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COMMENT ON "IDENTITIES AND CONGRUENCES FOR THE GENERAL PARTITION AND RAMANUJAN'S TAU FUNCTIONS"

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This short commentary serves as a correction of the paper by Baruah and Sarmah [N. D. Baruah and B. K. Sarmah, Identities and congruences for the general partition and Ramanujan's tau functions, *Indian J. Pure. Appl. Math.*, **44**(5) (2013), 643-671].

Key words : Congruences; Ramanujan's tau functions

1. Corrections

Let

$$\sum_{n=0}^{\infty} p_r(n)q^n = (q;q)_\infty^r,$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

In [1], Baruah and Sarmah introduced the following identity (which is identity (1.7) in [1])

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^6}{(q^5;q^5)_\infty^6}\Big(F^4(q^5) + qF^3(q^5) + 2q^2F^2(q^5) + 3q^3F(q^5) + 5q^4 - 3q^5F^{-1}(q^5) + 2q^6F^{-2}(q^5) - q^7F^{-3}(q^5) + q^8F^{-4}(q^5)\Big),$$

where

$$R(q) := \frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}\,\frac{q^3}{+1}+\cdots \quad |q|<1$$

is the Rogers-Ramanujan continued fraction and

$$F(q) := q^{-1/5}R(q).$$

However, the identity of this version is wrong. The correct version should be as follows. For $F(q)$ to be defined as above, then

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\left(\frac{1}{F^4(q^5)} + \frac{q}{F^3(q^5)} + \frac{2q^2}{F^2(q^5)} + \frac{3q^3}{F(q^5)} + 5q^4 - 3q^5F(q^5) + 2q^6F^2(q^5) - q^7F^3(q^5) + q^8F^4(q^5)\right). \quad (1)$$

There are several proofs of this famous identity, see for example [2, 3]. Actually, this identity plays a central role in proving one of the famous Ramanujan congruences, say

$$p(5n+4) \equiv 0 \pmod 5.$$

Squaring the identity (1), we obtain

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^{25};q^{25})_\infty^{10}}{(q^5;q^5)_\infty^{12}}\left(\frac{1}{F^8(q^5)} + \frac{2q}{F^7(q^5)} + \frac{5q^2}{F^6(q^5)} + \frac{10q^3}{F^5(q^5)} + \frac{20q^4}{F^4(q^5)} + \frac{16q^5}{F^3(q^5)} + \frac{27q^6}{F^2(q^5)} + \frac{20q^7}{F(q^5)} + 15q^8 - 20q^9F(q^5) + 27q^{10}F^2(q^5) - 16q^{11}F^3(q^5) + 20q^{12}F^4(q^5) - 10q^{13}F^5(q^5) + 5q^{14}F^6(q^5) - 2q^{15}F^7(q^5) + q^{16}F^8(q^5)\right). \quad (2)$$

This identity is the correct version of the identity (5.20) in [1], by which Baruah and Sarmah obtained

$$p_{-2}(5n+2) \equiv p_{-2}(5n+3) \equiv p_{-2}(5n+4) \equiv 0 \pmod 5.$$

We say that the congruences above are correct though Baruah and Sarmah used a wrong version of the important identity we showed above.

Furthermore, by using (1), one can prove that

$$p_{-4}(5n+3) \equiv p_{-4}(5n+4) \equiv 0 \pmod 5,$$

which means (5.5) in [1] is true. However, one should note we should use the correct version of the identity to obtain the conclusion.

References

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