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SOME FAMILY OF DIOPHANTINE PAIRS WITH FIBONACCI NUMBERS

Jinseo Park* and June Bok Lee**

**Catholic Kwandong University, Department of Mathematics Education,*
Gangneung 25601, *S. Korea*

***Yonsei University, Department of Mathematical Sciences, Sinchon-Dong,*
Seodaemun-Gu, Seoul, S. Korea

e-mails: jspark@cku.ac.kr; leejb@yonsei.ac.kr

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A set $\{a_1, a_2, \ldots, a_m\}$ of positive integers is called a Diophantine m-tuple if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. In this paper, we prove that if $\{F_{2k}, F_{2k+4}, c\}$ is a Diophantine triple then there is a unique positive integer d such that $c < d$ and $\{F_{2k}, F_{2k+4}, c, d\}$ is a Diophantine quadruple.

Key words : Diophantine m-tuple; Fibonacci numbers, Pell equation.

1. INTRODUCTION

A Diophantine m-tuple is a set of m positive integers such that the product of any two of them increased by 1 gives a perfect square. The first set which satisfies the above property with four positive rationals was found by Diophantus. However, the first Diophantine quadruple $\{1, 3, 8, 120\}$ was found by Fermat. In 1969, Baker and Davenport [2] proved that the above set cannot be extended to a Diophantine quintuple, namely, if the set $\{1, 3, 8, d\}$ is a Diophantine quadruple then $d = 120$. Many famous mathematicians like Diophantus, Fermat, Euler and Fields medalist Alan Baker made important contributions to problems related with Diophantine m-tuples, but lots of problems still remain open until now. We can find the more history and many papers about the Diophantine m-tuple in the webpage of Dujella [10].

For any Diophantine triple $\{a, b, c\}$, the set $\{a, b, c, d_\pm\}$ is a Diophantine quadruple, where

$$d_\pm = a + b + c + 2abc \pm 2rst$$

and r, s, t are the positive integers satisfying

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2.$$

The conjecture that there does not exist a Diophantine quintuple has been proved recently [19]. The stronger version of this conjecture states that if $\{a, b, c, d\}$ is a Diophantine quadruple and $d > \max\{a, b, c\}$ then $d = d_+$. These Diophantine quadruples are called regular.

We can find that the extendibility of Diophantine m-tuple is related to the elliptic curves. We have to solve the equations

$$ax + 1 = \square,\ bx + 1 = \square,\ cx + 1 = \square$$

to extend the Diophantine triple $\{a, b, c\}$ to Diophantine quadruple. Hence, we get the equation

$$E : y^2 = (ax + 1)(bx + 1)(cx + 1),$$

which is the elliptic curve from the product of three equations. There are always integer points

$$(0, \pm 1),\ (d_+, \pm(at + rs)(bs + rt)(cr + st)),\ (d_-, \pm((at - rs)(bs - rt)(cr - st))),$$

and also $(-1, 0)$ if $1 \in \{a, b, c\}$ on E. The conjecture suggests that it is possible to prove that there are no other integer points on E for some family of Diophantine triples. For example, Dujella [5] proved that the elliptic curve

$$E_k : y^2 = ((k - 1)x + 1)((k + 1)x + 1)(4kx + 1)$$

has four integer points

$$(0, \pm 1),\ (16k^3 - 4k, \pm(128k^6 - 112k^4 - 20k^2 - 1))$$

under assumption that rank$(E_k(\mathbb{Q})) = 1$. Similar results [7] and [17] were proved for the equation

$$y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4} + 1)$$

and

$$y^2 = (F_{2k+1}x + 1)(F_{2k+3}x + 1)(F_{2k+5} + 1),$$

respectively, where F_n is the n-th Fibonacci number.

Let $\{a, b, c\}$ be a Diophantine triple and F_n be the n-th Fibonacci number, defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. In 1977, Hoggatt and Bergum conjectured that if $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ is a Diophantine quadruple then d is a unique [20]. The conjecture was proved by Dujella in

1999 [4]. There are many papers which contain generalizations of the result of Hoggatt and Bergum [9, 17, 23]. Filipin, Fujita, and Togbé proved that Diophantine pairs $\{F_{2k}, F_{2k+2}\}$ cannot be extended to Diophantine quintuples [14, Theorem 1.7]. It is natural to ask what is the next number which satisfies the Diophantine property where $a = F_{2k}$ and how large these sets.

In this paper, we prove that the sets $\{F_{2k}, F_{2k+4}\}$ can be extended only to regular quadruples. Since $F_{2k}F_{2k+4} + 1 = \square$ suggests that $4F_{2k}F_{2k+4} + 4 = \square$, this result can be considered as a generalization of the result of [15]. This result is proved using system of simultaneous Pellian equation and Baker's theory of linear form in logarithms, namely, standard method which is used in solving similar problems.

2. Preliminaries

2.1 *The third elements of Diophantine pairs* $\{F_{2k}, F_{2k+4}\}$

Let $\{a, b, c\}$ be a Diophantine triple, and r, s, t be the positive integers satisfying $ab+1 = r^2,\ ac+1 = s^2,\ bc+1 = t^2$. Then we have

$$at^2 - bs^2 = a - b.$$

We easily find the form of solutions of the equation above is

$$(t\sqrt{a} + s\sqrt{b}) = (t_0\sqrt{a} + s_0\sqrt{b})(r + \sqrt{bc})^{\nu}.$$

If (t_0, s_0) belongs to the same class as either of the solutions $(\pm 1, 1)$ then s can be expressed as $s = s_\nu^\tau$, where $\tau \in \{\pm 1\}$ and

$$s_0 = s_0^\tau = 1,\ s_1^\tau = r + \tau a,\ s_{\nu+2}^\tau = 2rs_{\nu+1}^\tau - s_\nu^\tau.$$

Define $c_\nu^\tau = ((s_\nu^\tau)^2 - 1)/a$. Then, we obtain

$$c = c_\nu^\tau = \frac{1}{4ab}[(a + b \pm 2\sqrt{ab})(2ab + 1 + 2r\sqrt{ab})^{\nu}$$

$$+(a + b \mp 2\sqrt{ab})(2ab + 1 - 2r\sqrt{ab})^{\nu} - 2(a + b)].$$

Using the following theorem, we can find the form of third element c in the Diophantine triple $\{a, b, c\}$.

Theorem 1 — *[21, Theorem 8]. If $a < b < 4a$, and b are in $\mathbb{Z}^+$, and*

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2$$

holds then $c = c_k^+ = c_k^+(a,b)$ *for some* k *or* $c = c_j^- = c_j^-(a,b)$ *for some* j. *The set* c_j^- *is omitted if* $b = a+2$.

Since the upper bound of b in the above theorem is too small, the theorem can not be applied in many cases. Hence, there were a lot of researches to generalize the result. The following lemma generalizes the Theorem 1.

Lemma 1 — [13, Lemma 4.1]. Let $\{a,b,c\}$ be a Diophantine triple. Assume that $a < b \leq 8a$. Then $c = c_\nu^\tau$ for some ν and τ.

Next, the following theorem gives us the bound of third element c in the Diophantine triple $\{a,b,c\}$.

***Theorem* 2** — *[13, Theorem 1.2]. Let* $\{a,b,c\}$ *be a Diophantine triple with* $a < b$. *Suppose that* $\{a,b,c,d\}$ *is a Diophantine quadruple with* $d > d_+$ *and that* $\{a,b,c',c\}$ *is not a Diophantine quadruple for any* c' *with* $0 < c' < d_-$*, where* d_+ *and* d_- *are defined by*

$$d_\pm = a + b + c + 2abc \pm 2rst,$$

respectively.

If $b < 2a$ *then* $c < b^6$.

If $2a \leq b \leq 8a$ *then* $c < 9.5b^4$.

If $b > 8a$ *then* $c < b^5$.

If $c = c_\nu^\tau$ then we can find the upper bound of c more specific by the following theorem.

***Theorem* 3** — *[14, Theorem 1.4]. Suppose that* $\{a,b,c_\nu^\tau,d\}$ *is a Diophantine quadruple with* $d > c_{\nu+1}^\tau$ *and that* $\{a,b,c',c_\nu^\tau\}$ *is not a Diophantine quadruple for any* c' *with* $0 < c' < c_{\nu-1}^\tau$.

If $b < 2a$ *then* $c \leq c_3^+$.

If $2a \leq b \leq 8a$ *then* $c \leq c_2^+$.

2.2 *The Properties of solutions of Pell equation*

We have to solve the system

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2$$

to extend the Diophantine triple $\{a,b,c\}$ to the Diophantine quadruple $\{a,b,c,d\}$. One can eliminate

d to obtain the following system of Pell equations

$$ay^2 - bx^2 = a - b, \tag{2.1}$$

$$az^2 - cx^2 = a - c, \tag{2.2}$$

$$bz^2 - cy^2 = b - c. \tag{2.3}$$

Lemma 2 — [6, Lemma 1]. There exist positive integers i_0, j_0 and integers $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}, i = 1, \dots, i_0, j = 1, \dots, j_0$, with the following properties:

1) $(z_0^{(i)}, x_0^{(i)})$ and $(z_1^{(j)}, y_1^{(j)})$ are solutions of (2.2) and (2.3), respectively.

2) $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$ satisfy the following inequalities

$$0 < x_0^{(i)} \leq \sqrt{\frac{a(c-a)}{2(s-1)}} < \sqrt{\frac{s+1}{2}} < \sqrt[4]{ac},$$

$$0 \leq |z_0^{(i)}| \leq \sqrt{\frac{(s-1)(c-a)}{2a}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < \frac{c}{2},$$

$$0 < y_1^{(j)} \leq \sqrt{\frac{b(c-b)}{2(t-1)}} < \sqrt{\frac{t+1}{2}} < \sqrt[4]{bc},$$

$$0 \leq |z_1^{(j)}| \leq \sqrt{\frac{(t-1)(c-b)}{2b}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < \frac{c}{3}.$$

3) If (z, x) and (z, y) are positive integers of (2.2) and (2.3), respectively then there exist $i \in \{1, \dots, i_0\}$, $j \in \{1, \dots, j_0\}$ and integers $m, n \geq 0$ such that

$$z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})(s + \sqrt{ac})^m, \tag{2.4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t + \sqrt{bc})^n. \tag{2.5}$$

From now on, we omit the superscripts (i) and (j). By (2.4), we may write $z = v_m$, where

$$v_0 = z_0,\ v_1 = sz_0 + cx_0,\ v_{m+2} = 2sv_{m+1} - v_m, \tag{2.6}$$

and by (2.5), we may write $z = w_n$, where

$$w_0 = z_1,\ w_1 = tz_1 + cy_1,\ w_{n+2} = 2tw_{n+1} - w_n. \tag{2.7}$$

Lemma 3 — [8, Lemma 3]. If $v_m = w_n$ then $n - 1 \leq m \leq 2n + 1$.

2.3 *Congruence relation between solutions of Pell equations*

We give the congruence relations between v_m and w_n, and properties of initial terms of (2.6) and (2.7).

Lemma 4 — [6, Lemma 4]. We have the following properties of v_m and w_n.

$$v_{2m} \equiv z_0 + 2c[az_0m^2 + sx_0m] \pmod{8c^2},$$

$$v_{2m+1} \equiv sz_0 + c[2asz_0m(m+1) + x_0(2m+1)] \pmod{4c^2},$$

$$w_{2n} \equiv z_1 + 2c[bz_1n^2 + ty_1n] \pmod{8c^2},$$

$$w_{2n+1} \equiv tz_1 + c[2btz_1n(n+1) + y_1(2n+1)] \pmod{4c^2}.$$

It is natural to ask when does $v_m = w_n$ have a solution and if there exists a solution of $v_m = w_n$ then what are the values possible for the solution. The following lemma gives us the answer.

Lemma 5 — [8, Lemma 8]. We have the following results.

1) If the equation $v_{2m} = w_{2n}$ has a solution then $z_0 = z_1$. Furthermore, $|z_0| = 1$ or $|z_0| = cr - st$ or $|z_0| < \min\{0.869a^{-5/14}c^{9/14}, 0.972b^{-0.3}c^{0.7}\}$.

2) If the equation $v_{2m+1} = w_{2n}$ has a solution then $|z_0| = t, |z_1| = cr - st$ and $z_0z_1 < 0$.

3) If the equation $v_{2m} = w_{2n+1}$ has a solution then $|z_0| = cr - st, |z_1| = s$ and $z_0z_1 < 0$.

4) If the equation $v_{2m+1} = w_{2n+1}$ has a solution then $|z_0| = t, |z_1| = s$ and $z_0z_1 > 0$.

Furthermore, the solution of $v_m = w_n$ is more specific when $c = c_\nu^\tau \leq c_3^+$ by the following lemma.

Lemma 6 — [14, Lemma 3.1]. Assume that $a < b \leq 8a$.

(i) Assume that $b < 3a$. In the case of $c = c_1^-$, we have $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$ then $z_0 = z_1 = 1$.

(ii) In the case of $c = c_1^+$, we have $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$ then $z_0 = z_1$ and $|z_0| = 1$.

(iii) In the case of $c = c_2^-$, we have $v_{2m+1} \neq w_{2n}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, we have the following:

1) If $v_{2m} = w_{2n}$ then $z_0 = z_1$ and $|z_0| = 1$ or $cr - st$.

2) If $v_{2m} = w_{2n+1}$ then $|z_0| = cr - st$ and $|z_1| = s$ with $z_0z_1 < 0$.

Furthermore, (2) occurs if and only if (1) with $|z_0| = cr - st$ occurs.

(iv) In the case of $c \in \{c_2^+, c_3^-, c_3^+\}$, we have $v_{2m+1} \neq w_{2n}$ and $v_{2m} \neq w_{2n+1}$. Moreover, we get the following:

1) If $v_{2m} = w_{2n}$ then $z_0 = z_1$ and $|z_0| = 1$.

2) If $v_{2m+1} = w_{2n+1}$ then $|z_0| = t$ and $|z_1| = s$ with $z_0 z_1 > 0$.

2.4 *Some theorems for applying the reduction method*

From (2.4), (2.5), we get

$$v_m = \frac{1}{2\sqrt{a}}[(z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m + (z_0\sqrt{a} - x_0\sqrt{c})(s - \sqrt{ac})^m],$$

$$w_n = \frac{1}{2\sqrt{b}}[(z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n + (z_1\sqrt{b} - y_1\sqrt{c})(t - \sqrt{bc})^n],$$

respectively. Hence, we transform the equation $v_m = w_n$ into the following inequality.

Lemma 7 — [6, Lemma 5]. Assume that $c > 4b$. If $v_m = w_n$ and $m, n \neq 0$ then

$$0 < m\log(s + \sqrt{ac}) - n\log(t + \sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c} + z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c} + z_1\sqrt{b})} < \frac{8}{3}ac(s + \sqrt{ac})^{-2m}.$$

We use the following theorem and lemma to obtain the upper bound for m.

***Theorem* 4** — *[3, p. 20]. For a linear form*

$$\Lambda = \beta_1 \log\alpha_1 + \cdots + \beta_l \log\alpha_l \neq 0$$

in logarithms of l algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$ with rational coefficients $\beta_1, \beta_2, \ldots, \beta_l$, we have

$$\log|\Lambda| \geq -18(l+1)!l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log\beta,$$

where $\beta := \max\{|\beta_1|, \ldots, |\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1, \cdots, \alpha_l) : \mathbb{Q}]$ and

$$h'(\alpha) = \frac{1}{d}\max\{h(\alpha), |\log\alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α.

Lemma 8 — [11, Lemma 5]. Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.

1) If $\epsilon > 0$ then there is no solution of the inequality

$$0 < m\kappa - n + \mu < AB^{-m} \tag{2.8}$$

in integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \le m \le M.$$

2) Let $r = \lfloor \mu q + \frac{1}{2} \rfloor$. If $p - q + r = 0$ then there is no solution of inequality (2.8) in integers m and n with

$$\max\left\{\frac{\log(3Aq)}{\log B}, 1\right\} < m \le M.$$

2.5 *Strategies of the proof*

Our strategies of the possibility of extending a Diophantine pairs with Fibonacci numbers to Diophantine quadruple are to combine the techniques used in [11] and properties of Fibonacci numbers. The definition and another form of Fibonacci numbers is very useful to prove some lemmas or theorems which need to prove the extendibility of Diophantine pairs. We prove the extendibility of Diophantine pairs under the hypothesis of Theorem 3 in Section 3. We find the initial terms of the sequences v_m and w_n using Lemma 6, and find the lower bounds of m for each c_ν^τ using Lemma 4. If the lower bounds of m is small then we have lots of cases k which we have to check in the reduction methods. In this paper, we get good lower bounds of m using the definition of Fibonacci numbers. Finally, we find the inequality similar to Lemma 7 which does not satisfy the condition of Lemma 7. Combining these results, we apply the theorem of Baker-Wüstholz and the reduction methods. Then we get $m \le 1$ in all cases, and this implies Theorem 5 by the Lemma 6 and the following Remark 1.

Remark 1 : For small values of c_ν^τ, the cases of $m \le 1$ imply following results.

- The case of c_1^+

  1) $z_0 = z_1 = 1$: We get the solution of $v_{2m} = w_{2n}$ when $m = n = 0$. This means $d = 0$, since $cd + 1 = z^2$. Also, we easily find that $v_2 < w_2$.

  2) he case of $m = 0$ then it is the same as above. Also, we get the solution of $v_{2m} = w_{2n}$ when $m = n = 1$, that is $v_2 = w_2 = cr + st$. This means $d = c_2^+ = d_+$.

- The case of c_2^- with $v_{2m} = w_{2n}$

  1) $z_0 = z_1 = 1$: We get only $d = 0$ when $m = 0$, and we can easily show that $v_2 < w_2$ when the case of $m = 1$.

2) $z_0 = z_1 = -1$: It is same as above when $m = 0$, and we easily find that $v_2 \neq w_2$.

- The case of c_2^- with $v_{2m} = w_{2n+1}$

1) $z_0 = st - cr, z_1 = s$: We get the solution of $v_{2m} = w_{2n+1}$ when $m = 1, n = 0$. Also, $v_2 = cr + st$ means $d = c_3^- = d_+$.
2) $z_0 = cr - st, z_1 = -s$: We get the solution of $v_{2m} = w_{2n+1}$ when $m = n = 0$, that is $v_0 = cr - st$. This means $d = c_1^- = d_-$.

- The case of c_2^+ with $v_{2m} = w_{2n}$

1) $z_0 = z_1 = 1$: We get only trivial case, that is $d = 0$.
2) $z_0 = z_1 = -1$: It is same as above when $m = 0$, and we easily find that $v_2 \neq w_2$.

- The case of c_2^+ with $v_{2m+1} = w_{2n+1}$

1) $z_0 = t, z_1 = s$: We get the solution of $v_{2m+1} = w_{2n+1}$ when $m = n = 0$. Also, $v_1 = cr + st$ means $d = c_3^+ = d_+$.
2) $z_0 = -t, z_1 = -s$: We get the solution of $v_{2m+1} = w_{2n+1}$ when $m = n = 0$. Also, $v_1 = cr - st$ means $d = c_1^+ = d_-$.

3. The extendibility of $\{F_{2k}, F_{2k+4}\}$

In this section, we prove the set $\{F_{2k}, F_{2k+4}, c\}$ can be extended only to regular using the theorem of Baker-Wüstholz and the reduction method based on the Baker-Davenport lemma. By Lemma 1 and Theorem 3, we only have to check the extendibility of the set $\{F_{2k}, F_{2k+4}, c\}$, where $c = c_1^-, c_1^+, c_2^-$ and c_2^+, since $F_{2k+4} \leq 8F_{2k}$. However, the case of c_1^- was proved by Dujella [4], so there remain only three cases.

Remark 2 : The three cases of c_ν^τ are listed below:

- $c_1^+ = 5F_{2k+2}$:
  $s_1^+ = F_{2k} + F_{2k+2}, \quad t_1^+ = F_{2k+2} + F_{2k+4},$

- $c_2^- = 4F_{2k+2}(F_{2k+2}^2 + 1)$:
  $s_2^- = 2F_{2k+1}F_{2k+2} - 1, \quad t_2^- = 2F_{2k+2}F_{2k+3} + 1,$

- $c_2^+ = 4F_{2k+2}(5F_{2k+2}^2 - 1)$:
  $s_2^+ = F_{2k+2}(F_{2k-2} + 2F_{2k+2}) + F_{2k}F_{2k+3},$
  $t_2^+ = F_{2k+2}(F_{2k+6} + F_{2k-2}) + F_{2k}F_{2k+1},$

where $s_\nu^\tau = \sqrt{ac_\nu^\tau + 1},\ t_\nu^\tau = \sqrt{bc_\nu^\tau + 1}$.

3.1 *Bounds for m and k*

First, we consider the case of $v_{2m} = w_{2n}$ with $c = c_2^-$. If we put $d_0 := (z_1^2 - 1)/c$ then the set $\{a, b, c, d_0\}$ satisfies the property of Diophantine quadruple. Suppose that $|z_1| > 1$, then we have $d_0 = F_{2k+2} = a + b - 2r = c_1^-$. That means,

$$F_{2k+2} = (z_1^2 - 1)/c < \frac{\frac{c\sqrt{c}}{2\sqrt{b}}}{c} = \sqrt{\frac{F_{2k+2}(F_{2k+2}^2 + 1)}{F_{2k+4}}},$$

which is a contradiction. So, we do not need to check the case of $v_{2m} = w_{2n}$ with $|z_0| = cr - st$. We have the relation between m and n for $\{F_{2k}, F_{2k+4}, c\}$.

Lemma 9 — If $v_{2m} = w_{2n}$ then $n \le m \le 2n$. If $v_{2m} = w_{2n+1}$ or $v_{2m+1} = w_{2n+1}$ then $n \le m \le 2n + 1$.

PROOF : By Lemma 3, if $v_m = w_n$ then we have $n - 1 \le m \le 2n + 1$. In even case, we have $2n - 1 \le 2m \le 4n + 1$. In other cases, $2n \le 2m \le 4n + 3$ or $2n \le 2m + 1 \le 4n + 3$. Hence, the result is obtained. □

Let us find the lower bounds of m for each cases. If we do not have a good lower bound then we have to check many cases in the reduction method. Hence, these lower bounds have significant meaning.

Lemma 10 — Suppose that $m, n \ge 2$.

1) If $v_{2m} = w_{2n}$ then

$$m \ge \begin{cases} (\sqrt{2F_{2k+2} + 1} - 1)/2, & \text{if } c_1^+, \\ \sqrt[4]{F_{2k+2}}/2, & \text{if } c_2^- \text{ or } c_2^+. \end{cases}$$

2) If $v_{2m} = w_{2n+1}$ or $v_{2m+1} = w_{2n+1}$ then

$$m \ge \frac{\sqrt{F_{2k+2} + 1} - 1}{2}.$$

PROOF : Since all cases of c have divisor $r = F_{2k+2}$, and by Remark 2,

- In the case of c_1^+,
  $s_1^+ \equiv F_{2k} \pmod{F_{2k+2}}$ and $t_1^+ \equiv -F_{2k} \pmod{F_{2k+2}}$.

- In the case of c_2^-,
  $s_2^- \equiv -1 \pmod{F_{2k+2}}$ and $t_2^- \equiv 1 \pmod{F_{2k+2}}$.

- In the case of c_2^+,
  $s_2^+ \equiv F_{2k}F_{2k+3} \equiv -(F_{2k})^2 \pmod{F_{2k+2}}$
  and $t_2^+ \equiv F_{2k}F_{2k+1} \equiv -(F_{2k})^2 \pmod{F_{2k+2}}$.

1) The case of $v_{2m} = w_{2n}$:
From Lemma 4, we see that

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}.$$

- In the case of $c = c_1^+$, we have

$$(m^2 + n^2 \pm m \pm n) \equiv 0 \pmod{F_{2k+2}},$$

since $F_{2k+4} \equiv -F_{2k} \pmod{F_{2k+2}}$ and $\gcd(F_{2k}, F_{2k+2}) = 1$. However, $2(m^2+m) \geq (m^2+n^2) \pm m \pm n > 0$. Hence, we have $2(m^2+m) \geq F_{2k+2}$.

- In the case of $c = c_2^-$, we have

$$\pm F_{2k}(m^2 + n^2) \equiv m + n \pmod{F_{2k+2}}.$$

Squaring both sides, we have $(m^2+n^2)^2-(m+n)^2 \equiv 0 \pmod{F_{2k+2}}$, since $F_{2k+2}F_{2k-2}-1 = F_{2k}^2$. However, $(2m^2)^2 > (m^2+n^2)^2-(m+n)^2 > 0$. Hence, we have $4m^4 \geq F_{2k+2}$.

- In the case of $c = c_2^+$, we have

$$\pm F_{2k}(m^2 + n^2) \equiv m - n \pmod{F_{2k+2}}.$$

Squaring both sides, we have $(m^2+n^2)^2 - (m-n)^2 \equiv 0 \pmod{F_{2k+2}}$. Hence, we also have $4m^4 \geq F_{2k+2}$.

2) The cases of $v_{2m} = w_{2n+1}$ and $v_{2m+1} = w_{2n+1}$:

- In the case of $c = c_2^-$ with $v_{2m} = w_{2n+1}$, we have

$$2F_{2k}(m^2 + n^2 \pm m + n) \equiv 0 \pmod{F_{2k+2}},$$

since $F_{2k+4} \equiv -F_{2k} \pmod{F_{2k+2}}$.

- In the case of $c = c_2^-$ with $v_{2m+1} = w_{2n+1}$, we have
$$\pm 2F_{2k}(m(m+1) + n(n+1)) \equiv 0 \pmod{F_{2k+2}},$$
since $(F_{2k})^2 \equiv -1 \pmod{F_{2k+2}}$.

Both cases imply that $4m^2 + 4m \geq F_{2k+2}$, since $\gcd(F_{2k}, F_{2k+2}) = 1$ and $4(m^2+m) \geq 2(m^2+n^2) \pm 2(m+n) > 0$.

Therefore, we have the desired result. □

There is a condition $c > 4b$ in Lemma 7. However, the case of $c = c_1^+$ do not satisfy the condition, so we find the logarithmic inequality for $c = c_1^+$.

Lemma 11 — If $v_{2m} = w_{2n}$ with c_1^+ and $m, n \neq 0$ then
$$\begin{aligned} 0 &< 2m\log(s+\sqrt{ac}) - 2n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})} \\ &< 2.15(s+\sqrt{ac})^{-4m}. \end{aligned}$$

PROOF : Put
$$P = \frac{1}{\sqrt{a}}(x_0\sqrt{c}+z_0\sqrt{a})(s+\sqrt{ac})^m, \quad Q = \frac{1}{\sqrt{b}}(y_1\sqrt{c}+z_1\sqrt{b})(t+\sqrt{bc})^n.$$
Then
$$P^{-1} = \frac{\sqrt{a}(x_0\sqrt{c}-z_0\sqrt{a})}{c-a}(s-\sqrt{ac})^m, \quad Q^{-1} = \frac{\sqrt{b}(y_1\sqrt{c}-z_1\sqrt{b})}{c-b}(t-\sqrt{bc})^n.$$
Therefore, the relation $v_m = w_n$ becomes
$$P - \frac{c-a}{a}P^{-1} = Q - \frac{c-b}{b}Q^{-1}.$$
Since $P > 0,\ Q > 0$ and
$$P - Q > \frac{c-a}{a}(Q-P)P^{-1}Q^{-1},$$
it follows that $P > Q$. Furthermore, we have
$$\frac{P-Q}{P} < \frac{c-a}{a}P^{-2} < \frac{1}{a(c-a)} \leq \frac{1}{14}.$$
Hence,
$$\begin{aligned} 0 < \log\frac{P}{Q} &= -\log(1 - \frac{P-Q}{P}) < \frac{15}{14}(\frac{c-a}{a})P^{-2} \\ &< \frac{15}{14}\frac{c-a}{(\sqrt{c}-\sqrt{a})^2}(s+\sqrt{ac})^{-2m}. \end{aligned}$$

Since $\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} < 2$, we obtain the result. □

Now we apply theorem of Baker and Wüstholz.

(i) Let us first consider the equation $v_{2m} = w_{2n}$ with $n \neq 0$. We assume that $k \geq 2$, because the extendibility of $\{1, 8\}$ was proved by Filipin, Fujita and Togbé [14]. Using Lemma 7 and Lemma 11, and applying Theorem 4, then we have $l = 3, d = 4, \beta = 2m$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_3 = \frac{(\sqrt{c} \pm \sqrt{a})\sqrt{b}}{(\sqrt{c} \pm \sqrt{b})\sqrt{a}}.$$

Let α_3' and α_3'' be the conjugates of α_3 whose absolute values are greater than one. Then

$$h'(\alpha_1) = \frac{1}{2}\log(\alpha_1) < \frac{1}{2}\log(2s), \quad h'(\alpha_2) = \frac{1}{2}\log(\alpha_2) < \frac{1}{2}\log(2t),$$

$$\begin{aligned} h'(\alpha_3) &\leq \frac{1}{4}\{\log(a^2(c-b)^2) + \log(\alpha_3\alpha_3'\alpha_3'')\} \\ &= \frac{1}{4}\{\log(b\sqrt{ab}(\sqrt{c}+\sqrt{a})(\sqrt{c}+\sqrt{b})(c-a))\} < \log(1.42c) \end{aligned}$$

and

$$\log|\Lambda| \geq -18 \cdot 4!\ 3^4(32 \cdot 4)^5 \frac{1}{2}\log(2s)\frac{1}{2}\log(2t)\log(1.42c) \cdot \log(24) \cdot \log(2m).$$

Since

$$\log(\frac{8}{3}ac(s+\sqrt{ac})^{-4m}) < (-2m+1)\log(4ac)$$

and

$$\log(2.15(s+\sqrt{ac})^{-4m}) < (-2m+1)\log(4ac),$$

we have

$$\frac{2m-1}{\log(2m)} < 9.556 \cdot 10^{14}\log(2c)\log(1.42c). \tag{3.1}$$

Let $x = F_{2k+2}$.

- If $c = c_1^+$ then

$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14}\log(10x) \cdot \log(7.1x) \cdot \log(2x+1).$$

  Hence, $x < 8.91 \cdot 10^{40}$ and $c_1^+ < 4.46 \cdot 10^{41}$.

- If $c = c_2^-$ then

$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14}\log(2.09x)^3 \cdot \log(1.89x)^3 \cdot \log x.$$

  Hence, $x < 1.22 \cdot 10^{89}$ and $c_2^- < 7.27 \cdot 10^{267}$.

- If $c = c_2^+$ then

$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14} \log(3.42x)^3 \cdot \log(3.06x)^3 \cdot \log x.$$

Hence, $x < 1.25 \cdot 10^{89}$ and $c_2^+ < 3.91 \cdot 10^{268}$.

Since $F_{2k+2} = (\alpha^{2k+2} - \overline{\alpha}^{2k+2})/\sqrt{5}$, where $\alpha = (1+\sqrt{5})/2 > 1.618$, we get an upper bound of k from the following inequality

$$(1.618)^{2k+2} < \alpha^{2k+2} < \overline{\alpha}^{2k+2} + \sqrt{5} \cdot (1.25 \cdot 10^{89}).$$

Also from (3.1) and an upper bound of c, we obtain an upper bound of m, that is,

- If $c = c_1^+$ then $k \leq 97$ and $m < 2.11 \cdot 10^{20}$.
- If $c = c_2^-$ then $k \leq 212$ and $m < 9.34 \cdot 10^{21}$.
- If $c = c_2^+$ then $k \leq 213$ and $m < 9.39 \cdot 10^{21}$.

(ii) Let $v_{2m} = w_{2n+1}$ with $n \neq 0$. We have $l = 3, d = 4, \beta = 2m+1$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_4 = \frac{((sr-ta)\sqrt{c} \pm (cr-st)\sqrt{a})\sqrt{b}}{(r\sqrt{c} \mp s\sqrt{b})\sqrt{a}}.$$

Let α_4' and α_4'' be the conjugates of α_4 whose absolute values are greater than one. Then

$$\begin{aligned} h'(\alpha_1) &= \frac{1}{2}\log(\alpha_1) < \frac{1}{2}\log(2s), \quad h'(\alpha_2) = \frac{1}{2}\log(\alpha_2) < \frac{1}{2}\log(2t), \\ h'(\alpha_4) &\leq \frac{1}{4}\{\log(a^2(c-b)^2) + \log(\alpha_4\alpha_4'\alpha_4'')\} \\ &= \frac{1}{4}\{\log(b\sqrt{ab}((sr-ta)\sqrt{c} + (cr-st)\sqrt{a})(r\sqrt{c}+s\sqrt{b})(c-a))\} \\ &< \log(1.42\sqrt{r}c), \end{aligned}$$

since $(sr-ta)\sqrt{c} > (cr-st)\sqrt{a}$ and $sr - ta = F_{2k+1} < r = F_{2k+2}$. Hence, we have

$$\frac{2m-1}{\log(2m+1)} < 9.556 \cdot 10^{14} \log(2c)\log(1.42\sqrt{r}c). \tag{3.2}$$

If $c = c_2^-$ then

$$\sqrt{x+1} - 2 < 4.778 \cdot 10^{14} \log(2.09x)^3 \cdot \log(1.73x)^{7/2} \cdot \log(x+1),$$

where $x = F_{2k+2}$. Hence, we get $x < 2.58 \cdot 10^{43}$, $c_2^- < 6.87 \cdot 10^{130}$. Hence $k \leq 103$, since $(1.618)^{2k+2} < \alpha^{2k+2} < \overline{\alpha}^{2k+2} + \sqrt{5} \cdot (2.58 \cdot 10^{43})$. From (3.2) and the upper bound of c, we obtain $m < 2.54 \cdot 10^{21}$.

(iii) Let $v_{2m+1} = w_{2n+1}$ with $n \neq 0$. We have $l = 3, d = 4, \beta = 2m + 1$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_5 = \frac{(r\sqrt{c} \pm t\sqrt{a})\sqrt{b}}{(r\sqrt{c} \pm s\sqrt{b})\sqrt{a}}.$$

Let α_5' and α_5'' be the conjugates of α_5 whose absolute values are greater than one. Then

$$\begin{aligned} h'(\alpha_1) &= \frac{1}{2}\log(\alpha_1) < \frac{1}{2}\log(2s), \quad h'(\alpha_2) = \frac{1}{2}\log(\alpha_2) < \frac{1}{2}\log(2t), \\ h'(\alpha_5) &\leq \frac{1}{4}\{\log(a^2(c-b)^2) + \log(\alpha_5\alpha_5'\alpha_5'')\} \\ &= \frac{1}{4}\{\log(b\sqrt{ab}(r\sqrt{c} + t\sqrt{a})(r\sqrt{c} + s\sqrt{b})(c-a))\} \\ &< \log(1.42\sqrt{r}c). \end{aligned}$$

Hence, we have

$$\frac{2m}{\log(2m+1)} < 9.556 \cdot 10^{14} \log(2c) \log(1.42\sqrt{r}c). \tag{3.3}$$

If $c = c_2^+$ then

$$\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(3.42x)^3 \cdot \log(2.61x)^{7/2} \cdot \log(x+1),$$

where $x = F_{2k+2}$. Hence, we get $x < 2.63 \cdot 10^{43}$ and $c_2^+ < 3.64 \cdot 10^{131}$. Hence $k \leq 103$, since $(1.618)^{2k+2} < \alpha^{2k+2} < \overline{\alpha}^{2k+2} + \sqrt{5} \cdot (2.63 \cdot 10^{43})$. From (3.3) and the upper bound of c, we obtain $m < 2.57 \cdot 10^{21}$.

3.2 *The reduction method*

Now dividing logarithmic inequalities from Lemma 7 and Lemma 11 by $\log \alpha_2$, respectively leads us to the inequalities

$$0 < m_1\kappa - n_1 + \mu_1 < A_1 B^{m_1},$$

$$0 < m_1\kappa - n_1 + \mu_1 < A_2 B^{m_1},$$

$$0 < m_1\kappa - n_2 + \mu_2 < A_1 B^{m_1},$$

$$0 < m_2\kappa - n_2 + \mu_3 < A_1 B^{m_2},$$

where $m_1 := 2m,\ m_2 := 2m+1,\ n_1 := 2n,\ n_2 := 2n+1$ and

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2},\ \mu_1 = \frac{\log \alpha_3}{\log \alpha_2},\ \mu_2 = \frac{\log \alpha_4}{\log \alpha_2},\ \mu_3 = \frac{\log \alpha_5}{\log \alpha_2},$$

$$A_1 = \frac{(8/3)ac}{\log \alpha_2},\ A_2 = \frac{2.15}{\log \alpha_2},\ B = \alpha_1^2.$$

We apply Lemma 8 to the logarithmic inequalities with $M_1 := 2m \leq 1.88 \cdot 10^{22}$ and $M_2 := 2m+1 \leq 5.14 \cdot 10^{21}$. We have to examine $2 \cdot 97 + 2 \cdot 212 + 2 \cdot 213 + 2 \cdot 103 + 2 \cdot 103 = 1456$ cases. The program was developed in **PARI/GP** running with 150 digits. For the computations, if the first convergent such that $q > 6M_i$ with $i = 1, 2$ does not satisfy the condition $\epsilon > 0$ then we use the next convergent until we find the one that satisfies the conditions. Then we have the results as the following Table 1.

Table 1: Results from **PARI/GP** running

Case of c	Initial values	Use the next convergent
c_1^+	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	78 cases $(k = 20, \ldots, 97)$
c_2^-	$z_0 = z_1 = 1$	139 cases $(k = 74, \ldots, 212)$
	$z_0 = z_1 = -1$	136 cases $(k = 77, \ldots, 212)$
c_2^-	$z_0 = cr - st, z_1 = -s$	66 cases $(k = 38, \ldots, 103)$
	$z_0 = st - cr, z_1 = s$	91 cases $(k = 13, \ldots, 103)$
c_2^+	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	0 case
c_2^+	$z_0 = t, z_1 = s$	0 case
	$z_0 = -t, z_1 = -s$	0 case

In the case of c_1^+ and c_2^-, we have $m \leq 8$ and 5, respectively. If we take $M = 2m$, and run the program again then we obtain $m \leq 1$. Other cases, namely the cases of c_2^+ with $z_0 = z_1 = \pm 1$ and $z_0 = \pm t,\ z_1 = \pm s$, we have $m \leq 4$ and 3, respectively. Hence, we take $M = 2m$ or $2m+1$, and run the program again to obtain $m \leq 1$. Hence, we get the following theorem.

***Theorem* 5** — *Let k be a positive integer. If the set $\{F_{2k}, F_{2k+4}, c, d\}$ is a Diophantine quadruple with $c < d$, then $d = d_+$.*

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REFERENCES

1. Lj. Baćić and A. Filipin, The extendibility of $D(4)$-pairs $\{F_{2k}, F_{2k+6}\}$ and $\{P_{2k}, P_{2k+4}\}$, *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.*, **20** (2016), 27-35.

2. A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser.*, **20**(2) (1969), 129-137.

3. A. Baker and G. Wüstholz, Logarithmic forms and group varieties, *J. Reine Angew. Math.*, **442** (1993), 19-62.

4. A. Dujella, A proof of the Hoggatt-Bergum conjecture, *Proc. Amer. Math. Soc.*, **127**(7) (1999), 1999-2005.

5. A. Dujella, A parametric family of elliptic curves, *Acta Arith.*, **94**(1) (2000), 87-101.

6. A. Dujella, An absolute bound for the size of Diophantine m-tuple, *J. Number theory*, **89**(1) (2001), 126-150.

7. A. Dujella, Diophantine m-tuples and elliptic curves, *21st Journées Arithmétiques (Rome, 2001), J. Théor. Nombres Bordeaux*, **13**(1) (2001), 111-124.

8. A. Dujella, There are only finitely many Diophantine quintuples, *J. Reine Angew. Math.*, **566** (2004), 183-214.

9. A. Dujella, Diophantine quadruples and Fibonacci numbers, *Bull. Kerala Math. Assoc.*, **1**(2) (2004), 133-147.

10. A. Dujella, What is a Diophantine m-tuples, *Notices Amer. Math. Soc.*, **63** (2016), 772-774.

11. A. Dujella and A. Pethö, Generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser. (2)*, **49**(195) (1998), 291-306.

12. A. Filipin, The extendibility of $D(4)$-pair $\{F_{2k}, 5F_{2k}\}$, *Fibonacci Quart.*, **53** (2015), 124-129.

13. A. Filipin, Y. Fujita, and A. Togbé, The extendibility of Diophantine pairs I: The general case, *Glas. Mat. Ser. III*, **49**(1) (2014), 25-36.

14. A. Filipin, Y. Fujita, and A. Togbé, The extendibility of Diophantine pairs II: Examples, *J. Number theory*, **145** (2014), 604-631.

15. A. Filipin, B. He, and A. Togbé, On the $D(4)$-triple $\{F_{2k}, F_{2k+6}, 4F_{2k+4}\}$, *Fibonacci Quart.*, **48**(3) (2010), 219-227.

16. Y. Fujita, The extensibility of Diophantine pair $\{k-1, k+1\}$, *J. Number Theory*, **128**(2) (2008), 323-353.

17. Y. Fujita, The Hoggatt-Bergum conjecture on $D(-1)$-triples $\{F_{2k+1}, F_{2k+3}, F_{2k+5}\}$ and integer points on the attached elliptic curves, *Rocky Mountain J. Math.*, **39**(6) (2009), 1907-1932.

18. B. He, F. Luca, and A. Togbé, Diophantine triples of Fibonacci numbers, *Acta Arith.*, **175** (2016), 57-70.

19. B. He, A. Togbé, and V. Ziegler, There is no Diophantine quintuple, *Trans. Amer. Math. Soc.*, to appear.

20. V. E. Hoggatt and G. E. Bergum, A problem of Fermat and the Fibonacci sequence, *Fibonacci Quart*., **15**(4) (1977), 323-330.

21. B. W. Jones, A second variation on a problem of Diophantus and Davenport, *Fibonacci Quart*., **16**(2) (1978), 155-165.

22. F. Luca and Y. Fujita, On Diophantine quadruples of Fibonacci numbers, *Glas. Mat. Ser. III*, **52** (2017), 221-234.

23. J. Morgado, Generalization of a result of Hoggatt and Bergum on Fibonacci numbers, *Portugaliae Math*., **42**(4) (1983/84), 441-445.