Indian J. Pure Appl. Math., **50**(2): 403-412, June 2019

DOI: 10.1007/s13226-019-0334-7

TWISTED CONJUGACY AND QUASI-ISOMETRIC RIGIDITY OF IRREDUCIBLE LATTICES IN SEMISIMPLE LIE GROUPS

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(*Received* 14 *April* 2018; *accepted* 13 *June* 2018)

Let G be a non-compact semisimple Lie group with finite centre and finitely many connected components. We show that any finitely generated group Γ which is quasi-isometric to an irreducible lattice in G has the R_∞-property, namely, that there are infinitely many ϕ-twisted conjugacy classes for every automorphism ϕ of Γ. Also, we show that any lattice in G has the R_∞-property, extending our earlier result for irreducible lattices.

Key words : Twisted conjugacy; lattices in semisimple Lie groups;, quasi-isometry

1. INTRODUCTION

An automorphism $\phi : \Gamma \to \Gamma$ of an infinite group Γ induces an action of Γ on itself defined as $g.x = gx\phi(g^{-1})$. The orbits of this action are called ϕ-twisted conjugacy classes; elements of Γ belonging to the same orbits are said to be ϕ-twisted conjugates. The orbit space is denoted $\mathcal{R}(\phi)$ and its cardinality, denoted $R(\phi)$, is called the Reidemeister number of ϕ. We write $R(\phi) = \infty$ if $\mathcal{R}(\phi)$ is infinite. The group Γ is said to have the R_∞*-property* if $R(\phi) = \infty$ for all automorphisms ϕ of Γ. The notion of Reidemeister number arose in Nielsen fixed point theory; see [10]. The problem of classifying (finitely generated) groups that have the R_∞-property was proposed by Fel'shtyn and Hill [8].

The R_∞-property does not behave well with respect to finite index subgroup, as has first been observed by Gonçalves and Wong [9] who showed that the infinite dihedral group D_∞ has the R_∞-property although the infinite cyclic group does not. Thus R_∞-property is a property that is not *geometric*—that is, it is not a quasi-isometry invariant among the class of all finitely generated groups. On the other hand, the work of Levitt and Lustig [11] shows that any torsionless non-elementary hyperbolic group has the R_∞-property. This has been extended by Fel'shtyn [6] who removed the restriction of torsionlessness. He also showed in [7] that R_∞-property holds for relatively hyperbolic groups.

The purpose of this note is to show that the R_∞-property is geometric for the class of all finitely generated groups that are quasi-isometric to irreducible lattices in real semisimple Lie groups with finite centre and finitely many connected components. The classification of such groups is a deep result that has been achieved by the effort of several mathematicians including Eskin, Schwartz, Farb, Pansu, Kleiner and Leeb. See [5] for a survey on this result. The definition of quasi-isometry will be recalled in §2.

Recall that a *lattice* in a non-compact real Lie group G with finitely many connected components is a discrete subgroup $\Lambda \subset G$ such that G/Λ has finite volume with respect to the G-invariant measure associated to a Haar measure on G. A lattice is *uniform* if G/Λ is compact.

A lattice Λ in G is said to be *irreducible* if, for any *non-compact* closed normal subgroup H of G, the image of Λ under the quotient $G \to G/H$ is dense. This definition of irreducibility differs from the definition used in [16], which assumes that G has no compact factors. (So, according to that definition, if G admits a non-trivial compact factor, then no lattice in G is irreducible.) If M is the maximal compact normal subgroup of G then $\Lambda/\Lambda \cap M$ is an irreducible lattice in $\bar{G} := G/M$. Since $\Lambda \cap M$ is finite, Λ is quasi-isometric to $\Lambda/\Lambda \cap M$, and so either definition leads to the *same* quasi-isometry class of groups. Note that, by a theorem of Mostow, $\bar{G}$ is connected. Since M contains the centre of G, $\bar{G}$ is of adjoint type. A lattice which is not irreducible will be called *reducible*.

We now state the main result of this note.

***Theorem* 1** — *Let Γ be a finitely generated group which is quasi-isometric to an irreducible lattice Λ in a non-compact semisimple real Lie group G with finite centre and finitely many connected components. Then Γ has the R_∞-property.*

When Λ is uniform, it turns out that it is quasi-isometric to any other uniform lattice in G. So the above theorem implies that *any* uniform lattice in G has the R_∞-property. However, we need to first establish the theorem in this special case first, when G is connected and has trivial centre. The R_∞-property for irreducible lattices was proved in [14] and will be an ingredient of the proof of the main theorem in the case Λ is nonuniform. The major result that will be needed to prove the above theorem, besides the results on the R_∞-property for lattices, is the classification theorem for groups quasi-isometric to irreducible lattices in G. (See Theorem 2.) Perhaps, since very few classes of groups are known for which the R_∞-property is geometric, the above result is surprising, since R_∞-property is known to be not even an invariant under passage to a finite index subgroup.

2. Quasi-isometric Rigidity of Irreducible Lattices

Let (X, d_X) and (Y, d_Y) be metric spaces and let $\lambda \geq 1, \epsilon > 0$. A (λ, ϵ)-*quasi-isometric embedding* is a set-map $f : X \to Y$ such that, for all $x_0, x_1 \in X$, the following coarse bi-Lipschitz condition holds:

$$-\epsilon + (1/\lambda)d_X(x_0, x_1) \leq d_Y(f(x_0), f(x_1)) \leq \lambda d_X(x_0, x_1) + \epsilon.$$

If, in addition, there exists a $C > 0$ such that, for any $y \in Y$, there exists an $x \in X$ such that $d_Y(f(x), y) \leq C$, one says that f is a *quasi-isometry*. In this case there exists a quasi-isometry $g : Y \to X$ for possibly a different set of constants λ', ϵ', C'. The map g is called a *quasi-inverse* of f. One says that $(X, d_X), (Y, d_Y)$ are *quasi-isometrically equivalent* if there exists a (λ, ϵ, C)-quasi-isometry $f : X \to Y$ for some set of constants λ, ϵ, C. This is an 'equivalence relation' on the class of all metric spaces.

If Γ is a group with a finite generating set S, then Γ becomes a metric space (Γ, d_S) where d_S is the *word metric* defined as $d_S(x, y) = \ell_S(x^{-1}y)$, the length of the shortest expression for $x^{-1}y$ in the 'alphabet' $S \cup S^{-1}$. Changing the (finite) generating set changes the word metric, but not the quasi-isometry equivalence class of (Γ, d_S). It is an important problem, for a given group Γ, to classify all groups which are quasi-isometrically equivalent to Γ with a word metric. It is easy to see that, if there is a homomorphism $\phi : \Gamma \to \Lambda$ whose kernel and cokernel are finite, then ϕ is a quasi-isometry. When Λ is an irreducible lattice in a semisimple Lie group, the converse is also true. More precisely, we have the following classification theorem for groups that are quasi-isometric to irreducible lattices; see [5, Theorem I].

***Theorem* 2** — *(QI rigidity of irreducible lattices). Let Γ be a finitely generated group which is quasi-isometric to an irreducible lattice Λ in a non-compact semisimple connected Lie group G with trivial centre. Then there is an exact sequence*

$$1 \to F \to \Gamma \to \Lambda_0 \to 1 \tag{1}$$

where F is finite and Λ_0 is a lattice in G.

Next we state the following companion theorem which gives the quasi-isometry equivalence classes among lattices in G. See [5, Theorem II].

***Theorem* 3** — *(QI classification of lattices). Let G be a non-compact semisimple Lie group with finite centre and without compact factors. Then:*

(a) All uniform lattices in G form a single quasi-isometry class.

(b) If G is not locally isomorphic to $SL(2,\mathbb{R})$, two irreducible nonuniform lattices Γ_0, Γ_1 are quasi-isometrically equivalent to each other if and only if there exists a $g \in G$ such that $\Gamma_0 \cap g\Gamma_1 g^{-1}$ is a finite index subgroup of both Γ_0 and $g\Gamma_1 g^{-1}$. If G is locally isomorphic to $SL(2,\mathbb{R})$, then all *nonuniform lattices of G form a single quasi-isometry class.*

Note that part (a) of the above theorem is classical: By Svarc-Milnor Lemma [1, Ch. I.8], any uniform lattice in G is quasi-isometric to G where G is endowed with a G-invariant Riemannian metric. Also, when G is locally isomorphic to $SL(2,\mathbb{R})$, any nonuniform lattice contains a finite index subgroup which is isomorphic to a free group of finite rank. Thus any such lattice is quasi-isometric to the free group of rank 2.

Remark 4 : (i) The assumption that G has trivial centre seems to be required in Theorem 2, although it is not explicitly stated in [5, Theorem I]. Indeed, suppose that Λ is a certain arithmetic nonuniform lattice in an appropriate covering group of $G = Spin(n,2)$ as constructed by Millson [12] (cf. Deligne [4, 17]) so that Λ does not contain any torsionless finite index subgroup. Let $\Gamma := \Lambda/Z(\Lambda)$. Then $\Gamma \subset G/Z(G)$ is a torsionless lattice which is evidently quasi-isometric to Λ. But there can be no surjection $\eta : \Gamma \to \Lambda_0$ with finite kernel for *any* lattice Λ_0 in G. For, Λ_0 would then be commensurable with $g\Lambda g^{-1}$ for some $g \in G$ by Theorem 3 and hence Λ_0 would have torsion elements. Since Γ is torsionless, $\ker(\eta)$ cannot be finite.

When the lattice Λ in Theorem 2 is nonuniform (resp. uniform), so is Λ_0. This follows from part (a) of Theorem 3, and by part (b) of the same theorem, if Λ is nonuniform, Λ_0 is *irreducible*. However, this is not the case in general when Λ is uniform, since any two uniform lattices are quasi-isometric.

3. Twisted Conjugacy in Reducible Lattices

Suppose that G is a connected semisimple Lie group with trivial centre and having no compact factors. Suppose that $\Lambda \subset G$ is a reducible lattice. Then there exists a factorization $G = H_1 \times \cdots \times H_n$ into semisimple Lie groups H_j where $\Lambda \cap H_j =: \Lambda_j$ is an irreducible lattice in $H_j, 1 \leq j \leq n$, $\Lambda_j \subset \Lambda$ is a normal subgroup of Λ, and $\Lambda_i \cap \Lambda_j$ is trivial if $i \neq j$. Moreover $\Lambda_0 := \prod_{1\leq j\leq n} \Lambda_j$ is a finite index normal subgroup of Λ. Also, the j-th projection $G \to H_j$ maps Λ onto an irreducible lattice $\tilde{\Lambda}_j$ which contains Λ_j as a finite index normal subgroup. Evidently Λ is a finite index subgroup of $\prod_{1\leq j\leq n} \tilde{\Lambda}_j$.

We have the following lemma.

Lemma 5 — We keep the above notation. Assume that Λ is a torsionless lattice in a connected semisimple Lie group G with trivial centre and without compact factors.

(i) Suppose that $\phi : \Lambda \to \Lambda$ is an automorphism. Then there exists a permutation $\sigma \in S_n$ such that $\phi(\Lambda_i) \subset \tilde{\Lambda}_{\sigma(i)}\ \forall i \leq n$.

(ii) There exists a finite index characteristic subgroup Γ_0 of Λ of the form $\Gamma_0 = \Gamma_1 \times \cdots \times \Gamma_n$ where Γ_j is an irreducible lattice in H_j for $1 \leq j \leq n$.

PROOF : (i). Denote by π_j the natural surjection $\pi_j : \Lambda \to \tilde{\Lambda}_j$ and by ϕ_j the composition $\Lambda \xrightarrow{\phi} \Lambda \xrightarrow{\pi_j} \tilde{\Lambda}_j$. Fix $i \leq n$. Then there exists a j such that $\phi_j(\Lambda_i)$ is nontrivial. Set $A := \phi_j(\Lambda_i), B := \phi_j(\prod_{k\neq i} \Lambda_k)$. Then both A and B are normal subgroups of $\tilde{\Lambda}_j$ and $A.B = \phi_j(\Lambda_0)$ is a finite index normal subgroup of $\tilde{\Lambda}_j$. Clearly $ab = ba, \forall a \in A, b \in B$ since $xy = yx$ for $x \in \Lambda_i, y \in \prod_{k\neq i} \Lambda_k$. So $A \cap B$ is contained in the centre of $\phi_j(\Lambda_0)$. By the Borel density theorem, since H_j has trivial centre the same is true of $\phi_i(\Lambda_0)$ as well. Hence $A \cap B$ is trivial. Therefore $\phi_j(\Lambda_0) = A \times B$. If both A and B are infinite, then $\phi_j(\Lambda_0)$, and hence $\tilde{\Lambda}_j$, would be reducible. Therefore one of A, B must be finite and the other a finite index subgroup of $\tilde{\Lambda}_j$.

Suppose that A is finite and B has finite index in $\tilde{\Lambda}_j$. Since every element of A commutes with every element of B, this implies, again by Borel density theorem, that A is contained in the centre of H_j and so A is trivial—contrary to our choice of j. Therefore A must have finite index in $\tilde{\Lambda}_j$ and B must be finite. The same argument shows that B is trivial.

Thus, given any j, we see that there is at most one i so that $\phi_j(\Lambda_i)$ is nontrivial. The same argument applied to ϕ^{-1} shows that for any i, there is at most one j such that $\phi_j(\Lambda_i)$ is nontrivial. Therefore there is a permutation $\sigma \in S_n$ such that $\phi_j(\Lambda_i)$ is nontrivial if and only if $j = \sigma(i)$ for $1 \leq i \leq n$, and hence $\phi(\Lambda_i) \subset \tilde{\Lambda}_{\sigma(i)}, 1 \leq i \leq n$.

(ii). By (i) $\phi(\Lambda_0)$ is a product $\prod_{1\leq j\leq n} \Lambda'_j$, which has finite index in Λ. It follows that taking intersection of all subgroups $\phi(\Lambda_0)$ as ϕ varies over all automorphisms of Λ we obtain a finite index characteristic subgroup $\Gamma_0 \subset \Lambda_0$ of the form $\Gamma_1 \times \cdots \times \Gamma_n$. □

Before proceeding further, we need the following lemma, which was proved in [13, Lemma 2.2]. For the convenience of the reader we have included the short proof.

Lemma 6 — Consider an exact sequence of groups

$$1 \to N \overset{j}{\hookrightarrow} \Gamma \xrightarrow{\eta} \Lambda \to 1$$

where N is characteristic in Γ. (i) If Λ has the R_∞-property, then so does Γ. (ii) If Λ is finite and N has the R_∞-property, then so does Γ.

PROOF : Let $\phi : \Gamma \to \Gamma$ be any automorphism. Since N is characteristic, ϕ restricts to an automorphism of N and hence induces an automorphism $\bar{\phi} : \Lambda \to \Lambda$.

(i) Since ϕ-twisted conjugacy classes are mapped to $\bar{\phi}$-twisted conjugacy classes, it follows that $R(\phi) \geq R(\bar{\phi})$. So $R(\phi) = \infty$ if $R(\bar{\phi}) = \infty$, which proves the assertion.

(ii) Let $\theta = \phi|_N$ and suppose that $R(\theta) = \infty$. Suppose that $x_k \in N, k \geq 0$, are in pairwise distinct θ-twisted conjugacy classes in N, but that they are all in the same ϕ-twisted conjugacy class. Let $\gamma_k \in \Gamma, k \geq 1$, be such that $x_k = \gamma_k x_0 \phi(\gamma_k^{-1})$. Since $\Gamma/N \cong \Lambda$ is finite, there exist $k, l \geq 1$ such that $g := \gamma_k \gamma_l^{-1} \in N$. Substituting $x_0 = \gamma_l^{-1} x_l \phi(\gamma_l)$, in $x_k = \gamma_k x_0 \phi(\gamma_k^{-1})$ we obtain $x_k = g x_l \phi(g^{-1}) = g x_l \theta(g^{-1})$ showing that x_k and x_l are in the same θ-twisted conjugacy class, contrary to our choice. Hence $R(\phi) = \infty$. This completes the proof. □

We are now ready to prove the R_∞-property for lattices in the case when the Lie group is connected, has no compact factors, and has trivial centre.

***Theorem* 7** — *Suppose that $\Lambda \subset G$ is a lattice where G is a connected semisimple Lie group without compact factors and having trivial centre. Then Λ has the R_∞-property.*

PROOF : Let $\phi : \Lambda \to \Lambda$ be any automorphism. We need to show that $R(\phi) = \infty$. By the main result of [14], we may (and do) assume that Λ is reducible.

In view of Lemma 5, there exists a finite index characteristic subgroup $\Gamma_0 \subset \Lambda$ which is a product $\Gamma_1 \times \cdots \times \Gamma_n$ where each Γ_j is an irreducible lattice in a connected normal subgroup H_j of G and moreover ϕ permutes the subgroups Γ_j. The cycle decomposition of the permutation $\sigma \in S_n$ where $\phi(\Gamma_j) = \Gamma_{\sigma(j)}$ leads to a product decomposition $\Lambda = L_1 \times \cdots \times L_r$ where each L_i is the product of those Γ_j which form an orbit of σ. Then ϕ restricts to an automorphism of L_i for each i. By relabelling if necessary, we assume that $L_1 = \Gamma_1 \times \cdots \times \Gamma_p, \phi(\Gamma_j) = \Gamma_{j+1}, 1 \leq j < p, \phi(\Lambda_p) = \Lambda_1$. In particular $\Gamma_i \cong \Gamma_1, 1 \leq i \leq p$. Let $\gamma_k = (\gamma_{k,1}, \cdots, \gamma_{k,n}) \in \Lambda, k \geq 1$, be a sequence of elements in $\Gamma_0 = \prod_{1 \leq i \leq n} \Gamma_i$ satisfying the following conditions, where we denote by $\phi_i : \Gamma_i \to \Gamma_{i+1}, 1 \leq i < p$ and $\phi_p : \Gamma_p \to \Gamma_1$, the isomorphisms obtained from ϕ via restriction.

(i) $\gamma_{ki} = 1, 1 < i \leq n$, for all $k \in \mathbb{N}$, and,

(ii) the $\gamma_{k,1} \in \Gamma_1$ are in pairwise distinct ψ-twisted conjugacy classes where $\psi := \phi^p|_{\Gamma_1} = \phi_p \circ \cdots \circ \phi_2 \circ \phi_1 \in \mathrm{Aut}(\Gamma_1)$. Since Γ_1 is an irreducible lattice, such a sequence exists by the main theorem of [14] when the real rank of H_1 is at least 2, by [11] when Γ_1 is uniform and the real rank of H_1 equals 1, and by the result of [6] when Γ_1 is nonuniform and the rank of H_1 equals 1.

We claim that the γ_k are in pairwise distinct ϕ-twisted conjugacy classes. Observe that the claim implies that $R(\phi) = \infty$, thereby establishing the theorem.

First, suppose that $y = zx\phi(z^{-1})$ where $y = (y_1, 1, \ldots, 1), x = (x_1, 1, \ldots, 1) \in \Gamma_0$ and $z = (z_1, \ldots, z_n) \in \Gamma_0$. This is equivalent to the following set of equations:

$$y_1 = z_1 x_1 \phi_p(z_p^{-1}); 1 = z_i \phi_{i-1}(z_{i-1}^{-1}), 1 < i \leq p.$$

It follows that $y_1 = z_1 x_1 \phi_p \phi_{p-1}(z_{p-1}^{-1}) = \cdots = z_1 x_1 \phi_p \phi_{p-1} \cdots \phi_1(z_1^{-1}) = z_1 x_1 \psi(z_1^{-1})$. Hence, if γ_k and γ_l are in the same ϕ-twisted conjugacy class, it follows from (i) that $\gamma_{k,1}$ and $\gamma_{l,1}$ are ψ-twisted conjugates, contradicting (ii). Hence $\gamma_k \in \Lambda, k \geq 1$, are in pairwise distinct ϕ-twisted conjugacy classes as was to be shown. □

Remark 8 : As already observed in [14, Remark 3.3], the method of proof of [14, Theorem 1.2] applies to show that any irreducible lattice Λ in a semisimple Lie group G of rank one with trivial centre and finitely many components has the R_∞-property when the identity component G^0 of G is not locally isomorphic to $SL(2, \mathbb{R})$. When G^0 is locally isomorphic to $SL(2, \mathbb{R})$ and Λ is nonuniform, it has a finite index subgroup isomorphic to a free group of finite rank. Applying Lemma 6, it suffices to consider the case $G^0 = SL(2, \mathbb{R})$ and $\Lambda = SL(2, \mathbb{Z})$. There are several proofs available for the R_∞-property of $SL(2, \mathbb{Z})$ that do not rely on the work of Levitt and Lustig [11]. See, for example, [13, Theorem 3.1] and also [15]. Also Gonçalves and Wong [9, Theorem 3.4] have given a purely algebraic proof that a free group of rank 2 has the R_∞-property. Dekimpe and Gonçalves [3] have obtained a purely algebraic proof of the R_∞-property of the non-abelian free groups of finite ranks and the fundamental group of a compact oriented surface of genus $g \geq 2$—which is the same as a torsionless uniform lattice in $SL(2, \mathbb{R})$. Thus one has algebraic proofs available in the case rank 1 lattices as an alternative to the geometric group theoretic proofs due to Levitt and Lustig [11] and to Fel'shtyn [6].

4. Proof of Theorem 1

We are now ready to establish the main result of this note.

Proof of Theorem 1 : Let Λ be an irreducible lattice in a non-compact semisimple Lie group G with finite centre and finitely many connected components. Let M be the maximal connected compact normal subgroup of G. As remarked in §1, $\bar{\Lambda} := \Lambda/(M \cap \Lambda)$ is an irreducible lattice in the connected semisimple Lie group $\bar{G} = G/M$, which has trivial centre. Suppose that Γ is a finitely generated group which is quasi-isometric to Λ. Then it is quasi-isometric to $\bar{\Lambda}$. By Theorem 2, there exists an exact sequence

$$1 \to F \hookrightarrow \Gamma \stackrel{\eta}{\to} \Lambda_0 \to 1$$

where Λ_0 is a lattice in $\bar{G}$. By Theorem 3, when Λ is nonuniform, so is Λ_0. When Λ is uniform, however, Λ_0 is not necessarily irreducible.

In general F is not characteristic in Γ. In order to overcome this, we proceed as follows. Since $\bar{G}$ is linear, any finitely generated subgroup of $\bar{G}$ is residually finite; in particular, Λ_0 is residually finite. Again, linearity of $\bar{G}$ implies that there exists a finite index torsionless sublattice $\Lambda_1 \subset \Lambda_0$. Let $\Gamma_1 = \eta^{-1}(\Lambda_1)$. Note that Γ_1 has finite index in Γ. Let Γ_0 be a finite index *characteristic* subgroup of Γ that is contained in Γ_1. (For instance, we may take Γ_0 to be the intersection of all subgroups of Γ having index equal to the index of Γ_1 in Γ.)

To show that Γ has the R_∞-property, by Lemma 6(ii), it suffices to show that Γ_0 has the R_∞-property. Set $F_0 := F \cap \Gamma_0, \eta_0 := \eta|_{\Gamma_0}$, and $\Lambda_0' := \eta(\Gamma_0)$. Then we have an exact sequence

$$1 \to F_0 \hookrightarrow \Gamma_0 \stackrel{\eta_0}{\to} \Lambda_0' \to 1. \tag{2}$$

We claim that F_0 is a *characteristic* subgroup of Γ_0. Note that F_0 consists of *all* finite order elements of Γ_0 since Λ_0' has no (nontrivial) torsion elements. Since $F_0 \subset F$ is finite, it follows that F_0 *equals* the set of all finite order elements of Γ_0. Since any automorphism of Γ_0 permutes the set of finite order elements of Γ_0 among themselves, we conclude that F_0 is characteristic in Γ_0.

Applying Lemma 6, we see that, Γ_0 has the R_∞-property since Λ_0' has the same property by Theorem 7. This completes the proof. □.

As a corollary of the above *proof*, we obtain the following

***Theorem* 9** — *Any lattice Γ in a semisimple real Lie group G with finite centre and finitely many connected components has the R_∞-property.*

Outline of Proof : We need only consider the case when Γ is reducible. One has an exact sequence (1) with F finite and Λ a lattice in $\bar{G} = G/M$ where M is a maximal compact normal subgroup of G. Also one has a characteristic subgroup Γ_0 of Γ and an exact sequence (2) where F_0 is a finite characteristic subgroup of Γ_0 and Λ_0' is a lattice in $\bar{G}$. Since $\bar{G}$ has trivial centre and no compact factors, Λ_0' has the R_∞-property by Theorem 7. By Lemma 6, Γ_0 has the R_∞-property. Hence, by the same lemma, Γ also has the R_∞-property. □

Remark 10 : If a group is quasi-isometric to a uniform lattice in a semisimple Lie group with finite centre and finitely many components, then it is also quasi-isometric to an *irreducible* uniform lattice and hence has the R_∞-property. *We do not know if every finitely generated group quasi-isometric to a* nonuniform *lattice has the R_∞-property.*

Acknowledgement

The authors thank the referee for a careful reading of the paper. This project was initiated during the visit of the first named author to the Institute of Mathematical Sciences, Chennai, in May 2016. She thanks the Institute for her visit and for its hospitality. The second named author acknowledges the Department of Atomic Energy, Government of India, for partially supporting his research under a XII Plan Project.

References

1. Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften **319**, Springer-Verlag, Berlin, 1999.
2. K. Dekimpe and D. L. Gonçalves, The R_∞-property for free groups, free nilpotent groups, and free solvable groups, *Bull. London Math. Soc.*, **46** (2014), 737-746.
3. K. Dekimpe and D. L. Gonçalves, The R_∞-property for nilpotent quotients of surface groups, *Trans. London Math. Soc.*, **3**(1) (2016), 28-46.
4. P. Deligne, Extension centrales non-residuellement finis de groupes arithmétiques, *C. R. Acad. Paris. Sér A-B*, **287** (1978), A203-A208.
5. B. Farb, The quasi-isometry classification of lattices in semisimple Lie groups, *Math. Res. Lett.*, **4** (1997), 705-717.
6. A. Fel'shtyn, The Reidemeister number of any automorphism of a Gromov hyperbolic group is infinite, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 279 (2001), *Geom. i Topol.*, **6** 229-240.
7. A. Fel'shtyn, New directions in Nielsen-Reidemeister theory, *Topology Appl.*, **157** (2010), 1724-1735.
8. A. Fel'shtyn and R. Hill, Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion, *K-Theory*, **8** (1994), 367-393.
9. D. L. Gonçalves and P. Wong, Twisted conjugacy classes in nilpotent groups, *J. Reine Angew. Math.*, **633** (2009), 11-27.
10. Bo Ju Jiang, Lectures on Nielsen fixed point theory, *Contemp. Math.*, **14**, AMS, Providence, RI. 1983.
11. G. Levitt and M. Lustig, Most automorphisms of a hyperbolic group have simple dynamics, *Ann. Sci. Ecole. Norm. Sup.*, **33** (2000), 507-517.
12. J. J. Millson, Real vector bundles with discrete structure group, *Topology*, **18** (1979), 83-89.

13. T. Mubeena and P. Sankaran, Twisted conjugacy in abelian extensions of certain linear groups, *Canadian Math. Bull*.

14. T. Mubeena and P. Sankaran, Twisted conjugacy in lattices in semisimple Lie groups, *Transformation Groups*, **19** (2014), 159-169.

15. T. Nasibullov, Twisted conjugacy classes in general and special linear groups, *Algebra and Logic*, **51** (2012), 220-231.

16. M. S. Raghunathan, Discrete subgroups of Lie groups, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, **68**, Springer-Verlag, Berlin, 1972.

17. M. S. Raghunathan, Torsion in cocompact lattices in coverings of $Spin(2, n)$, *Math. Ann.*, **266** (1984), 403-419.