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# NORMAL COMPLETIONS OF BLOCK PARTIAL MATRICES

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Bhatia and Choi asked when can two matrices $B$ and $C$ be the off-diagonal blocks of a partitioned normal matrix. Based on their study, we give some further results on this problem. The similar problem for conjugate-normal matrices is also considered.

**Key words** : Partial matrix; normal completion; block matrix; norm; conjugate-normal matrix.

## 1. Introduction

A *partial matrix* is a matrix in which some entries are specified while the others are free to be chosen. A *completion* of a partial matrix is a specific choice of values for its unspecified entries. A typical matrix completion problem asks whether a given partial matrix can be completed to a matrix with prescribed properties [2].

A square matrix $A$ is called *normal* if $AA^* = A^*A$, where $A^*$ denotes the conjugate transpose of $A$. Unitary matrices and Hermitian matrices are special normal matrices. If a partial matrix can be completed to a normal matrix, then we say that the partial matrix has a *normal completion.* In [1], Bhatia and Choi asked the following question:

Given a block partial matrix

$$N = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix},$$

when does $N$ have a normal completion?

This problem is interesting, difficult and thus worthy of investigation. Bhatia and Choi have made some achievements on it. In Section 2, we further discuss this problem and give some more results. In Section 3, we consider the similar completion problem for conjugate-normal matrices.

Denote by $M_{m,n}$ the set of $m \times n$ complex matrices and $M_{n,n}$ is abbreviated as $M_n$. A norm $||\cdot||$ on $M_{m,n}$ is called *unitarily invariant* if $||UAV|| = ||A||$ for any $A \in M_{m,n}$ and any unitary $U \in M_m$, $V \in M_n$. Denote by $||\cdot||_F$ the Frobenius norm on $M_{m,n}$, i.e., $||A||_F = \sqrt{\mathrm{tr}(A^*A)}$, $A \in M_{m,n}$. Let $\lambda_i(A)$ be the $i$-th eigenvalue of $A \in M_n$, and let $s_i(A) = \sqrt{\lambda_i(A^*A)}$ be the $i$-th singular value of $A \in M_{m,n}$. Denote by $I_n$ the identity matrix of order $n$ and by $J_{m,n}$ the $m \times n$ matrix with each entry being 1. For simplicity, we use 0 to denote the zero matrix whose size will be clear from the context.

## 2. Normal Completion

First we show that the converse of statement in [1, Proposition] is also true.

***Theorem* 2.1** — *Let $B \in M_{n,m}$, $C \in M_{m,n}$. Then the block partial matrix $N = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a completion that is a scalar multiple of a unitary matrix if and only if $||B^*|| = ||C||$ for every unitarily invariant norm $||\cdot||$ on $M_{m,n}$.*

PROOF : For the necessity, suppose $\widehat{N} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a scalar multiple of a unitary matrix, where $A \in M_n$, $D \in M_m$.

Then there exists a complex number $a$ such that $\widehat{N}\widehat{N}^* = \widehat{N}^*\widehat{N} = |a|^2 I_{m+n}$, which implies $AA^* + BB^* = A^*A + C^*C = |a|^2 I_n$. Thus $s_i^2(B^*) = \lambda_i(BB^*) = |a|^2 - \lambda_i(AA^*) = |a|^2 - \lambda_i(A^*A) = \lambda_i(C^*C) = s_i^2(C)$. Using the fact that each unitarily invariant norm of a matrix is a symmetric gauge function of its singular values [5], the necessity is proved.

Conversely, if $||B^*|| = ||C||$ for every unitarily invariant norm $||\cdot||$ on $M_{m,n}$, then $B^*$ and $C$ have the same singular values. When $m \le n$, we consider the singular value decompositions of $B$ and $C$:

$$B = U_1 \begin{bmatrix} S \\ 0 \end{bmatrix} V_1, \quad C = U_2 \begin{bmatrix} S & 0 \end{bmatrix} V_2,$$

where $U_1, V_2 \in M_n, U_2, V_1 \in M_m$ are unitary, $S \in M_m$ is diagonal.

Since $\mathrm{tr}(S^2)I_m - S^2$ is positive semidefinite, it has a unique positive semidefinite square root, denoted by $[\mathrm{tr}(S^2)I_m - S^2]^{\frac{1}{2}}$. Now let

$$\widetilde{N} = \begin{bmatrix} -[\mathrm{tr}(S^2)I_m - S^2]^{\frac{1}{2}} & 0 & S \\ 0 & \sqrt{\mathrm{tr}(S^2)}I_{n-m} & 0 \\ S & 0 & [\mathrm{tr}(S^2)I_m - S^2]^{\frac{1}{2}} \end{bmatrix}.$$

Direct computation shows $\widetilde{N}\widetilde{N}^* = \widetilde{N}^*\widetilde{N} = \mathrm{tr}(S^2)I_{m+n}$, which implies that $\widetilde{N}$ is a scalar multiple of a unitary matrix. Let $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}, V = \begin{bmatrix} V_2 & 0 \\ 0 & V_1 \end{bmatrix}$. Then $U, V$ are unitary and $U\widetilde{N}V$ is a scalar multiple of a unitary matrix whose off-diagonal blocks are $B$ and $C$.

When $m > n$, the argument is similar. This proves the sufficiency. □

*Corollary* 2.2 — Let $B$ be a row (column) vector of dimension $n$, and let $C$ be a column (row) vector of dimension $n$. Then the following statements are equivalent:

(i) $N = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a completion that is a scalar multiple of a unitary matrix.

(ii) $N = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a normal completion.

(iii) $||B^*||_F = ||C||_F$, i.e., $B$ and $C$ have the same length.

PROOF : (i)⇒(ii). Clear.

(ii)⇒(iii). Suppose $\widehat{N} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a normal completion of $N$.

Then $AA^* + BB^* = A^*A + C^*C$ implies $\mathrm{tr}(BB^*) = \mathrm{tr}(C^*C)$. Thus $||B^*||_F = ||C||_F$.

(iii)⇒(i). If $||B^*||_F = ||C||_F = 0$, then $B^* = C = 0$ and the conclusion is clear. If $||B^*||_F = ||C||_F \neq 0$, then $B$ and $C$ are nonzero vectors. In this case, $||B^*||_F$ ($||C||_F$) is the unique nonzero singular value of $B^*$ ($C$). Thus $B^*$ and $C$ have the same singular values, which implies that $||B^*|| = ||C||$ for every unitarily invariant norm $||\cdot||$. By Theorem 2.1, $N$ has a completion that is a scalar multiple of a unitary matrix. □

Let $B \in M_{n,m}, C \in M_{m,n}$. Then $||B^*||_F = ||C||_F$ is a necessary condition for $N = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ to have a normal completion. Corollary 2.2 shows that when $m = 1$ or $n = 1$, the condition is also sufficient. Thus for block partial matrix $N$ of order $m + n \leq 3$, we only need to compute the Frobenius norms of the off-diagonal blocks $B$ and $C$ to tell whether $N$ has a normal completion or not.

Bhatia and Choi gave an example to show that $B, C \in M_2$ having the same Frobenius norm is not sufficient for $N$ to have a normal completion. However, that example does not serve the purpose.

Consider the matrices

$$B = \begin{bmatrix} 1 & \varepsilon \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}.$$

If $\varepsilon = 0$, it is clear that $N$ has a normal completion. If $\varepsilon \neq 0$, let

$$A = \begin{bmatrix} 0 & 0 \\ \frac{\bar{\varepsilon}^2}{|\varepsilon|}i & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \frac{|\varepsilon|}{2}i & -\frac{\varepsilon}{|\varepsilon|}i \\ \frac{\bar{\varepsilon}}{|\varepsilon|}i & 0 \end{bmatrix},$$

where $i = \sqrt{-1}$. Direct computation shows that $\widehat{N} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is normal.

When $B, C \in M_n$ with $n \geq 4$, Theorem 1 in [1] gave us hints to show that $||B||_F = ||C||_F$ is not sufficient for $N$ to have a normal completion. For example, let $B = \mathrm{diag}(\sqrt{n}, 0, \ldots, 0)$, $C = I_n$. Then $||B||_F = ||C||_F$. But by Theorem 1 in [1], $N$ cannot be completed to any normal matrix.

Next we discuss normal completions for the case $B, C \in M_2$.

***Theorem* 2.3** — *Let $B, C \in M_2$ be diagonal matrices. Then the block partial matrix $N = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a normal completion if and only if $||B||_F = ||C||_F$.*

PROOF : The necessity is clear. Now we prove the sufficiency.

Suppose $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$, $C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ with $||B||_F = ||C||_F$. We need to find $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$ such that $\widehat{N} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is normal.

If $B$ and $C$ have the same singular values, then by Theorem 2.1, $N$ has a normal completion. Next suppose $B$ and $C$ have different singular values. Since $|b_1|^2 + |b_2|^2 = |c_1|^2 + |c_2|^2$, we have $|b_i| \neq |c_j|$ for $i, j = 1, 2$.

By the normality of $\widehat{N}$, the entries of $A$ and $D$ satisfy the following conditions:

(i) $|a_2|^2 + |b_1|^2 = |a_3|^2 + |c_1|^2$;

(ii) $|c_1|^2 + |d_2|^2 = |b_1|^2 + |d_3|^2$;

(iii) $a_1\overline{a_3} + a_2\overline{a_4} = \overline{a_1}a_2 + \overline{a_3}a_4$;

(iv) $a_1\overline{c_1} + b_1\overline{d_1} = \overline{a_1}b_1 + \overline{c_1}d_1$;

(v) $a_2\overline{c_2} + b_1\overline{d_3} = \overline{a_3}b_2 + \overline{c_1}d_2$;

(vi) $a_3\overline{c_1} + b_2\overline{d_2} = \overline{a_2}b_1 + \overline{c_2}d_3$;

(vii) $a_4\overline{c_2} + b_2\overline{d_4} = \overline{a_4}b_2 + \overline{c_2}d_4$;

(viii) $d_1\overline{d_3} + d_2\overline{d_4} = \overline{d_1}d_2 + \overline{d_3}d_4$.

First let $a_1 = a_4 = d_1 = d_4$. Then (iii), (iv), (vii) and (viii) hold.

Note that $|b_1| \neq |c_1|, |c_2|$. We distinguish two cases.

*Case* 1 : $|b_1| < |c_1|$. Let $d_2 = 0$.

If $\overline{b_1}\overline{c_1} = \overline{b_2}\overline{c_2}$, then $|b_1|^2(|c_1|^2 + |c_2|^2) = |c_2|^2(|b_1|^2 + |b_2|^2) = |c_2|^2(|c_1|^2 + |c_2|^2)$. This implies $|b_1| = |c_2|$, a contradiction. Thus $\overline{b_1}\overline{c_1} \neq \overline{b_2}\overline{c_2}$.

By (v) and (vi),

$$a_3 = \frac{\overline{a_2}(|b_1|^2 - |c_2|^2)}{\overline{b_1}\overline{c_1} - \overline{b_2}\overline{c_2}}, \quad d_3 = \frac{\overline{a_2}(b_1\overline{b_2} - \overline{c_1}c_2)}{\overline{b_1}\overline{c_1} - \overline{b_2}\overline{c_2}}.$$

Then

$$\begin{aligned} |a_3|^2 - |a_2|^2 &= \frac{|a_2|^2(|b_1|^2 - |c_2|^2)^2}{|b_1c_1 - b_2c_2|^2} - |a_2|^2 \\ &= \frac{|a_2|^2(|b_1|^4 + |c_2|^4 - 2|b_1|^2|c_2|^2 - |b_1|^2|c_1|^2 - |b_2|^2|c_2|^2 + b_1c_1\overline{b_2}\overline{c_2} + \overline{b_1}\overline{c_1}b_2c_2)}{|b_1c_1 - b_2c_2|^2} \\ &= \frac{|a_2|^2[|b_1|^2(|b_1|^2 - |c_1|^2 - |c_2|^2) + |c_2|^2(|c_2|^2 - |b_1|^2 - |b_2|^2) + b_1c_1\overline{b_2}\overline{c_2} + \overline{b_1}\overline{c_1}b_2c_2]}{|b_1c_1 - b_2c_2|^2} \\ &= \frac{|a_2|^2(-|b_1|^2|b_2|^2 - |c_1|^2|c_2|^2 + b_1\overline{b_2}c_1\overline{c_2} + \overline{b_1}b_2\overline{c_1}c_2)}{|b_1c_1 - b_2c_2|^2} \\ &= \frac{-|a_2|^2|b_1\overline{b_2} - \overline{c_1}c_2|^2}{|b_1c_1 - b_2c_2|^2} = |d_2|^2 - |d_3|^2. \end{aligned}$$

If $b_1\overline{b_2} = \overline{c_1}c_2$, then $|b_1|^2(|c_1|^2 + |c_2|^2) = |b_1|^2(|b_1|^2 + |b_2|^2) = |b_1|^4 + |c_1|^2|c_2|^2$, i.e., $(|b_1|^2 - |c_1|^2)(|b_1|^2 - |c_2|^2) = 0$. This implies $|b_1| = |c_1|$ or $|c_2|$, a contradiction. Thus $b_1\overline{b_2} \neq \overline{c_1}c_2$.

Let $a_2 = \dfrac{|b_1c_1 - b_2c_2|\sqrt{|c_1|^2 - |b_1|^2}}{|b_1\overline{b_2} - \overline{c_1}c_2|}$. Then (i) and (ii) hold.

*Case* 2 : $|b_1| > |c_1|$. Let $d_3 = 0$.

Since $|b_2| \neq |c_1|$, by (v) and (vi),

$$a_3 = \frac{\overline{a_2}(b_1c_1 - b_2c_2)}{|c_1|^2 - |b_2|^2}, \quad d_2 = \frac{a_2(c_1\overline{c_2} - \overline{b_1}b_2)}{|c_1|^2 - |b_2|^2}.$$

Then

$$
\begin{aligned}
|a_3|^2-|a_2|^2 &= \frac{|a_2|^2|b_1c_1-b_2c_2|^2}{(|c_1|^2-|b_2|^2)^2}-|a_2|^2\\
&= \frac{|a_2|^2(|b_1|^2|c_1|^2+|b_2|^2|c_2|^2-b_1c_1\overline{b_2}\overline{c_2}-\overline{b_1}\overline{c_1}b_2c_2-|c_1|^4-|b_2|^4+2|b_2|^2|c_1|^2)}{(|c_1|^2-|b_2|^2)^2}\\
&= \frac{|a_2|^2[|c_1|^2(|b_1|^2+|b_2|^2-|c_1|^2)+|b_2|^2(|c_1|^2+|c_2|^2-|b_2|^2)-b_1c_1\overline{b_2}\overline{c_2}-\overline{b_1}\overline{c_1}b_2c_2]}{(|c_1|^2-|b_2|^2)^2}\\
&= \frac{|a_2|^2(|c_1|^2|c_2|^2+|b_1|^2|b_2|^2-c_1\overline{c_2}b_1\overline{b_2}-\overline{c_1}c_2\overline{b_1}b_2)}{(|c_1|^2-|b_2|^2)^2}\\
&= \frac{|a_2|^2|c_1\overline{c_2}-\overline{b_1}b_2|^2}{(|c_1|^2-|b_2|^2)^2}=|d_2|^2-|d_3|^2.
\end{aligned}
$$

Note that $\overline{b_1}b_2\neq c_1\overline{c_2}$, as shown above. Let $a_2=\dfrac{||c_1|^2-|b_2|^2|\sqrt{|b_1|^2-|c_1|^2}}{|c_1\overline{c_2}-\overline{b_1}b_2|}$. Then (i) and (ii) hold. This completes the proof. □

*Corollary* 2.4 — Let $B, C\in M_2$ be commuting normal matrices. Then $N=\begin{bmatrix} ? & B\\ C & ?\end{bmatrix}$ has a normal completion if and only if $||B||_F=||C||_F$.

PROOF : The necessity is clear. We prove the sufficiency.

Since $B$ and $C$ are commuting normal matrices, there exists a unitary matrix $U\in M_2$ and diagonal matrices $\Lambda_1,\Lambda_2\in M_2$ such that $B=U\Lambda_1U^*$, $C=U\Lambda_2U^*$. Since $||\Lambda_1||_F=||U\Lambda_1U^*||_F=||B||_F=||C||_F=||U\Lambda_2U^*||_F=||\Lambda_2||_F$, by Theorem 2.3, there exist $A, D\in M_2$ such that $\widehat{N}=\begin{bmatrix} A & \Lambda_1\\ \Lambda_2 & D\end{bmatrix}$ is normal. Let $W=\begin{bmatrix} U & 0\\ 0 & U\end{bmatrix}$. Then $W$ is unitary and $W\widehat{N}W^*=\begin{bmatrix} UAU^* & B\\ C & UDU^*\end{bmatrix}$ is a normal completion of $N$. □

*Lemma* 2.5 — [6]. Let $A, B\in M_n$. Then there exist unitary matrices $U, V$ such that $UAV$ and $UBV$ are diagonal matrices if and only if $AB^*$ and $B^*A$ are normal matrices.

*Corollary* 2.6 — Let $B, C\in M_2$ satisfying $BC$ and $CB$ are normal matrices. Then $N=\begin{bmatrix} ? & B\\ C & ?\end{bmatrix}$ has a normal completion if and only if $||B||_F=||C||_F$.

PROOF : It suffices to prove the sufficiency.

Since $BC$ and $CB$ are normal, by Lemma 2.5, there exist unitary $U, V\in M_2$ and diagonal $\Lambda_1,\Lambda_2\in M_2$ such that $B=U\Lambda_1V^*$, $C=V\Lambda_2U^*$. Since $||\Lambda_1||_F=||B||_F=||C||_F=||\Lambda_2||_F$, by Theorem 2.3, there exist $A, D\in M_2$ such that $\widehat{N}=\begin{bmatrix} A & \Lambda_1\\ \Lambda_2 & D\end{bmatrix}$ is normal. Let $W=\begin{bmatrix} U & 0\\ 0 & V\end{bmatrix}$.

Then $W$ is unitary and $W\widehat{N}W^*$ is a normal completion of $N$. □

We complete this section by giving a result that relates to the Kronecker product. Recall that for two matrices $A = (a_{ij}) \in M_{m,n}$ and $B \in M_{s,t}$, the *Kronecker product* of $A$ and $B$ is denoted by $A \otimes B$ and is defined to be the block matrix $A \otimes B = (a_{ij}B) \in M_{ms,nt}$.

*Lemma* 2.7 — [4]. Let $B$ and $X$ be square matrices, and let $m$ be any positive integer. If the matrix $\begin{bmatrix} B & U \\ V & X \end{bmatrix}$ is normal, then the matrix

$$\begin{bmatrix} B & \frac{1}{\sqrt{m}}U & \frac{1}{\sqrt{m}}U & \cdots & \frac{1}{\sqrt{m}}U \\ \frac{1}{\sqrt{m}}V & \frac{1}{m}X & \frac{1}{m}X & \cdots & \frac{1}{m}X \\ \frac{1}{\sqrt{m}}V & \frac{1}{m}X & \frac{1}{m}X & \cdots & \frac{1}{m}X \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{m}}V & \frac{1}{m}X & \frac{1}{m}X & \cdots & \frac{1}{m}X \end{bmatrix}$$

of block size $(m+1) \times (m+1)$ is normal.

The following lemma is easy to prove.

*Lemma* 2.8 — Let $B$ and $X$ be square matrices. If the matrix $\begin{bmatrix} B & U \\ V & X \end{bmatrix}$ is normal, then the matrix $\begin{bmatrix} X^* & U^* \\ V^* & B^* \end{bmatrix}$ is normal.

***Theorem* 2.9** — *Suppose the block partial matrix* $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ *has a normal completion, where the diagonal blocks are square matrices. Then for any positive integers* $m$ *and* $n$, *the block partial matrix*

$$\begin{bmatrix} ? & J_{m,n} \otimes B \\ J_{n,m} \otimes C & ? \end{bmatrix}$$

*has a normal completion.*

PROOF : Suppose $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a normal completion $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where $A$ and $D$ are square matrices. By Lemma 2.7, the matrix

$$\begin{bmatrix} A & \frac{1}{\sqrt{n}}J_{1,n}\otimes B \\ \frac{1}{\sqrt{n}}J_{n,1}\otimes C & \frac{1}{n}J_{n,n}\otimes D \end{bmatrix} = \begin{bmatrix} A & \frac{1}{\sqrt{n}}B & \frac{1}{\sqrt{n}}B & \cdots & \frac{1}{\sqrt{n}}B \\ \frac{1}{\sqrt{n}}C & \frac{1}{n}D & \frac{1}{n}D & \cdots & \frac{1}{n}D \\ \frac{1}{\sqrt{n}}C & \frac{1}{n}D & \frac{1}{n}D & \cdots & \frac{1}{n}D \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n}}C & \frac{1}{n}D & \frac{1}{n}D & \cdots & \frac{1}{n}D \end{bmatrix}$$

is normal. Note that $(U\otimes V)^* = U^*\otimes V^*$ for any matrices $U$ and $V$. Then by Lemma 2.8, the matrix

$$\begin{bmatrix} \frac{1}{n}J_{n,n}\otimes D^* & \frac{1}{\sqrt{n}}J_{n,1}\otimes B^* \\ \frac{1}{\sqrt{n}}J_{1,n}\otimes C^* & A^* \end{bmatrix}$$

is normal. Since $J_{1,m}\otimes J_{n,1} = J_{n,m}$ and $J_{m,1}\otimes J_{1,n} = J_{m,n}$, again by Lemma 2.7, the matrix

$$\begin{bmatrix} \frac{1}{n}J_{n,n}\otimes D^* & \frac{1}{\sqrt{m}}J_{1,m}\otimes (\frac{1}{\sqrt{n}}J_{n,1}\otimes B^*) \\ \frac{1}{\sqrt{m}}J_{m,1}\otimes (\frac{1}{\sqrt{n}}J_{1,n}\otimes C^*) & \frac{1}{m}J_{m,m}\otimes A^* \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{n}J_{n,n}\otimes D^* & \frac{1}{\sqrt{mn}}J_{n,m}\otimes B^* \\ \frac{1}{\sqrt{mn}}J_{m,n}\otimes C^* & \frac{1}{m}J_{m,m}\otimes A^* \end{bmatrix}$$

is normal. Then again by Lemma 2.8, the matrix

$$\begin{bmatrix} \frac{1}{m}J_{m,m}\otimes A & \frac{1}{\sqrt{mn}}J_{m,n}\otimes B \\ \frac{1}{\sqrt{mn}}J_{n,m}\otimes C & \frac{1}{n}J_{n,n}\otimes D \end{bmatrix}$$

is normal. Thus the matrix

$$\begin{bmatrix} \sqrt{\frac{n}{m}}J_{m,m}\otimes A & J_{m,n}\otimes B \\ J_{n,m}\otimes C & \sqrt{\frac{m}{n}}J_{n,n}\otimes D \end{bmatrix}$$

is a normal completion of $\begin{bmatrix} ? & J_{m,n}\otimes B \\ J_{n,m}\otimes C & ? \end{bmatrix}$. □

## 3. Conjugate-normal Completion

A square matrix $A$ is called *conjugate-normal* if $AA^* = \overline{A^*A}$, where $\overline{A}$ denotes the entrywise conjugate of $A$. Unitary matrices and complex symmetric matrices are special conjugate-normal matrices. Conjugate-normal matrices play the same important role in the theory of unitary congruence as the normal matrices do with respect to unitary similarities. For more properties and characterizations of this kind of matrices, see the survey paper [3].

If a partial matrix can be completed to a conjugate-normal matrix, then we say that the partial matrix has a *conjugate-normal completion.* Naturally we have the following question:

Given a block partial matrix

$$N_c = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix},$$

when does $N_c$ have a conjugate-normal completion?

Let $B \in M_{n,m}$, $C \in M_{m,n}$. Suppose $\widehat{N_c} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a conjugate-normal completion of $N_c$. Then $AA^* + BB^* = \overline{A^*A} + \overline{C^*C}$ implies $\mathrm{tr}(BB^*) = \mathrm{tr}(C^*C)$. Thus $||B^*||_F = ||C||_F$, where $||\cdot||_F$ is the Frobenius norm on $M_{m,n}$. Since a scalar multiple of a unitary matrix is conjugate-normal, we have an analogue of Corollary 2.2 for conjugate-normal completions. Thus when $m = 1$ or $n = 1$, $||B^*||_F = ||C||_F$ is a necessary and sufficient condition for $N_c$ to have a conjugate-normal completion.

Now we discuss conjugate-normal completions for the case $B, C \in M_2$. First we found that $||B||_F = ||C||_F$ is not sufficient for $N_c = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ to have a conjugate-normal completion. Consider the matrices

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $||B||_F = ||C||_F$. Assume that there exist matrices $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$ such that $\widehat{N_c} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is conjugate-normal. Direct computation shows that the entries of $A$ and $D$ satisfy the following conditions:

(i) $|a_2| = |a_3|$;

(ii) $a_1\overline{a_3} + a_2\overline{a_4} = a_1\overline{a_2} + a_3\overline{a_4} + 1$;

(iii) $a_2 = 0$;

(iv) $\overline{d_3} = a_3 + \overline{d_2}$;

(v) $a_3 + a_4 + \overline{d_2} = a_2 + \overline{d_1}$;

(vi) $\overline{d_4} = a_4 + \overline{d_2}$;

(vii) $1 + |d_2|^2 = |d_3|^2$;

(viii) $d_1\overline{d_3} + d_2\overline{d_4} = d_1\overline{d_2} + d_3\overline{d_4}$.

Note that (i) and (iii) imply $a_2 = a_3 = 0$, which contradicts (ii). Thus $N_c$ cannot be completed to any conjugate-normal matrix.

For conjugate-normal completions, we also have analogues of Theorems 2.3 and 2.9.

***Theorem* 3.1** — *Let $B, C \in M_2$ be diagonal matrices. Then the block partial matrix $N_c = \begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ has a conjugate-normal completion if and only if $||B||_F = ||C||_F$.*

PROOF : The necessity is clear. Now we prove the sufficiency.

Suppose $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$, $C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ with $|b_1|^2 + |b_2|^2 = |c_1|^2 + |c_2|^2$. We need to find $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $D = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix}$ such that $\widehat{N_c} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is conjugate-normal.

If $B$ and $C$ have the same singular values, then by Theorem 2.1, $N_c$ has a conjugate-normal completion. Next suppose $B$ and $C$ have different singular values, i.e., $|b_i| \neq |c_j|$ for $i, j = 1, 2$. We distinguish two cases.

*Case* 1 : $|b_1| < |c_1|$. If $b_1\overline{c_1} = \overline{b_2}c_2$, then $|b_1|^2(|c_1|^2+|c_2|^2) = |c_2|^2(|b_1|^2+|b_2|^2) = |c_2|^2(|c_1|^2+|c_2|^2)$. This implies $|b_1| = |c_2|$, a contradiction. Thus $b_1\overline{c_1} \neq \overline{b_2}c_2$.

If $\overline{b_1 b_2} = \overline{c_1 c_2}$, then $|b_1|^2(|c_1|^2 + |c_2|^2) = |b_1|^2(|b_1|^2 + |b_2|^2) = |b_1|^4 + |c_1|^2|c_2|^2$, i.e., $(|b_1|^2 - |c_1|^2)(|b_1|^2 - |c_2|^2) = 0$. This implies $|b_1| = |c_1|$ or $|c_2|$, a contradiction. Thus $\overline{b_1 b_2} \neq \overline{c_1 c_2}$.

Let $a_1 = 0$, $a_2 = \dfrac{|b_1\overline{c_1} - \overline{b_2}c_2|\sqrt{|c_1|^2 - |b_1|^2}}{|\overline{b_1 b_2} - \overline{c_1 c_2}|}$, $a_3 = \dfrac{a_2(|b_1|^2 - |c_2|^2)}{b_1\overline{c_1} - \overline{b_2}c_2}$, $a_4 = 0$,
$d_1 = 0$, $d_2 = 0$, $d_3 = \dfrac{a_2(\overline{b_1 b_2} - \overline{c_1 c_2})}{b_1\overline{c_1} - \overline{b_2}c_2}$, $d_4 = 0$.

Direct computation shows that $\widehat{N_c} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is conjugate-normal.

*Case* 2 : $|b_1| > |c_1|$. Note that $b_1 b_2 \neq c_1 c_2$, as shown above.

Let $a_1 = 0$, $a_2 = \dfrac{||c_1|^2 - |b_2|^2|\sqrt{|b_1|^2 - |c_1|^2}}{|c_1 c_2 - b_1 b_2|}$, $a_3 = \dfrac{a_2(\overline{b_1}c_1 - b_2\overline{c_2})}{|c_1|^2 - |b_2|^2}$, $a_4 = 0$,
$d_1 = 0$, $d_2 = \dfrac{\overline{a_2}(c_1 c_2 - b_1 b_2)}{|c_1|^2 - |b_2|^2}$, $d_3 = 0$, $d_4 = 0$.

Direct computation shows that $\widehat{N_c} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is conjugate-normal. This completes the proof. □

The following lemma is easy to prove.

*Lemma* 3.2 — Let $B$ and $X$ be square matrices, and let $m$ be any positive integer. If the matrix $\begin{bmatrix} B & U \\ V & X \end{bmatrix}$ is conjugate-normal, then the matrices $\begin{bmatrix} B & \frac{1}{\sqrt{m}} J_{1,m} \otimes U \\ \frac{1}{\sqrt{m}} J_{m,1} \otimes V & \frac{1}{m} J_{m,m} \otimes X \end{bmatrix}$ and $\begin{bmatrix} X^* & U^* \\ V^* & B^* \end{bmatrix}$ are conjugate-normal.

***Theorem* 3.3** — *Suppose the block partial matrix* $\begin{bmatrix} ? & B \\ C & ? \end{bmatrix}$ *has a conjugate-normal completion, where the diagonal blocks are square matrices. Then for any positive integers* $m$ *and* $n$, *the block partial matrix*

$$\begin{bmatrix} ? & J_{m,n} \otimes B \\ J_{n,m} \otimes C & ? \end{bmatrix}$$

*has a conjugate-normal completion.*

PROOF : The argument is similar to that of Theorem 2.9. □

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## REFERENCES

1. R. Bhatia and M. Choi, Corners of normal matrices, *Proc. Indian Acad. Sci. Math. Sci*., **116** (2006), 393-399.
2. G. Cravo, Matrix completion problems, *Linear Algebra Appl*., **430** (2009), 2511-2540.
3. H. Faßender and Kh.D. Ikramov, Conjugate-normal matrices: A survey, *Linear Algebra Appl*., **429** (2008), 1425-1441.
4. Z. Li, F. Hall, and F. Zhang, Sign patterns of nonnegative normal matrices, *Linear Algebra Appl*., **254** (1997), 335-354.
5. J. von Neumann, Some matrix-inequalities and metrization of matric-space, *Tomsk. Univ. Rev*., **1** (1937), 286-300.
6. J. Williamson, Note on a principal axis transformation for non-hermitian matrices, *Bull. Amer. Math. Soc*., **45** (1939), 920-922.