and $s \geq 1$, then $(u_t, u_s) = u_{(t,s)}$.

Let $(u_n(\alpha, -1))$ be the generalized Fibonacci sequence, we state the following theorem.
Lemma 2.2 — [5]. Let $\alpha$ be an integer with $\alpha > 2$. If $u_n(\alpha, -1) = cx^2$ with $x \in \mathbb{Z}$ and $c \in \{1, 2, 3, 6\}$ and $n > 3$, then $(n, \alpha, c) = (4, 338, 1)$ or $(6, 3, 1)$.

The following identities for $(u_n)$ and $(v_n)$ are well-known:

(A) $u_{2n} = u_nv_n$

(B) $v_n = u_{n+1} - u_{n-1}$

(C) $u_{2t+1} - 1 = u_tv_{t+1}$

Moreover, if $\alpha$ is even, then:

(D) $u_n$ is even $\iff$ $n$ is even.

(E) $u_n$ is odd $\iff$ $n$ is odd.

For more information about generalized Fibonacci and Lucas sequences, we refer the reader to [6, 8].

Proposition 2.3 — [4, Theorem 5.1]. Let $E$ be an elliptic curve given by Weierstrass equation with coefficients in $\mathbb{Z}$. If $p$ is an odd prime such that $p \nmid \Delta$, then the restriction to $E(\mathbb{Q})_{\text{tors}}$ of the reduction homomorphism $r_p : E(\mathbb{Q}) \rightarrow E_p(\mathbb{F}_p)$ is one-one.

Proposition 2.4 — [7, Theorem 104]. If $D$ is a natural number which is not a perfect square, the equation

$$x^2 - Dy^2 = 1$$

(2)

has infinitely many solutions $x + y\sqrt{D}$. All solutions with positive $x$ and $y$ are obtained by the formula

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n$$

(3)

where $x_1 + y_1\sqrt{D}$ is the fundamental solution of (2), where $n$ runs through all natural numbers.

3. PROOF OF THEOREM 1.1

We would like point out that the idea used here is similar to that in [3].

We assume that $k \geq 1$ in the following. By the equation of the elliptic curve $E_{pq}$, we have:

$$E_{pq} : y^2 = (x - 2)(x^2 + 2x + (pq - 8)).$$

(4)

Since $x^2 + 2x + pq - 8 > 0$, the elliptic curve $E_{pq}$ has only the integral point $(x, y) = (2, 0)$ with $y = 0$. 
We now assume that \((x, y)\) is an integral point of (4) with \(y \neq 0\). Since \(y^2 > 0\) and \(x^2 + 2x + pq - 8 > 0\), by (4), we have \(x - 2 > 0\). Let \(l = x - 2 > 0\). Substituting this value of \(l\) into (4), we obtain:

\[ E_{pq} : y^2 = l(l^2 + 6l + pq). \]  

Let \(d = (l, l^2 + 6l + pq)\). It’s clear that \(d \in \{1, q, p, pq\}\). From (5), we obtain

\[ l = da^2, \quad l^2 + 6l + pq = db^2, \quad y = \pm dab, \quad (a, b) = 1 \]

for some positive integers \(a\) and \(b\).

**Case 1:** Assume \(d = 1\). By (5), we obtain:

\[ a^4 + 6a^2 + pq = b^2. \]

Hence \(a\) and \(b\) have different parity.

(i) If \(a\) is odd and \(b\) is even, then \(b = 2m\) for some \(m \in \mathbb{N}^*\) and

\[ 6 \equiv a^4 + 6a^2 + pq \equiv b^2 \equiv 4m^2 \pmod{8} \]

i.e.,

\[ 6 \equiv 4m^2 \equiv 0 \pmod{4} \]

a contradiction.

(ii) If \(a\) is even and \(b\) is odd, then

\[ -1 \equiv a^4 + 6a^2 + pq \equiv b^2 \equiv 1 \pmod{8}, \]

a contradiction.

**Case 2:** Assume \(d = p, q\). By (5), we obtain:

\[ pa^4 + 6a^2 + q = b^2, \quad \text{or} \quad qa^4 + 6a^2 + p = b^2. \]

It is an analogy to that of case 1. By using modulo 8, we shall reduce a contradiction.

**Case 3:** Assume \(d = pq\). By (5), we obtain:

\[ (pqa^2 + 3)^2 + pq - 9 = pqb^2. \]

This equation is of the form:

\[ u^2 - pqv^2 = -(pq - 9). \]  

(6)
It is easy to see that the fundamental solution of \( x^2 - pqy^2 = 1 \) is \( \eta = (p - 1) + \sqrt{pq} \). Hence all positive solutions of \( x^2 - pqy^2 = 1 \) are of the form

\[
x_n + y_n \sqrt{pq} = \eta^n, \quad n \in \mathbb{N},
\]

and \( x_n = \frac{\eta^n + \theta^n}{2} \) and \( y_n = \frac{\eta^n - \theta^n}{2 \sqrt{pq}} \), where \( \theta = (p - 1) - \sqrt{pq} \).

By Binet’s formulas (6), and \( \eta = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2} \), \( \theta = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2} \), we have \( \alpha = 2(p - 1) \), \( \beta = -1 \), hence, \( x_n = \frac{\eta^n + \theta^n}{2} = v_n(2(p - 1), -1) \) and \( y_n = \frac{\eta^n - \theta^n}{2 \sqrt{pq}} = (\eta - \theta)(u_n(2(p - 1), -1)) \)

\[
= u_n(2(p - 1), -1). \quad \text{Since the fundamental solutions of (6) are } 3 + \sqrt{pq} \text{ and } 3 - \sqrt{pq}, \text{ by Proposition 2.4 and } P_{204-212} \text{ of [7], the equation (6) has exactly two solution classes given by:}
\]

\[
a_n + b_n \sqrt{pq} = (3 - \sqrt{pq})(x_n + y_n \sqrt{pq}) = 3x_n - pqy_n + (3y_n - x_n) \sqrt{pq}
\]

and

\[
a_n + b_n \sqrt{pq} = (3 + \sqrt{pq})(x_n + y_n \sqrt{pq}) = 3x_n + pqy_n + (x_n + 3y_n) \sqrt{pq}
\]

with \( n \geq 1 \). Since \((pq^2 + 3, b)\) is a solution of (6), we have \((pq^2 + 3, b) = (a_n, b_n)\) for some \( n_0 \in \mathbb{N}^* \). Hence

\[
pqa^2 + 3 = 3x_{n_0} - pqy_{n_0}, \quad \text{or} \quad 3x_{n_0} + pqy_{n_0}.
\]

On the one hand, by \( x_n = \frac{v_p}{2}, y_n = u_n \) and \( v_n = u_{n+1} - u_{n-1} \), we have

\[
3x_{n_0} - pqy_{n_0} = (-4p^2 + 11p - 3)u_{n_0} - 3u_{n_0-1}
\]

and

\[
3x_{n_0} + pqy_{n_0} = 3u_{n_0+1} + (4p^2 - 11p + 3)u_{n_0}.
\]

Since \(-4p^2 + 11p - 3 < 0 \) and \( u_{n_0} \geq 0 \), we obtain

\[
pqa^2 + 3 = 3u_{n_0+1} + (4p^2 - 11p + 3)u_{n_0}.
\]

On the other hand, by induction on \( n \), it is easy to show that:

\[
u_n \equiv \begin{cases} 
-n \pmod{p}, & \text{if } n \text{ even,} \\
\quad n \pmod{p}, & \text{if } n \text{ odd,}
\end{cases}
\]

and

\[
u_n \equiv n \pmod{2p - 4}.
\]
If \( n_0 \) is odd, by using (9), we have

\[
3 \equiv pqa^2 + 3 \equiv 3u_{n_0 + 1} + (4p^2 - 11p + 3)u_n_0 \equiv -3(n_0 + 1) + (4p^2 - 11p + 3)n_0 \equiv -3 \pmod p.
\]

This is a contradiction. Hence \( n_0 \) is even, i.e., \( n_0 = 2r \) for some \( r > 0 \).

(i) Assume \( a \) is odd. By (10), we have

\[
pqa^2 + 3 \equiv 3u_{n_0 + 1} + (4p^2 - 11p + 3)u_n_0 \equiv (p^2 - 5p)n_0 + 6n_0 + 3 \pmod {2p - 4}.
\]

Hence

\[
pqa^2 \equiv 0 \pmod 2.
\]

This is a contradiction since \( p, q \) and \( a \) are odd.

(ii) Assume \( a \) is even. Then we have

\[
pqa^2 = 3u_{n_0 + 1} + (4p^2 - 11p + 3)u_n_0 - 3
\]

\[
= 3u_{2r + 1} + (4p^2 - 11p + 3)u_{2r} - 3
\]

\[
= 3(u_rv_{r+1}) + (4p^2 - 11p + 3)u_ru_r \text{ (by (A) and (C))}
\]

\[
= u_r[3v_{r+1} + (4p^2 - 11p + 3)v_r]
\]

\[
= u_r[3u_{r+2} - 3u_r + (4p^2 - 11p + 3)u_{r+1} - (4p^2 - 11p + 3)u_{r-1}] \text{ (by (B))}
\]

\[
= u_r[(2p^3 - 8p)u_r - (2p^2 - 4p)u_{r-1}]
\]

\[
= u_r[2pq(p + 2)u_r - 2pqu_{r-1}].
\]

Hence

\[
a^2 = u_r[2(p + 2)u_r - 2u_{r-1}].
\]

This is a contradiction by the following Lemma 3.1.

To sum up, we complete the proof of the Theorem 1.1. \(\square\)

**Lemma 3.1** — Let \( a \in \mathbb{N}^* \) and \( r \in \mathbb{N}^* \). If \( a \) and \( r \) are even, then

\[
a^2 \neq u_r[2(p + 2)u_r - 2u_{r-1}].
\]

**Proof**: Assume there exist even integers \( a > 0, r > 0 \) such that

\[
a^2 = u_r[2(p + 2)u_r - 2u_{r-1}]. \tag{11}
\]

By (D), \( u_r \) is even, i.e., \( u_r = 2A_r \) for some integer \( A_r \). Let \( a = 2b \) for some \( b \in \mathbb{N}^* \). Then the equality (11) is changed to

\[
4b^2 = 2A_r[2(p + 2)u_r - 2u_{r-1}],
\]
i.e.,

\[ b^2 = A_r[(p + 2)u_r - u_{r-1}] . \]  

(12)

Hence \( b \) and \( A_r \) have the same parity.

(i) If \( b \) is even and \( A_r \) is even, i.e., \( b = 2c \) for some \( c \in \mathbb{N}^* \), and \( A_r = 2B_r \) for some \( B_r \in \mathbb{N} \). Then the equality (12) is changed to

\[ 4c^2 = 2B_r[(p + 2)u_r - u_{r-1}] . \]  

(13)

i.e.,

\[ 2c^2 = B_r[(p + 2)u_r - u_{r-1}] . \]  

(14)

By Lemma 2.1, we have

\[ (4B_r, u_{r-1}) = (u_r, u_{r-1}) = u_{(r,r-1)} = 1 . \]

Hence

\[ (B_r, (p + 2)u_r - u_{r-1}) = (B_r, u_{r-1}) = 1 . \]

By (15), we have

\[
\begin{align*}
&\begin{cases}
  c = ef, (e, f) = 1, \\
  B_r = 2e^2, \\
  (p + 2)u_r - u_{r-1} = f^2.
\end{cases}
\end{align*}
\]

Hence

\[ u_r = 2(2e)^2 . \]

(ii) If \( b \) is odd and \( A_r \) is odd, then By Lemma 2.2, we have

\[ (2A_r, u_{r-1}) = (u_r, u_{r-1}) = u_{(r,r-1)} = 1 . \]

Hence

\[ (A_r, (p + 2)u_r - u_{r-1}) = (A_r, u_{r-1}) = 1 . \]

By (11), we have

\[
\begin{align*}
&\begin{cases}
  b = ef, (e, f) = 1, \\
  A_r = e^2, \\
  (p + 2)u_r - u_{r-1} = f^2.
\end{cases}
\end{align*}
\]
Hence
\[ u_r = (2e)^2 \]

By the above, we have \( u_r = 2(2e)^2 \) or \( u_r = (2e)^2 \), since \( r \) is even and \( \alpha = 2(p - 1) > 2 \), by Lemma 2.2, we have \( r = 2 \). By (11), we have
\[ a^2 = u_2[2(p + 2)u_2 - 2u_1] = 8p^3 - 28p + 20 = 8q^3 + 48q^2 + 68q + 28. \]

Hence
\[ a^2 \equiv 20 \pmod{p} \text{ and } a^2 \equiv 28 \pmod{q} \]
i.e.,
\[ b^2 \equiv 5 \pmod{p} \text{ and } b^2 \equiv 7 \pmod{q} \]

This is a contradiction, since
\[ \left( \frac{5}{p} \right) = -1 \text{ or } \left( \frac{7}{q} \right) = -1. \]

This completes the proof of the lemma. \( \square \)

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