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SYMMETRIC BIDERIVATIONS ON BANACH ALGEBRAS

Choonkil Park

Research Institute for Natural Sciences, Hanyang University, Seoul 04763, *Korea*
e-mail: baak@hanyang.ac.kr

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In [25], Park introduced the following bi-additive s-functional inequalities

$$\begin{aligned}&\|f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\| \\ &\leq \left\|s\left(2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right)\right\|\end{aligned} \tag{1}$$

and

$$\begin{aligned}&\left\|2f\left(\frac{x+y}{2},z-w\right)+2f\left(\frac{x-y}{2},z+w\right)-2f(x,z)+2f(y,w)\right\| \\ &\leq \|s\left(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\right)\|,\end{aligned} \tag{2}$$

where s is a fixed nonzero complex number with $|s|<1$.

In this paper, we prove the Hyers-Ulam stability of symmetric biderivations and skew-symmetric biderivation on Banach algebras and unital C^*-algebras, associated with the bi-additive s-functional inequalities (1) and (2).

Key words : Symmetric biderivation on C^*-algebra; skew-symmetric biderivation on C^*-algebra; Hyers-Ulam stability; bi-additive s-functional inequality.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [26] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

Gilányi [9] showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(x+y)+f(x-y).$$

See also [27]. Fechner [6] and Gilányi [10] proved the Hyers-Ulam stability of the functional inequality (1.1). Park [21, 22] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations and functional inequalities have been extensively investigated by a number of authors (see [3-5, 7, 12, 15, 18, 23, 24]).

Maksa [16, 17] introduced and investigated biderivations and symmetric biderivations on rings. Öztürk and Sapanci [19], Vukman [29] and Yazarli [30] investigated some properties of symmetric biderivations on rings.

Definition 1.1 — [16, 17]. Let A be a ring. A bi-additive mapping $D : A \times A \to A$ is called a *symmetric biderivation* on A if D satisfies

$$\begin{aligned} D(xy,z) &= D(x,z)y + xD(y,z), \\ D(x,y) &= D(y,x) \end{aligned}$$

for all $x, y, z \in A$.

In this paper, we introduce a *symmetric biderivation* on a Banach algebra and a *skew-symmetric biderivation* on a Banach $*$-algebra.

Definition 1.2 — Let A be a complex Banach algebra. A $\mathbb{C}$-bilinear mapping $D : A \times A \to A$ is called a *symmetric biderivation* on A if D satisfies

$$\begin{aligned} D(xy,z) &= D(x,z)y + xD(y,z), \\ D(x,y) &= D(y,x) \end{aligned}$$

for all $x, y, z \in A$.

It is easy to show that if D is a symmetric biderivation, then

$$D(x,zw) = D(zw,x) = D(z,x)w + zD(w,x) = D(x,z)w + zD(x,w)$$

for all $x, z, w \in A$. So

$$\begin{aligned} D(xy, zw) &= D(x, zw)y + xD(y, zw) \\ &= D(x, z)wy + zD(x, w)y + xD(y, z)w + xzD(y, w) \end{aligned}$$

for all $x, y, z, w \in A$.

Definition 1.3 — Let A be a complex Banach $*$-algebra. A bi-additive mapping $D : A \times A \to A$ is called a *skew-symmetric biderivation* on A if D is $\mathbb{C}$-linear in the first variable and satisfies

$$\begin{aligned} D(xy, z) &= D(x, z)y + xD(y, z), \\ D(x, y) &= D(y, x)^* \end{aligned}$$

for all $x, y, z \in A$.

It is easy to show that if D is a skew-symmetric biderivation, then D is conjugate $\mathbb{C}$-linear in the second variable and

$$\begin{aligned} D(x, zw) &= D(zw, x)^* = (D(z, x)w + zD(w, x))^* = w^*D(z, x)^* + D(w, x)^*z^* \\ &= w^*D(x, z) + D(x, w)z^* \end{aligned}$$

for all $x, z, w \in A$. So

$$\begin{aligned} D(xy, zw) &= D(x, zw)y + xD(y, zw) \\ &= w^*D(x, z)y + D(x, w)z^*y + xw^*D(y, z) + xD(y, w)z^* \end{aligned}$$

for all $x, y, z, w \in A$.

This paper is organized as follows: In Section 2, we investigate symmetric biderivations on Banach algebras and unital C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2). In Section 3, we investigate skew-symmetric biderivations on Banach $*$-algebras and unital C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2).

Throughout this paper, let X be a complex normed space and Y a complex Banach space. Let A be a complex Banach algebra. Assume that s is a fixed nonzero complex number with $|s| < 1$.

2. Symmetric Biderivations on Banach Algebras

In [25], Park introduced and investigated the bi-additive s-functional inequalities (1) and (2) in complex Banach spaces.

***Theorem* 2.1** — *[25, Theorem 2.2]. Let $r > 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned} &\|f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\| \\ &\quad \le \left\| s\left(2f\left(\frac{x+y}{2}, z-w\right)+2f\left(\frac{x-y}{2}, z+w\right)-2f(x,z)+2f(y,w)\right)\right\| \\ &\quad +\theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r) \end{aligned} \tag{2.1}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that

$$\|f(x,z) - A(x,z)\| \le \frac{2\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{2.2}$$

for all $x, z \in X$.

***Theorem* 2.2** — *[25, Theorem 2.3]. Let $r < 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying (2.1) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that*

$$\|f(x,z) - A(x,z)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r\|z\|^r \tag{2.3}$$

for all $x, z \in X$.

***Theorem* 2.3** — *[25, Theorem 3.2]. Let $r > 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}, z-w\right)+2f\left(\frac{x-y}{2}, z+w\right)-2f(x,z)+2f(y,w)\right\| \\ &\quad \le \|s\left(f(x+y,z-w)+f(x-y,z+w)-2f(x,z)+2f(y,w)\right)\| \\ &\quad +\theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r) \end{aligned} \tag{2.4}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that

$$\|f(x,z) - A(x,z)\| \le \frac{2^{r-1}\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{2.5}$$

for all $x, z \in X$.

***Theorem* 2.4** — *[25, Theorem 3.3]. Let $r < 1$ and θ be nonnegative real numbers and let $f : X^2 \to Y$ be a mapping satisfying (2.4) and $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique bi-additive mapping $A : X^2 \to Y$ such that*

$$\|f(x,z) - A(x,z)\| \le \frac{\theta}{2(2 - 2^r)}\|x\|^r\|z\|^r \tag{2.6}$$

for all $x, z \in X$.

Now, we investigate symmetric biderivations on complex Banach algebras and unital C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2).

Lemma 2.5 — [2, Lemma 2.1]. Let $f : X^2 \to Y$ be a bi-additive mapping such that $f(\lambda x, \mu z) = \lambda\mu f(x, z)$ for all $x, z \in X$ and $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} \ : \ |\nu| = 1\}$. Then f is $\mathbb{C}$-bilinear.

***Theorem* 2.6** — *Let A be a complex Banach algebra. Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(\lambda(x+y), \mu(z+w)) + f(\lambda(x+y), \mu(z-w)) + f(\lambda(x-y), \mu(z+w)) \\ &\qquad + f(\lambda(x-y), \mu(z-w)) - 4\lambda\mu f(x, z)\| \\ &\quad \leq \left\| s\left(4f\left(\frac{x+y}{2}, z-w\right) + 4f\left(\frac{x-y}{2}, z+w\right) - 4f(x, z) + 4f(y, w)\right)\right\| \\ &\qquad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{2.7}$$

for all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} \ : \ |\nu| = 1\}$ and all $x, y, z, w \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that

$$\|f(x, z) - D(x, z)\| \leq \frac{\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{2.8}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\|f(xy, z) - f(x, z)y - xf(y, z)\| \leq \theta(\|x\|^r + \|y\|^r)\|z\|^r, \tag{2.9}$$

$$\|f(x, z) - f(z, x)\| \leq \theta\|x\|^r\|z\|^r \tag{2.10}$$

for all $x, y, z \in A$, then the mapping $f : A^2 \to A$ is a symmetric biderivation.

PROOF : Let $\lambda = \mu = 1$ in (2.7). By Theorem 2.1, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (2.8) defined by

$$D(x, z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

Letting $y = w = 0$ in (2.7), we get $f(\lambda x, \mu z) = \lambda\mu f(x, z)$ for all $x, z \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. By Lemma 2.5, the bi-additive mapping $D : A^2 \to A$ is $\mathbb{C}$-bilinear.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $D(x, z) = f(x, z)$ for all $x, z \in A$.

It follows from (2.9) that

$$\begin{aligned}\|D(xy,z)-D(x,z)y-xD(y,z)\| &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{xy}{2^n\cdot 2^n},z\right)-f\left(\frac{x}{2^n},z\right)\frac{y}{2^n}-\frac{x}{2^n}f\left(\frac{y}{2^n},z\right)\right\| \\ &\le \lim_{n\to\infty}\frac{4^n\theta}{2^{rn}}(\|x\|^r+\|y\|^r)\|z\|^r=0\end{aligned}$$

for all $x, y, z \in A$. Thus

$$D(xy,z)=D(x,z)y+xD(y,z)$$

for all $x, y, z \in A$.

It follows from (2.10) that

$$\|D(x,z)-D(z,x)\|=\lim_{n\to\infty} 2^n\left\|f\left(x,\frac{z}{2^n}\right)-f\left(\frac{z}{2^n},x\right)\right\|\le\lim_{n\to\infty}\frac{2^n\theta}{2^{rn}}\|x\|^r\|z\|^r=0$$

for all $x, z \in A$. Thus

$$D(x,z)=D(z,x)$$

for all $x, z \in A$. Hence the mapping $f : A^2 \to A$ is a symmetric biderivation. □

***Theorem* 2.7** — *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (2.7) and $f(x,0) = f(0,z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ such that*

$$\|f(x,z)-D(x,z)\|\le\frac{\theta}{2-2^r}\|x\|^r\|z\|^r \tag{2.11}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (2.9), (2.10) and $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a symmetric biderivation.

PROOF : The proof is similar to the proof of Theorem 2.6. □

Similarly, we can obtain the following results.

***Theorem* 2.8** — *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ and*

$$\begin{aligned}&\left\|4f\left(\lambda\frac{x+y}{2},\mu(z-w)\right)+4f\left(\lambda\frac{x-y}{2},\mu(z+w)\right)-4\lambda\mu f(x,z)+4\lambda\mu f(y,w)\right\| \\ &\le \|s\left(f(x+y,z+w)+f(x+y,z-w)+f(x-y,z+w)+f(x-y,z-w)-4f(x,z)\right)\| \\ &\quad+\theta(\|x\|^r+\|y\|^r)(\|z\|^r+\|w\|^r)\end{aligned} \tag{2.12}$$

for all $\lambda, \mu \in \mathbb{T}^1$ *and all* $x, y, z, w \in A$*. Then there exists a unique* $\mathbb{C}$*-bilinear mapping* $D : A^2 \to A$ *such that*

$$\|f(x,z) - D(x,z)\| \leq \frac{2^{r-2}\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{2.13}$$

for all $x, z \in A$*.*

If, in addition, the mapping $f : A^2 \to A$ *satisfies (2.9), (2.10) and* $f(2x,z) = 2f(x,z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a symmetric biderivation.*

***Theorem* 2.9** — *Let* $r < 1$ *and* θ *be nonnegative real numbers, and let* $f : A \to A$ *be a mapping satisfying (2.12) and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in A$*. Then there exists a unique* $\mathbb{C}$*-bilinear mapping* $D : A^2 \to A$ *such that*

$$\|f(x,z) - D(x,z)\| \leq \frac{\theta}{4(2 - 2^r)}\|x\|^r\|z\|^r \tag{2.14}$$

for all $x, z \in A$*.*

If, in addition, the mapping $f : A^2 \to A$ *satisfies (2.9), (2.10) and* $f(2x,z) = 2f(x,z)$ *for all* $x, z \in A$*, then the mapping* $f : A^2 \to A$ *is a symmetric biderivation.*

From now on, assume that A is a unital C^*-algebra with unit e and unitary group $U(A)$.

***Theorem* 2.10** — *Let* $r > 2$ *and* θ *be nonnegative real numbers, and let* $f : A^2 \to A$ *be a mapping satisfying (2.7) and* $f(x,0) = f(0,z) = 0$ *for all* $x, z \in A$*. Then there exists a unique* $\mathbb{C}$*-bilinear mapping* $D : A^2 \to A$ *satisfying (2.8).*

If, in addition, the mapping $f : A^2 \to A$ *satisfies (2.10),* $f(2x,z) = 2f(x,z)$ *and*

$$\|f(uy,z) - f(u,z)y - uf(y,z)\| \leq \theta(1 + \|y\|^r)\|z\|^r \tag{2.15}$$

for all $u, v \in U(A)$ *and all* $x, y, z \in A$*, then the mapping* $f : A^2 \to A$ *is a symmetric biderivation.*

PROOF : By the same reasoning as in the proof of Theorem 2.6, there is a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying (2.8) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x,z) = 2f(x,z)$ for all $x, z \in A$, then we can easily show that $D(x,z) = f(x,z)$ for all $x, z \in A$.

By the same reasoning as in the proof of Theorem 2.6, $D(uy, z) = D(u, z)y + uD(y, z)$ for all $u, v \in U(A)$ and all $y, z \in A$.

Since D is $\mathbb{C}$-linear in the first variable and each $x \in A$ is a finite linear combination of unitary elements (see [13]), i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in U(A))$,

$$\begin{aligned} D(xy, z) &= D(\sum_{j=1}^{m} \lambda_j u_j y, z) = \sum_{j=1}^{m} \lambda_j D(u_j y, z) = \sum_{j=1}^{m} \lambda_j (D(u_j, z)y + u_j D(y, z)) \\ &= (\sum_{j=1}^{m} \lambda_j) D(u_j, z)y + (\sum_{j=1}^{m} \lambda_j u_j) D(y, z) = D(x, z)y + xD(y, z) \end{aligned}$$

for all $x, y, z \in A$. So by the same reasoning as in the proof of Theorem 2.6, $D : A^2 \to A$ is a symmetric biderivation. Thus $f : A^2 \to A$ is a symmetric biderivation. □

***Theorem* 2.11** — *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (2.7) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying (2.11).*

If, in addition, the mapping $f : A \to A$ satisfies (2.10), (2.15) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a symmetric biderivation.

PROOF : The proof is similar to the proof of Theorem 2.10. □

Similarly, we can obtain the following results.

***Theorem* 2.12** — *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (2.12) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying (2.13).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (2.10), (2.15) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a symmetric biderivation.

***Theorem* 2.13** — *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (2.12) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique $\mathbb{C}$-bilinear mapping $D : A^2 \to A$ satisfying (2.14).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (2.10), (2.15) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a symmetric biderivation.

3. SKEW-SYMMETRIC BIDERIVATIONS ON BANACH $*$-ALGEBRAS

In this section, we investigate skew-symmetric biderivations on complex Banach $*$-algebras and unital

C^*-algebras associated with the bi-additive s-functional inequalities (1) and (2).

***Theorem* 3.1** — *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\|f(\lambda(x+y), z+w) + f(\lambda(x+y), z-w) + f(\lambda(x-y), z+w) \\ &\qquad + f(\lambda(x-y), z-w) - 4\lambda f(x,z)\| \\ &\leq \left\| s\left(4f\left(\frac{x+y}{2}, z-w\right) + 4f\left(\frac{x-y}{2}, z+w\right) - 4f(x,z) + 4f(y,w)\right)\right\| \\ &\qquad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned} \tag{3.1}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that

$$\|f(x,z) - D(x,z)\| \leq \frac{\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{3.2}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies $f(2x, z) = 2f(x, z)$ and

$$\|f(xy, z) - f(x, z)y - xf(y, z)\| \leq \theta(\|x\|^r + \|y\|^r)\|z\|^r, \tag{3.3}$$

$$\|f(x,z) - f(z,x)^*\| \leq \theta\|x\|^r\|z\|^r \tag{3.4}$$

for all $x, y, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetic biderivation.

PROOF : Let $\lambda = 1$ in (3.1). By Theorem 2.1, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (2.8) defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $D(x, z) = f(x, z)$ for all $x, z \in A$.

Letting $y = x$ and $w = 0$ in (3.1), we get

$$\|2f(2\lambda x, z) - 4\lambda f(x,z)\| \leq 2\theta\|x\|^r\|z\|^r$$

for all $x, z \in A$ and all $\lambda \in \mathbb{T}^1$. So

$$\|2D(2\lambda x, z) - 4\lambda D(x,z)\| = \lim_{n\to\infty} 2^n \left\|2f\left(2\lambda\frac{x}{2^n}, z\right) - 4\lambda f\left(\frac{x}{2^n}, z\right)\right\| \leq \lim_{n\to\infty}\frac{2^{n+1}}{2^{rn}}\theta\|x\|^r\|z\|^r = 0$$

for all $x, z \in A$ and all $\lambda \in \mathbb{T}^1$. Hence $2D(2\lambda x, z) = 4\lambda D(x,z)$ and so $D(\lambda x, z) = \lambda D(x,z)$ for all $x, z \in A$ and all $\lambda \in \mathbb{T}^1$. By [20, Theorem 2.1], the bi-additive mapping $D : A^2 \to A$ is $\mathbb{C}$-linear in the first variable.

It follows from (3.3) that

$$\begin{aligned} \|D(xy,z) - D(x,z)y - xD(y,z)\| &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{2^n \cdot 2^n}, z\right) - f\left(\frac{x}{2^n}, z\right)\frac{y}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}, z\right)\right\| \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{2^{rn}}(\|x\|^r + \|y\|^r)\|z\|^r = 0 \end{aligned}$$

for all $x, y, z \in A$. Thus

$$D(xy, z) = D(x,z)y + xD(y,z)$$

for all $x, y, z \in A$.

It follows from (3.4) that

$$\|D(x,z) - D(z,x)^*\| = \lim_{n\to\infty} 2^n \left\| f\left(x, \frac{z}{2^n}\right) - f\left(\frac{z}{2^n}, x\right)^* \right\| \le \lim_{n\to\infty} \frac{2^n\theta}{2^{rn}}\|x\|^r\|z\|^r = 0$$

for all $x, z \in A$. Thus

$$D(x,z) = D(z,x)^*$$

for all $x, z \in A$. Hence the mapping $f : A^2 \to A$ is a skew-symmetric biderivation. □

***Theorem* 3.2** — *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (3.1) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that*

$$\|f(x,z) - D(x,z)\| \le \frac{\theta}{2 - 2^r}\|x\|^r\|z\|^r \tag{3.5}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (3.3), (3.4) and $f(2x, z) = 2f(x,z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetric biderivation.

PROOF : The proof is similar to the proof of Theorem 3.1. □

Similarly, we can obtain the following results.

***Theorem* 3.3** — *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and*

$$\begin{aligned} &\left\| 4f\left(\lambda\frac{x+y}{2}, z - w\right) + 4f\left(\lambda\frac{x-y}{2}, z + w\right) - 4\lambda f(x,z) + 4\lambda f(y,w)\right\| \quad (3.6) \\ &\le \|s\left(f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) - 4f(x,z)\right)\| \\ &\quad + \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r) \end{aligned}$$

for all $\lambda \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that

$$\|f(x,z) - D(x,z)\| \leq \frac{2^{r-2}\theta}{2^r - 2}\|x\|^r\|z\|^r \tag{3.7}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (3.3), (3.4) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetric biderivation.

***Theorem* 3.4** — *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (3.6) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable, such that*

$$\|f(x,z) - D(x,z)\| \leq \frac{\theta}{4(2 - 2^r)}\|x\|^r\|z\|^r \tag{3.8}$$

for all $x, z \in A$.

If, in addition, the mapping $f : A^2 \to A$ satisfies (3.3), (3.4) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetric biderivation.

From now on, assume that A is a unital C^*-algebra with unitary group $U(A)$.

***Theorem* 3.5** — *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (3.1) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (3.2).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (3.4), $f(2x, z) = 2f(x, z)$ and

$$\|f(uy, z) - uf(y, z)\| \leq \theta(1 + \|y\|^r)\|z\|^r \tag{3.9}$$

for all $u \in U(A)$ and all $x, y, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetric biderivation.

PROOF : By the same reasoning as in the proof of Theorem 3.1, there is a unique bi-additive mapping $D : A^2 \to A$ satisfying (3.2), which is $\mathbb{C}$-linear in the first variable, defined by

$$D(x,z) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}, z\right)$$

for all $x, z \in A$.

If $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then we can easily show that $D(x, z) = f(x, z)$ for all $x, z \in A$.

By the same reasoning as in the proof of Theorem 3.1, $D(uy, z) = D(u, z)y + uD(y, z)$ for all $u \in U(A)$ and all $y, z \in A$.

By the same reasoning as in the proof of Theorem 2.10, $D(xy, z) = D(x, z)y + xD(y, z)$ for all $u \in U(A)$ and all $y, z \in A$. Thus $f : A^2 \to A$ is a skew-symmetric biderivation. □

***Theorem* 3.6** — *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (3.1) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (3.5).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (3.4), (3.9) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetric biderivation.

PROOF : The proof is similar to the proof of Theorem 3.5. □

Similarly, we can obtain the following results.

***Theorem* 3.7** — *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (3.6) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (3.7).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (3.4), (3.9) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetric biderivation.

***Theorem* 3.8** — *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A^2 \to A$ be a mapping satisfying (3.6) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in A$. Then there exists a unique bi-additive mapping $D : A^2 \to A$, which is $\mathbb{C}$-linear in the first variable and satisfies (3.8).*

If, in addition, the mapping $f : A^2 \to A$ satisfies (3.4), (3.9) and $f(2x, z) = 2f(x, z)$ for all $x, z \in A$, then the mapping $f : A^2 \to A$ is a skew-symmetric biderivation.

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