Indian J. Pure Appl. Math., **50**(2): 427-450, June 2019

DOI: 10.1007/s13226-019-0336-5

DYNAMICAL ANALYSIS OF A COMPLEX LOGISTIC-TYPE MAP

A. M. A. El-Sayed* and S. M. Salman**

Faculty of Science, Alexandria University, Alexandria, Egypt
***Faculty of Education, Alexandria University, Alexandria, Egypt*
e-mails: amasayed@gmail.com; samastars9@gmail.com

(*Received* 27 *April* 2018; *after final revision* 2 *June* 2018;
accepted 4 *July* 2018)

A complex logistic-type map is considered in the present work. The dynamic behavior of the underlined map is discussed in two different cases: the first case is when the parameter of the map being real, and the second case is when the parameter being complex. Existence and stability of fixed points are derived. The conditions for existence of a pitchfork bifurcation, flip bifurcation and Neimark-Sacker bifurcation are derived by using the center manifold theorem and bifurcation theory. Numerical simulations including Lyapunov exponent, phase plane, bifurcation diagrams is carried out using matlab to ensure theoretical results and to reveal more complex dynamics of the map. The results show that expressing the logistic map in terms of complex variables leads to more distinguished behaviors, which could not be achieved in the logistic map with real variables. In addition, considering the control parameter as a complex parameter shows more interesting dynamics if compared to the case when considering it as a real parameter. Existence of a snapback repeller is proved in the sense of Marotto. Finally, chaos is controlled using the OGY feedback method.

Key words :Logistic-type map; complex variables; fixed points; local stability; Lyapunov exponent; bifurcation; chaos; OGY feedback control.

1. INTRODUCTION

In the last years, studies of the effect of forcing on systems simulated by the logistic map $x_{n+1} = \rho x_n(1 - x_n)$ for choosing x_0 between 0 and 1 and $0 < \rho < 4$ have found a celebrated place in chaos, fractals and discrete dynamics which has been reported in [1-4]. However, since most of these interesting results have been obtained by changing single parameters, this subject is still far from being fully explored. In particular, it seems worthwhile to look for new dynamic properties regarding relevant parameter sets for which this system behaves similarly concerning one or several properties of interest, such as, for example, periodicity of attractors and basins of attraction. Hyperchaos is known as a chaotic attractor with more than one positive Lyapunov exponent and actually has more rich dynamical behaviors than the usual chaos. Over the past decades, hyperchaotic systems with real variables have been investigated. Since Fowler *et al*. [5] introduced a complex Lorenz model which generalizes the real Lorenz model in 1982, chaotic and hyperchaotic complex systems have attracted attention to the systems with complex variables which can be used to describe the physics of a detuned laser, rotating fluids, disk dynamos, electronic circuits, and particle beam dynamics in high energy accelerators. In [6], the dynamics of hyperchaotic complex Chen system are investigated. When applying the complex systems in communications, the complex variables will double the number of variables and can increase the content and security of the transmitted information.

Periodic solutions are important in the study of dynamical systems, since they represent stationary or repeatable behavior [7-9]. Non-linear differential equations with periodic coefficients are quite important from a periodical point of view. Especially, the search for periodic solutions of these equations is an essential problem. Many problems of physical interest are concerned with the motion of strongly non-linear oscillators, see for example [10]. The main thrust of the present paper is to study the main dynamic behavior of the logistic map with complex variables. Actually, considering the logistic difference equation with complex variables is because complex analysis in one variable represents a portion of mathematical beauty, the real beauty and excitement comes in higher dimensions and manifolds (Riemann surfaces, complex manifolds, several complex variables) [11]. There are many examples of considering systems with complex variables, for example, the complex Lorenz equations, complex Chen and Lü chaotic systems, complex logistic map, and complex Riccati map [12-16]. Indeed, the qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. As a matter of fact, many real life phenomena are modeled using difference equations. Examples from economy, biology, etc. can be found in [17-23]. Many non-linear systems have parameters which appear in the defining systems of equations. Changes may occur in the qualitative structure of the orbits for certain parameter values as the parameter varies [24]. The main problem in non-linear dynamics is that of determining how the properties of orbits may change and evolve as a parameter of a dynamical system is changed [25].

In this paper, we consider the logistic-type difference equation given in the form

$$z_{n+1} = \rho z_n(1 - z_n\bar{z}_n), \;\; n = 1, 2, 3, ... \tag{1.1}$$

where $\rho \in \mathbb{R}$, with initial condition

$$z(0) = z_o = x_0 + iy_0, \tag{1.2}$$

where $z = x + iy$, $\bar{z} = x - iy$, $|z_n| \leq 1$ and $i = \sqrt{-1}$. We are going to study the nonlinear dynamics of Eq. (1.1) such as stability of fixed points, local bifurcations analysis using bifurcation theory and center manifold theorem [26, 27], Lyapunov exponent, and chaos.

This paper is organized as follows, in Section 2 the existence and stability of the fixed points of the map (1.1) is presented when ρ is real. Qualitative behavior and local bifurcation analysis are discussed using center manifold theorem and bifurcation theory. Moreover, conditions for the existence of pitchfork bifurcation and flip bifurcation are derived. Section 3 represents qualitative behavior of map (1.1) in case ρ is complex. In addition, conditions for the existence of Neimark-Sacker (NS) bifurcation are given. Numerical simulations results are presented in Section 4 to verify our theoretical analysis and visualize the newly observed complex dynamics of the system. Finally, in Section 5 conclusion is given.

2. Existence and Local Stability of Fixed Points When ρ is Real

Rewrite Eq. (1.1) as

$$\begin{aligned} x_{n+1} &= \rho(x_n - x_n^3 - x_n y_n^2), \quad n = 1, 2, 3, ... \\ y_{n+1} &= \rho(y_n - y_n x_n^2 - y_n^3). \end{aligned} \tag{2.1}$$

To find fixed points of system (2.1) we solve the following system of algebraic equations

$$\begin{aligned} x &= \rho(x - x^3 - xy^2), \\ y &= \rho(y - yx^2 - y^3). \end{aligned}$$

So, fixed points of system (2.1) are as follows:

- For all values of ρ, there is one fixed point only which is the origin $fix_1(0, 0)$.
- For $\rho > 1$, there are infinite number of fixed points since they lie inside the circle

$$\left\{ \mathbb{Z} \in \mathbb{C} : |Z|^2 = \frac{\rho - 1}{\rho} \right\}.$$

For convenience, we will consider only here the obvious fixed points which are: $fix_2\left(0, \sqrt{\frac{\rho-1}{\rho}}\right)$, $fix_3\left(0, -\sqrt{\frac{\rho-1}{\rho}}\right)$, $fix_4\left(\sqrt{\frac{\rho-1}{\rho}}, 0\right)$, and $fix_5\left(-\sqrt{\frac{\rho-1}{\rho}}, 0\right)$. Note that fix_2 and fix_3 are symmetric about $x-$axis, while fix_4 and fix_5 are symmetric about $y-$axis.

In order to study the local stability of these fixed points, we need the moduli of the eigenvalues of the Jacobian matrix evaluated at each fixed points. The Jacobian matrix of system (2.1) at any fixed point (x^*, y^*) reads

$$J(x^*, y^*) = \begin{pmatrix} \rho(1 - 3x^{*2} - y^{*2}) & -2\rho x^* y^* \\ -2\rho x^* y^* & \rho(1 - 3y^{*2} - x^{*2}) \end{pmatrix}. \tag{2.2}$$

In order to discuss the stability of the fixed point (x^*, y^*), we need the following lemma.

Lemma 1 — Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0$, λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then

1. $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) < 0$, $Q < 1$,
2. $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$,
3. $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(1) > 0$ and $Q > 1$,
4. $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $P \neq 0, 2$,
5. λ_1 and λ_2 are complex and $|\lambda_1| = 1$ and $|\lambda_2| = 1$ if and only if $P^2 - 4Q < 0$ and $Q = 1$.

Let λ_1 and λ_2 be the two roots of the characteristic equation of the Jacobian matrix $J(x^*, y^*)$ at the fixed point (x^*, y^*) which are called eigenvalues of the fixed point (x^*, y^*). We recall some definitions of topological types for a fixed point (x^*, y^*). A fixed point (x^*, y^*) is called a sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, so the sink is locally asymptotically stable; it is called a source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, so the source is locally unstable; it is called a saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ (or $|\lambda_1| < 1$ and $|\lambda_2| > 1$); and (x, y) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.
According to Lemma 1 and (2.2), we have the following propositions.

Proposition 1 — $fix_1(0, 0)$ is:

A sink if $-1 < \rho < 1$,

A source if $\rho > 1$ or $\rho < -1$,

A non-hyperbolic if $\rho = 1$ or $\rho = -1$.

Note that fix_1 can not be a saddle.

Proposition 2 — $fix_2\left(0, \sqrt{\frac{\rho-1}{\rho}}\right)$ is a non-hyperbolic if $\rho = 1$ or $\rho = 2$.

The same proposition can be applied to $fix_4\left(\sqrt{\frac{\rho-1}{\rho}}, 0\right)$. Note that both fix_2 and fix_4 can not be a saddle, a sink, or a source.

2.1 *Bifurcation of* $fix_1(0,0)$

The Jacobian matrix at $fix_1(0,0)$ reads

$$J(fix_1) = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \tag{2.3}$$

which has two eigenvalues $\lambda_{1,2} = \rho$. If $\rho = 1$, then $\lambda_{1,2} = 1$.

Lemma 2 — If $\rho = 1$, system (2.1) undergoes a pitchfork bifurcation at $fix_1(0,0)$. Moreover, the system has only one fixed point.

PROOF : In the analysis of bifurcations we always take ρ as a bifurcation parameter. Let $\mu = \rho - 1$ such that parameter μ be a new and dependent variable, the system (2.1) is transformed into the following form

$$\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu \end{pmatrix} + \begin{pmatrix} \mu(x - x^3 - xy^2) - x^3 - xy^2 \\ -yx^2 - y^3 + \mu(y - yx^2 - y^3) \\ 0 \end{pmatrix}. \tag{2.4}$$

Then, there exists a center manifold for (2.4), which can be represented as follows:

$$W^c(fix_1) = \{(x, y, \mu) \in \mathbb{R}^3 | y = h(x, \mu), h(0,0) = Dh(0,0), |x| < \epsilon, |\mu| < \delta\},$$

for ϵ, δ sufficiently small.

To compute the center manifold W^c we assume

$$h(x, \mu) = c_0 x^2 + c_1 x\mu + c_2 \mu^2 + O((|x| + |\mu|)^3), \tag{2.5}$$

where $O((|x| + |\mu|)^3)$ is the sum of all terms whose order is great than 2.

The center manifold must satisfy

$$h(x + \mu(x - x^3) - \mu x h(x,\mu)^2 - x^3 - x h(x,\mu)^2, \mu) = h(x,\mu) - x^2 h(x,\mu) - h(x,\mu)^3. \tag{2.6}$$

Substituting (2.4) and (2.5) into (2.6) and then equating coefficients of like powers in (2.6), we get

$$c_0 = 0, \quad c_1 = 0, \quad c_2 = 0$$

Thus the map restricted to the center manifold is given by

$$F_1 : x_{n+1} = x_n + x_n\mu_n - x_n^3\mu_n - x_n^3 - O((|x_n| + |\mu_n|)^8).$$

Since

$$\begin{aligned}\left(\frac{\partial^2 F_1}{\partial x \partial \mu}\right)_{(0,0)} &= 1 \neq 0,\\ \left(\frac{\partial^3 F_1}{\partial x^3}\right)_{(0,0)} &= -6 \neq 0.\end{aligned}$$

Thus, system (2.1) undergoes a pitchfork bifurcation at $fix_1(0,0)$. This completes the proof. □

Lemma 3 — If $\rho = -1$, system (2.1) undergoes a flip bifurcation at $fix_1(0,0)$. Moreover, the stable periodic-2 point bifurcates from this fixed point.

PROOF : If $\rho = -1$, the two eigenvalues of (2.3) become $\lambda_{1,2} = -1$.

Let $\mu = \rho + 1$ be new and dependent variable, the system (2.1) is transformed into the following form

$$\begin{pmatrix} x \\ y \\ \mu \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu \end{pmatrix} + \begin{pmatrix} \mu(x - x^3 - xy^2) + x^3 + xy^2 \\ yx^2 + y^3 + \mu(y - yx^2 - y^3) \\ 0 \end{pmatrix}. \tag{2.7}$$

Then, there exists a center manifold for (2.7), which can be represented as follows:

$$W^c(fix_1) = \{(x, y, \mu) \in \mathbb{R}^3 | y = h(x, \mu), h(0,0) = Dh(0,0), |x| < \epsilon, |\mu| < \delta\},$$

for ϵ, δ sufficiently small.

To compute the center manifold W^c we assume

$$h(x, \mu) = a_0 x^2 + a_1 x\mu + a_2 \mu^2 + O((|x| + |\mu|)^3), \tag{2.8}$$

where $O((|x| + |\mu|)^3)$ is the sum of all terms whose order is great than 2.

The center manifold must satisfy

$$\begin{aligned} &h(-x + \mu(x - x^3) + \mu x h(x,\mu)^2 + x^3 + x h(x,\mu)^2, \mu) \\ &\qquad = (1-\mu)h(x,\mu) - \mu x^2 h(x,\mu) + h(x,\mu)^3. \end{aligned} \tag{2.9}$$

Substituting (2.7) and (2.8) into (2.9) and then equating coefficients of like powers in (2.9), we get

$$a_0 = 0, \;\; a_1 = 0, \;\; a_2 = 0$$

Thus the map restricted to the center manifold is given by

$$F_2 : x_{n+1} = -x_n + x_n\mu_n - x_n^3\mu_n + x_n^3 - O((|x_n| + |\mu_n|)^8).$$

Since

$$\begin{aligned}\alpha_1 &= \Big(2\frac{\partial^2 F_2}{\partial\mu\partial x}+\frac{\partial F_2}{\partial\mu}\frac{\partial^2 F_2}{\partial x^2}\Big)_{(0,0)} = 2\neq 0,\\ \alpha_2 &= \Big(\frac{1}{2}\Big(\frac{\partial^2 F_2}{\partial x^2}\Big)^2+\frac{1}{3}\Big(\frac{\partial^3 F_2}{\partial x^3}\Big)\Big)_{(0,0)} = 2\neq 0,\end{aligned}$$

Thus, system (2.1) undergoes a flip bifurcation at $fix_1(0,0)$. This completes the proof. □

2.2 *Bifurcation of* $fix_2\left(0,\sqrt{\frac{\rho-1}{\rho}}\right)$

In this part, we will investigate the local bifurcation analysis of the second fixed point of the system (2.1) using the center manifold theorem exactly as we did for the first fixed point. The Jacobian matrix of system (2.1) evaluated at fix_2 reads

$$J(fix_1)=\begin{pmatrix}1 & 0\\ 0 & -2\rho+3\end{pmatrix}, \tag{2.10}$$

which has two eigenvalues $\lambda_1=1$, and $\lambda_2=-2\rho+3$. If $\rho\neq 1,2$, then $|\lambda_2|\neq 1$, however, the conditions of the occurrence of pitchfork bifurcation at fix_2 are not satisfied. On the other hand, if $\rho=1$, we will have that $\lambda_{1,2}=1$. In the following lemma, we show that the conditions of occurrence of pitchfork bifurcation at fix_2 are satisfied when $\rho=1$.

Lemma 4 — If $\rho=1$, system (2.1) undergoes a pitchfork bifurcation at $fix_2\left(0,\sqrt{\frac{\rho-1}{\rho}}\right)$. Moreover, the system has only one fixed point.

PROOF : First of all we need to translate fix_2 to the origin using the translation

$$\xi_n=x_n,\quad \eta_n=y_n-\sqrt{\frac{\rho-1}{\rho}}.$$

Introduce $\mu=\rho-1$ as new and dependent variable, system (2.1) becomes

$$\begin{pmatrix}\xi\\ \eta\\ \mu\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}\xi\\ \eta\\ \mu\end{pmatrix}+\begin{pmatrix}f(\xi,\eta,\mu)\\ g(\xi,\eta,\mu)\\ 0\end{pmatrix}, \tag{2.11}$$

where $f(\xi,\eta,\mu)=-(\mu+1)\left(\xi^3+\xi\eta^2+\sqrt{\frac{\mu}{\mu+1}}\xi\eta\right)$,
$g(\xi,\eta,\mu)=-(\mu+1)\left(3\sqrt{\frac{\mu}{\mu+1}}\eta^2+\eta^3+\xi^2\eta+\sqrt{\frac{\mu}{\mu+1}}\xi^2\right)-2\mu\eta.$

Then, there exists a center manifold for (2.11), which can be represented as follows:

$$W^c(fix_2)=\{(\xi,\eta,\mu)\in\mathbb{R}^3|\eta=h(\xi,\mu),h(0,0)=Dh(0,0),|\xi|<\epsilon,|\mu|<\delta\},$$

for ϵ, δ sufficiently small.

To compute the center manifold W^c we assume

$$h(\xi,\mu) = d_0\xi^2 + d_1\xi\mu + d_2\mu^2 + O((|\xi| + |\mu|)^3), \tag{2.12}$$

where $O((|x| + |\mu|)^3)$ is the sum of all terms whose order is great than 2.

The center manifold must satisfy

$$\begin{aligned} h(\xi - (\mu+1)(\xi^3 + \xi h(\xi,\mu)^2 + \sqrt{\frac{\mu}{\mu+1}}\xi h(\xi,\mu)),\mu) \\ = h(\xi,\mu) - (\mu+1)\sqrt{\frac{\mu}{\mu+1}}\xi^2 - \\ - 3(\mu+1)\sqrt{\frac{\mu}{\mu+1}}h(\xi,\mu)^2 + \\ + h(\xi,\mu)^3 + (\xi^2 - 2\mu)h(\xi,\mu). \end{aligned} \tag{2.13}$$

Note that in order to equate coefficients in both sides, we should use binomial theorem for the term $\sqrt{\frac{\mu}{\mu+1}}$ as follows:

$$\sqrt{\frac{\mu}{\mu+1}} = \left(1 + \frac{1}{\mu}\right)^{\frac{-1}{2}} = 1 - \frac{1}{2\mu} + \frac{3}{8\mu^2} + ...$$

.

Substituting (2.11) and (2.12) into (2.13) and then equating coefficients of like powers in (2.13), we get

$$d_0 = 0, \;\; d_1(d_1^2 - 1) = 0, \;\; d_2 = 0.$$

For d_1, we have either $d_1 = 0$ or $d_1 = \pm 1$. After careful calculations, we found that pitchfork bifurcation conditions are not satisfied when $d_1 = \pm 1$. while when $d_1 = 0$, pitchfork bifurcation conditions are satisfied. Thus the map restricted to the center manifold is given by

$$F_3 : \xi_{n+1} = \xi_n - (\mu+1)\left(\xi_n^3 + \xi_n + \sqrt{\frac{\mu}{\mu+1}}\xi_n\right) - O((|x_n| + |\mu_n|)^6).$$

Now we have

$$\begin{aligned} \left(\frac{\partial^2 F_3}{\partial x \partial \mu}\right)_{(0,0)} &= -6 \neq 0, \\ \left(\frac{\partial^3 F_3}{\partial x^3}\right)_{(0,0)} &= \frac{-1}{2} \neq 0. \end{aligned}$$

Thus, system (2.1) undergoes a pitchfork bifurcation at $fix_2\left(0, \sqrt{1-\frac{1}{\rho}}\right)$. This completes the proof. □

We will not discuss the bifurcations of the fixed point fix_3 because it has symmetrical structure with fix_2. So, the process will be omitted here. The same thing can be said to fix_5 which has symmetrical structure with fix_4.

If one is interested in determining whether a dynamical system is chaotic or not, often just a few of the largest Lyapunov characteristic exponents (LCEs) may provide the answer. This actually because a positive LCE is a good indicator for chaos. Since for non-chaotic systems all LCEs are non-positive, the presence of a positive LCE has often been used to help determine if a system is chaotic or not. In this paper, we compute the (LCEs) via the Householder QR-based methods described in [28]. Figure 1 shows the LCEs for system (2.1) with the initial conditions $(x_o, y_o) = (0.1, 0.1)$. We find that LCE1 =0.6443 and LCE2 = 0.0031.

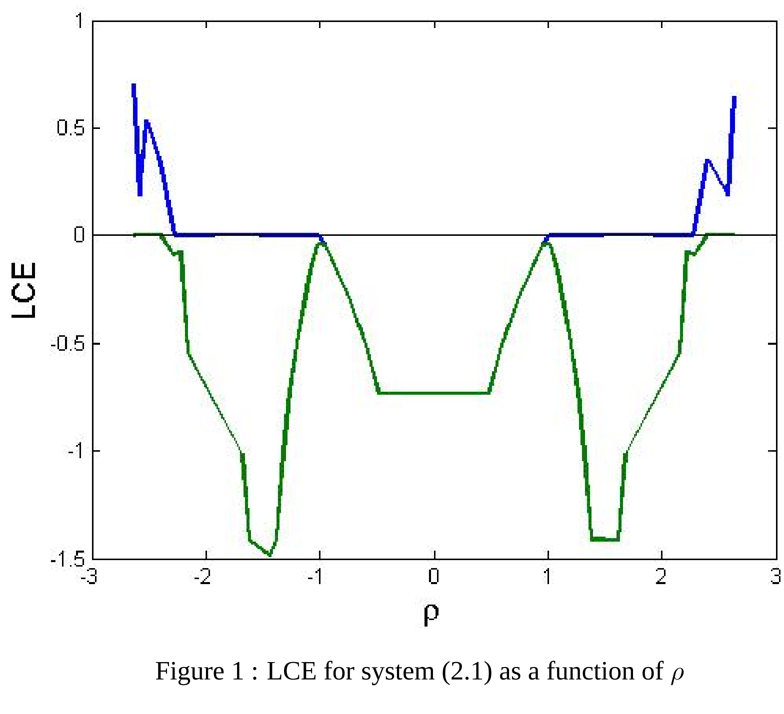

Figure 1 : LCE for system (2.1) as a function of ρ

3. Existence and Local Stability of Fixed Points When ρ is Complex

In this part, we consider $\rho = a + ib$, where $a, b \in \mathbb{R}$. Eq. (1.1) is rewritten as

$$\begin{aligned} x_{n+1} =& a(x_n - x_n^3 - x_n y_n^2) + b(-y_n + y_n x_n^2 + y_n^3), \\ y_{n+1} =& b(x_n - x_n^3 - x_n y_n^2) + a(y_n - y_n x_n^2 - y_n^3), \end{aligned} \tag{3.1}$$

where $n = 1, 2, 3, ...,$ $|x_n| \leq 1$, $|y_n| \leq 1$. Fixed points of system (3.1) are the solutions of the following system of algebraic equations

$$\begin{aligned} x &= a(x - x^3 - xy^2) + b(-y + yx^2 + y^3), \\ y &= b(x - x^3 - xy^2) + a(y - yx^2 - y^3). \end{aligned}$$

Thus, system (3.1) has one fixed point only for all values of a and b which is $fix(0, 0)$. In order to study the local stability of this fixed point, we need to analyze the eigenvalues associated to the Jacobian matrix of system (3.1) evaluated at $fix(0, 0)$ as shown in the following proposition.

Proposition 3 — The fixed point $fix(0, 0)$ of system (3.1) is:

A sink if $a^2 + b^2 < 1$,

A source if $a^2 + b^2 > 1$,

A non-hyperbolic if $a^2 + b^2 = 1$.

In what follows, we discuss the local bifurcation analysis at the fixed point $fix(0, 0)$ of system (3.1).

3.1 *Bifurcation of* $fix(0, 0)$

The Jacobian matrix of system (3.1) at any fixed point (x^*, y^*) reads

$$J(x^*, y^*) = \begin{pmatrix} a(1 - 3x^2 - y^2) + 2bxy & -2axy + b(-1 + x^2 + 3y^2) \\ -2axy + b(1 - 3x^2 - y^2) & a(1 - x^2 - 3y^2) - 2bxy \end{pmatrix}.$$

Now, the Jacobian matrix at $fix(0, 0)$ reads

$$J(0, 0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{3.2}$$

which has two eigenvalues, $\lambda_1 = a + ib$, $\lambda_2 = a - ib$. In the next lemma, we study the local bifurcation of $fix(0, 0)$ which loses stability at $a^2 + b^2 = 1$.

Lemma 5 — The fixed point $fix(0, 0)$ of system (3.1) loses stability via a Neimark-Sacker bifurcation when $a = \pm\sqrt{1 - b^2}$, $b \neq 1$. Moreover, an attracting invariant curve exists for $a > \sqrt{1 - b^2}$ and $a < -\sqrt{1 - b^2}$.

PROOF : From the Jacobian matrix (3.2), we can see that the characteristic equation reads

$$\lambda^2 - 2a\lambda + a^2 + b^2 = 0,$$

which has two roots, $\lambda_{1,2} = a \pm ib$. The two eigenvalues are complex conjugate with modulus equal to 1 if $a^2 + b^2 = 1$. We have the following:

- $|\lambda| = 1$ if $a_0 = a = \pm\sqrt{1-b^2}$.
- $\frac{d(|\lambda|)}{da}|_{a=a_0} = \pm\sqrt{1-b^2} \neq 0$, $b \neq 1$.
- $\lambda^n(-1)|_{a=a_0} \neq 1$, $n = 1, 2, 3, 4$, $b \neq 1$.

Construct an invertible matrix

$$T = \begin{pmatrix} -b & 0 \\ 0 & b \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix},$$

system (3.1) becomes

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \pm\sqrt{1-b^2} & b \\ -b & \pm\sqrt{1-b^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},$$

where

$$\begin{aligned} f(u,v) &= a_0(b^2u^3 - b^2uv^2) + b(-b^2vu^2 - b^2v^3), \\ g(u,v) &= (-b^3u^3 + b^3uv^2) + a_0(-b^2vu^2 - b^2v^3). \end{aligned}$$

Next, we study the Neimark-Sacker bifurcation of system (3.1). The coefficients are given as follows

$$\begin{aligned} l_1 &= -Re\left[\frac{(1-2\lambda)\overline{\lambda}^2}{1-\lambda}L_{11}L_{20}\right] - \frac{1}{2}|L_{11}|^2 - |L_{02}|^2 + Re(\overline{\lambda}L_{21}), \\ L_{20} &= \frac{1}{8}[(f_{uu} - f_{vv} + 2g_{uv}) + i((g_{uu} - g_{vv} - 2f_{uv})], \\ L_{11} &= \frac{1}{4}[(f_{uu} + f_{vv}) + i(g_{uu} + g_{vv})], \\ L_{02} &= \frac{1}{8}[(f_{uu} - f_{vv} - 2g_{uv}) + i((g_{uu} - g_{vv} + 2f_{uv})], \\ L_{21} &= \frac{1}{16}[(f_{uuu} + f_{uvv} + g_{uuv} + g_{vvv}) + i(g_{uuu} + g_{uvv} - f_{uuv} - f_{vvv})]. \end{aligned}$$

Thus, we have

$$\begin{aligned} l_1 = {} & Y(EF + GH) - X(AB - CD) - \frac{1}{2}\sqrt{M^2 + N^2} - \sqrt{P^2 + Q^2} + \frac{1}{16} \\ & [-2a_0b^3u + 2b^4 + 2b^4v] < 0, \end{aligned}$$

where

$$\begin{aligned}
f_{uu} &= 6ab^2u - 2b^3v, \;\; f_{vv} = -2ab^2u - b^3u^2 - 6b^3v, \\
f_{uv} &= -2ab^2v - 2b^3uv, \;\; f_{uuu} = 6ab^2, \\
f_{vvv} &= -6b^3, \;\; f_{uvv} = -2ab^2 - 2b^3u, \\
f_{uuv} &= -2b^3v, \;\; g_{uu} = -6b^3u - 2ab^2v, \\
g_{vv} &= 2b^3u - 6ab^2v, \;\; g_{uv} = 2b^3v - 2ab^2u, \\
g_{uuu} &= -6b^3, \;\; g_{vvv} = -6ab^2, \\
g_{uvv} &= 2b^3, \;\; g_{uuv} = -2ab^2, \\
A &= ab^2 - 2b^3v - \frac{1}{4}b^3u^2, \;\; B = \frac{1}{2}ab^2u + \frac{1}{8}b^3u^2, \\
C &= -b^3u - 2ab^2v, \;\; D = ab^2v - b^3u + \frac{1}{2}b^3uv, \\
E &= -b^3u - 2ab^2v, \;\; F = \frac{1}{2}ab^2u + \frac{1}{8}b^3u^2, \\
G &= ab^2u - 2b^3v - \frac{1}{4}b^3u^2, \;\; H = ab^2v - b^3u + \frac{1}{2}b^3uv, \\
M &= ab^2u - 2b^3v - \frac{1}{4}b^3u^2, \;\; N = -b^3u - 2ab^2v.
\end{aligned}$$

From the Neimark-sacker bifurcation theorem in [26, 27], the proof is completed. □

Again, we compute the (LCEs) for system (3.1) via the Householder QR-based method to detect chaos. Figure 2 shows the LCEs for system (2.1) with the initial conditions $(x_o, y_o) = (0.1, 0.1)$. We find that LCE1 =0.7009 and LCE2 = 0.0038.

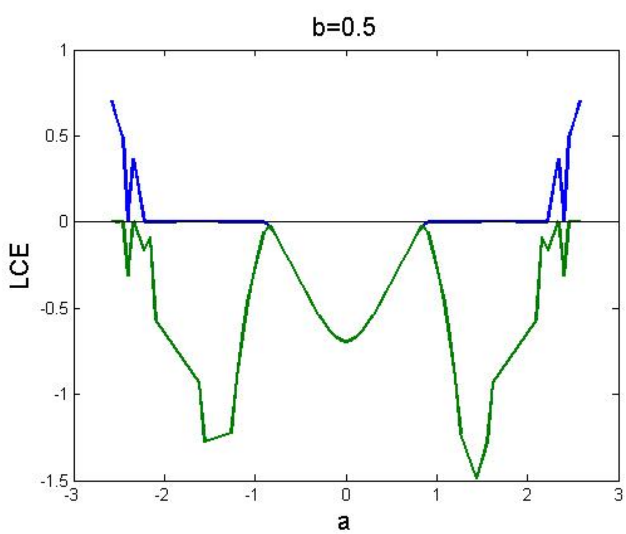

Figure 2 : LCE for system (2.1) as a function of a

4. EXISTENCE OF MARROTO'S CHAOS

In this section, we prove that the system (3.1) is chaotic in the sense of Marrotto [35].

Definition 1 — Let the function $F : R^n \to R^n$ be differentiable in $B_r(Z)$. The point $Z \in R^n$ is an expanding fixed point of F in $B_r(Z)$, if $F(Z) = Z$ and all eigenvalues of $DF(X)$ exceed 1 in norm for all $X \in B_r(Z)$.

Definition 2 — Assume that Z is an expanding fixed point of F in $B_r(Z)$ for some $r > 0$. Then Z is said to be a snapback repeller of F if there exists a point $X_0 \in B_r(Z)$ with $X_0 \neq Z, F^M(X_0) = Z$ and $DF^M(X_0) \neq 0$ for some positive integer M.

Firstly, we give the condition that $fix(0,0)$ of map (3.1) is an expanding fixed point of F. For map (3.1),

$$F(X_n) = \begin{pmatrix} a(x_n - x_n^3 - x_n y_n^2) + b(-y_n + y_n x_n^2 + y_n^3) \\ b(x_n - x_n^3 - x_n y_n^2) + a(y_n - y_n x_n^2 - y_n^3) \end{pmatrix}, \qquad X_n = (x_n \qquad y_n)^T.$$

The eigenvalues corresponding with the fixed point $fix(0,0)$ are given by

$$\lambda_{1,2} = \frac{-p(0,0) \pm \sqrt{p^2(0,0) - 4q(0,0)}}{2},$$

where

$$p(0,0) = -2a,$$

$$q(0,0) = a^2 + b^2,$$

Since the eigenvalues associated with the fixed point $fix(0,0)$ are a pair of complex eigenvalues λ and $\bar{\lambda}$, and the norm of them exceeds unity, which is equivalent to

$$\begin{cases} p^2(0,0) - 4q(0,0) < 0, \\ q(0,0) - 1 > 0. \end{cases}$$

Let

$$S_1(0,0) = p^2(0,0) - 4q(0,0) = -4b^2$$

if $b > 0$, then for

$$-4b^2 < 0,$$

we have

$$b^2 > 0.$$

Thus, $S_1(0,0) < 0$ if $(a,b) \in D_1 = \{(a,b)|b^2 > 0, b > 0\}$.

Let

$$S_2(0,0) = q(0,0) - 1 = a^2 + b^2 - 1,$$

So, we have

$$a < -ib \text{ or } a > ib.$$

Thus, $S_2(0,0) > 0$ if $(a,b) \in D_2 = \{(a,b)|a < -ib \text{ or } a > ib\}$.

In view of the above analysis, we may state the following lemma.

Lemma 6 — If $(a,b) \in D_1 \cap D_2$ and $b > 0$, then $p^2(0,0) - 4q(0,0) < 0$ and $q(0,0) - 1 > 0$. Moveover, if the fixed point $z^*(x^*, y^*)$ of map (3.1) satisfies

$$z^*(x^*, y^*) \in U_{z^*} = \{(x^*, y^*)|x \in D_1 \cap D_2,\ b > 0\}$$

then $z^*(x^*, y^*)$ is an expanding fixed point in U_{z^*}.

According to the definition of a snap-back repeller, one needs to find one point $z_1(x_1, y_1) \in U_{z^*}$ such that $z_1 \neq z^*$, $F^M(z_1) = z^*$, $|DF^M(z_1)| \neq 0$, for some positive integer M, where Map F is defined by (3.1).

To proceed, notice that

$$\begin{cases} a(x_1 - x_1^3 - x_1y_1^2) + b(-y_1 + y_1x_1^2 + y_1^3) = x_2 \\ b(x_1 - x_1^3 - x_1y_1^2) + a(y_1 - y_1x_1^2 - y_1^3) = y_2 \end{cases} \tag{4.1}$$

and

$$\begin{cases} a(x_2 - x_2^3 - x_2y_2^2) + b(-y_2 + y_2x_2^2 + y_2^3) = x^* \\ b(x_2 - x_2^3 - x_2y_2^2) + a(y_2 - y_2x_2^2 - y_2^3) = y^* \end{cases} \tag{4.2}$$

Now, a map F^2 has been constructed to map the point $z_1(x_1, y_1)$ to the fixed point $z^*(x^*, y^*)$ after two iterations if there are solutions different from z^* for Eqs. (4.1) and (4.2). The solutions different from z^* for Eq. (4.2) satisfy the following equation

$$\begin{cases} x_2 = \frac{x^* - by_2(-1+x_2^2+y_2^2)}{a(1-x_2^2-y_2^2)}, \\ y_2 = \frac{y^* - bx_2(1-x_2^2-y_2^2)}{a(1-x_2^2-y_2^2)}. \end{cases} \tag{4.3}$$

Substituting x_2 and y_2 into Eq. (4.1) and solving x_1, y_1, we have

$$\begin{cases} x_1 = \frac{x^* + by_2(1-x_2^2-y_2^2)}{a^2(1-x_2^2-y_2^2)(1-x_1^2-y_1^2)} + \frac{by_1}{a}, \\ y_1 = \frac{y^* - bx_2(1-x_2^2-y_2^2)}{a^2(1-x_2^2-y_2^2)(1-x_1^2-y_1^2)} - \frac{bx_1}{a}. \end{cases} \tag{4.4}$$

By simple calculations, we get

$$\begin{aligned}|DF^2(z_1)| &= [B + \frac{\delta}{2}\{-c\delta y - 2A^2B + B(1-A^2)\}] \\ &\times [D - \frac{c\delta^2}{2}C + \delta D(-s + cA)] \\ &- [c\delta y + c\delta BC + c\delta^2 y(-s + cA)] \\ &\times [-\frac{\delta}{2} + \frac{\delta}{2}\{-D + \delta A^2 - \frac{\delta}{2}(1-A^2)\}],\end{aligned}$$

where

$$A = x_1 + \frac{\delta}{2}(x_1(1-x_1^2) - y_1), \quad B = 1 + \frac{\delta}{2}(1 - 3x_1^2)$$
$$C = y_1 + \delta y_1(-s + cx_1), \quad D = 1 + \delta(-s + cx_1).$$

Obviously, if the condition in Lemma 4 is satisfied, the solutions of Eqs. (4.3) and (4.4) will be farther subject to $z_1(x_1, y_1)$, $z_2(x_2, y_2) \neq z^*(x^*, y^*)$, $z_1(x_1, y_1) \in U_{z^*}$ and $|DF^2(z_1)| \neq 0$, then z^* is a snap-back repeller in U_{z^*} . Thus, the following theorem is established.

***Theorem* 1** — *Assume that those conditions in Lemma 4 hold. If*

1. $\frac{\delta^2}{4}(1 - 3x^{*2})^2 - 2c\delta^2 y^* < 0$ *and* $\frac{\delta}{2}(1 - 3x^{*2}) + \frac{c\delta^2}{2}y^* > 0$,
2. *the solutions* (x_2, y_2) *and* (x_1, y_1) *of Eqs. (4.3) and (4.4) satisfy in addition* (x_1, y_1), $(x_2, y_2) \neq (x^*, y^*)$, $(x_1, y_1) \in U_{z^*}$, $(x_1, y_1) \neq (0, 0)$ *and* $|DF^2(z_1)| \neq 0$, *then* $z^*(x^*, y^*)$ *is a snap-back repeller of map (2.1), and hence map (2.1) is chaotic in the sense of Marotto.*

5. Numerical Simulations

This part is concerned with numerical simulations of Eq. (1.1) in both cases ρ being real and complex. First of all, to reveal the qualitative dynamical behaviors of system (2.1), we present a complete bifurcation sequence that is observed for different values of ρ.

As shown in Figure 3, when plotting the bifurcation diagram of x as a function of ρ, the fixed point $fix_1(0, 0)$ losses stability at $\rho = -1$ and a stable periodic solution of period 2 appears and as ρ decreases, the periodic solution of period 2 losses its stability. The numerical simulations shows that the period-doubling bifurcation may continue and go to chaos. The initial condition taken here is $(x_0, y_0) = (0.1, 0.1)$. In order to explore a larger portion of the behavior of system (2.1), we are interested in plotting the bifurcation diagram of $|z|$ as a function of ρ. Figure 4 shows that the region of stability of the fixed point $fix_1(0, 0)$ is the same as for x versus ρ. At $\rho = 1$, $fix_1(0, 0)$ losses stability and a stable periodic solution of period 2 appears, then the periodic solution of period

4 becomes unstable, and a periodic solution of period 8 appears and chaos happens. Second of all, to reveal qualitative dynamics of system (3.1), we fix the parameter $b = 0.5$, and assume that the parameter a is free. Now we give the specific values of the parameter in system (3.1). Let $b = 0.5$ and a range from 0.8 to 2.5, then system (3.1) becomes

$$\begin{aligned} x_{n+1} &= a(x_n - x_n^3 - x_n y_n^2) + 0.5(-y_n + y_n x_n^2 + y_n^3), \\ y_{n+1} &= 0.5(x_n - x_n^3 - x_n y_n^2) + a(y_n - y_n x_n^2 - y_n^3). \end{aligned}$$

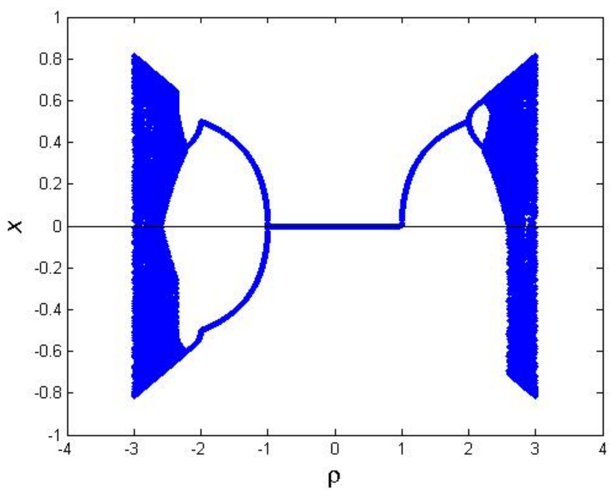

Figure 3: Bifurcation diagram of system (2.1): x vs. ρ.

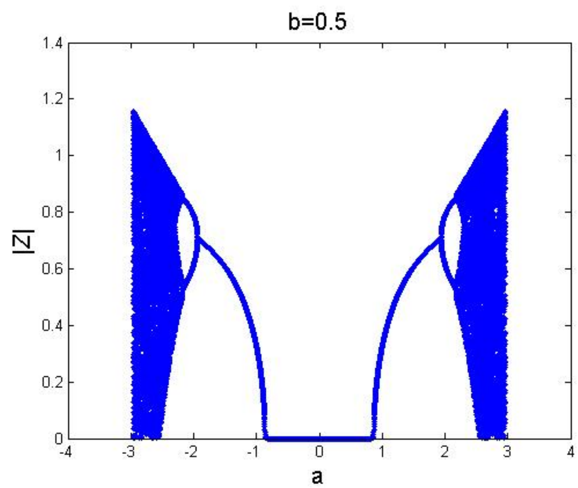

Figure 4: Bifurcation diagram of system (2.1): $|z|$ vs. ρ.

After simple calculations, one may discover that system (3.1) generates an invariant circle (quasi-period orbit) while parameter a goes through 0.9, which is the Neimark-Sacker bifurcation value of map (3.1) as shown in Figure 10. In fact, the Jacobian matrix of map (3.1) has a pair of complex conjugate eigenvalues: $\lambda_{1,2} = a \pm 0.5i$ and we can see that $\frac{d(|\lambda|)}{da}|_{0.9} \neq 0$, and $\lambda^n(-1)|_{0.9} \neq 1$, $n = 1, 2, 3, 4$. Figure 5 is the bifurcation diagram of system (3.1) showing the output of x component with respect to the parameter a. When plotting the bifurcation diagram to show $|z|$ as a function of a, one can see that the stability region of $fix(0, 0)$ has been shrunk as seen in Figure 6. The fixed point $fix(0, 0)$ of system (3.1) loses its stability at $a = 0.9$, as shown in Figure 8, on account of the norm of complex eigenvalues of its corresponding Jacobian matrix equal to 1, so there appears an attracting invariant circle when the parameter $a = 0.9$ as shown in Figure 10. System (3.1) before a Neimark-Sacker bifurcation is shown in Figure 9, while the breakdown of the invariant circle is shown in Figure 11. The phase portrait of the circle for $a = 2.3, 2.5$ is shown in Figures 12 and 13, respectively. Note that Figure 7 is just an amplification of Figure 5.

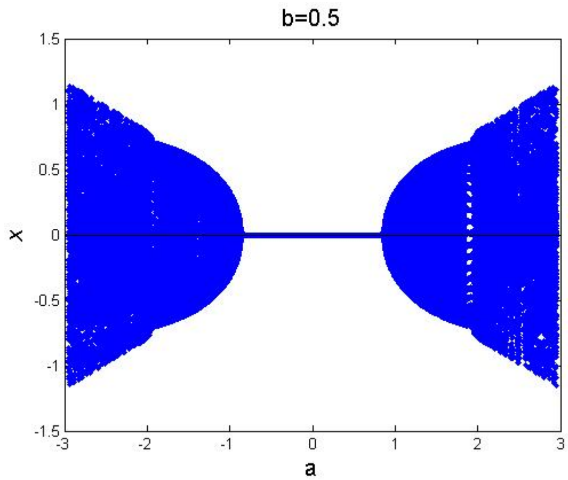

Figure 5: Bifurcation diagram of system (3.1): x vs. a.

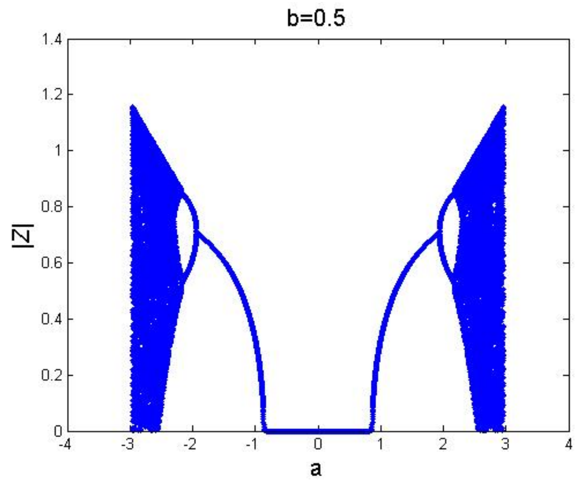

Figure 6: Bifurcation diagram of system (3.1): $|z|$ vs. a.

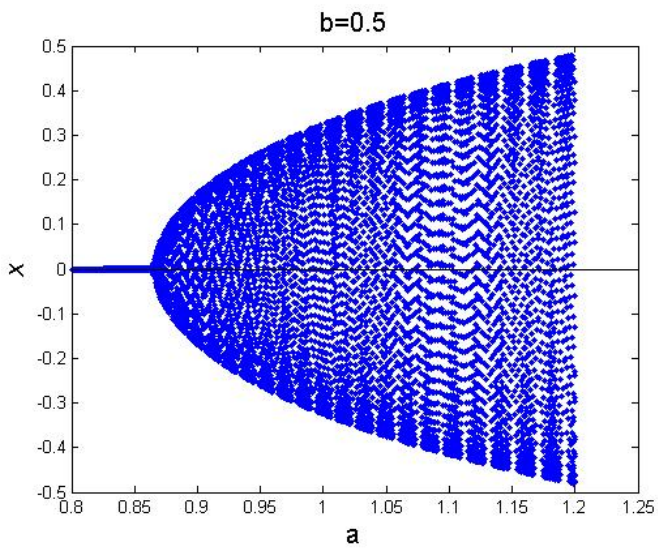

Figure 7: Bifurcation diagram of system (3.1): x vs. a.

Fractal Dimension allows us to measure the degree of complexity by evaluating how fast our measurements increase or decrease as our scale becomes larger or smaller. They were first applied as an index characterizing complicated geometric forms for which the details seemed more important than the gross picture [29]. For sets describing ordinary geometric shapes, the theoretical fractal dimension equals the set's familiar Euclidean or topological dimension. Unlike topological dimensions, the fractal index can take non-integer values indicating that a set fills its space qualitatively and quantitatively differently from how an ordinary geometrical set does. Indeed, many methods were proposed to measure the fractal dimension of a chaotic attractor, here we use Lyapunov dimension proposed by Kaplan and York [30, 31]. The fractal Lyapunov dimension of a chaotic attractor is given by [32]

$$d_L = j + \frac{\sum_{i=1}^{i=j} \Lambda_i}{|\Lambda_{j+1}|},$$

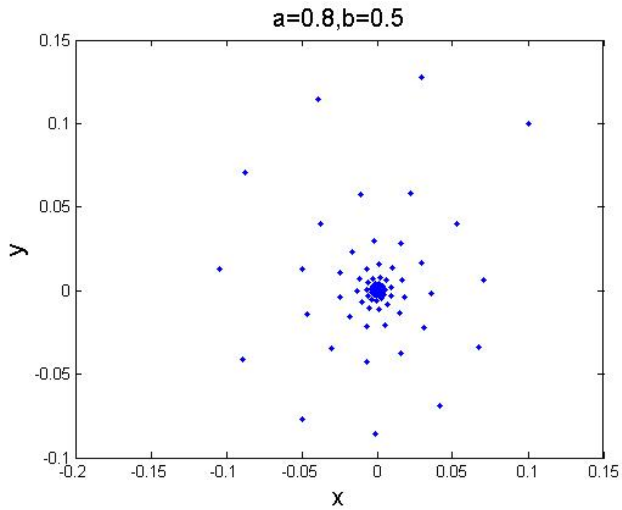

Figure 8 : A stable fixed point of system (3.1) for $a = 0.8, b = 0.5$.

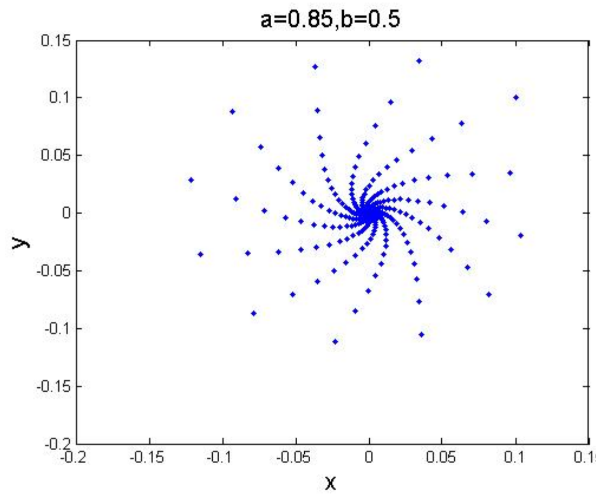

Figure 9 : Phase plane of system (3.1) before NS bifurcation for $a = 0.886, b = 0.5$.

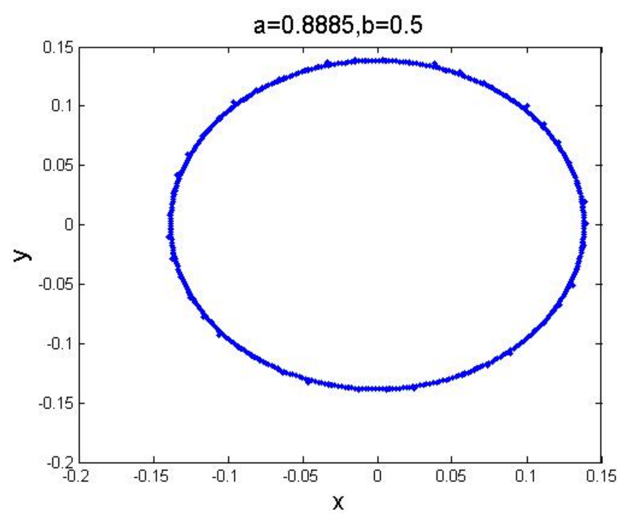

Figure 10 : An invariant circle of system (3.1) for $a = 0.9, b = 0.5$.

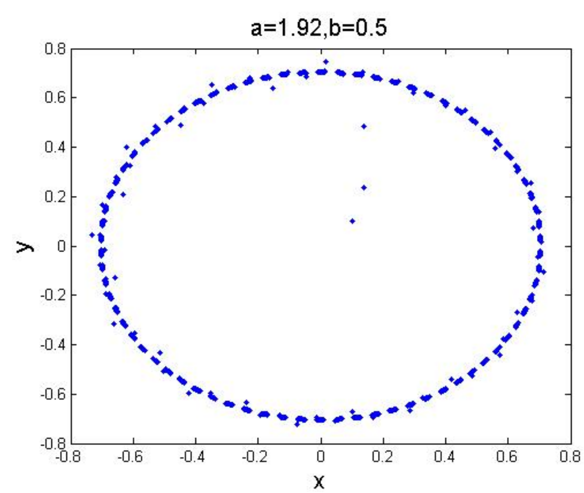

Figure 11 : The breakdown of the closed invariant circle of system (3.1) for $a = 1.98, b = 0.5$.

Roughly speaking, there are two kinds of ways to control chaos: feedback control and non-feedback control. The frame of a chaotic attractor is given by infinitely many unstable periodic orbits. The task is to use the unstable periodic orbits to control chaos. In this paper, we apply the Ott-Grebogi-Yorke method which uses this fact. Consider the unstable periodic orbit $\mathbf{X}_{p,i} \in \mathbb{R}^n$ with $i = 1, 2, ..., p$, where p is the period length. Consider the parameterized system

$$\mathbf{X}_{n+1} = \mathbf{f}(\mathbf{X}_n, \rho_n).$$

By changing ρ_n slightly, the periodic points are also shifted slightly. That is, $\mathbf{X}_{p,i}(\rho_n)$ for $i = 1, 2, ..., p$. We describe the method as applied to the stabilizing of period one orbits (i.e. fixed points) of the map (2.1).

Let $\mathbf{X}^*(\rho)$ denote an unstable fixed point on the attractor. For values of ρ close to ρ_0 and in the neighborhood of the fixed point $\mathbf{X}^*(\rho_0)$ the map can be approximated by the linear map

$$\mathbf{X}_{n+1} - \mathbf{X}^*(\rho_0) = A[\mathbf{X}_n - \mathbf{X}^*(\rho_0)] + B(\rho - \rho_0),$$

where A is an $n \times n$ jacobian matrix and B is an $n-$dimensional column vector given in the form

$$A := D_x\mathbf{f}(\mathbf{X}, \rho), \;\; B := D_\rho\mathbf{f}(\mathbf{X}, \rho).$$

We introduce the time-dependence of the parameter ρ by assuming that it is a linear function of the variable $\mathbf{X}_n$ of the form

$$\rho - \rho_0 = -K^T(\mathbf{X}_n - \mathbf{X}^*(\rho_0)).$$

The $1 \times n$ matrix K^T is to be determined so that the fixed point $\mathbf{X}^*(\rho_0)$ becomes stable.

We obtain

$$\mathbf{X}_{n+1} - \mathbf{X}^*(\rho_0) = (A - BK^T)(\mathbf{X}_n - \mathbf{X}^*(\rho_0)).$$

Which shows that the fixed point will be stable provided that the $n \times n$ matrix $A - BK^T$ is a asymptotically stable, i.e., all its eigenvalues have modulus less than one.

In this part we will control the chaotic behavior of system (2.1), that is ρ is real. Now we fix the parameter $\rho = 1.45$ such that the system (2.1) is chaotic. We take ρ as the control parameter which is restricted to lie in a small interval $|\rho - \rho_0| < \delta$, $\delta > 0$, around the nominal value $\rho_0 = 1.45$. The system (2.1) becomes

$$\begin{aligned} f : x_{n+1} =& 2.4(x_n - x_n^3 - x_n y_n^2), \\ g : y_{n+1} =& 2.4(y_n - y_n x_n^2 - y_n^3). \end{aligned} \tag{6.1}$$

That is

$$\begin{bmatrix} x_{n+1}-0 \\ y_{n+1}-0.7637 \end{bmatrix} \cong \begin{bmatrix} 1.0002 & 0 \\ -0.3183k_1 & -1.7993-0.3183k_2 \end{bmatrix} \begin{bmatrix} x_n-0 \\ y_n-0.7637 \end{bmatrix}.$$

The eigenvalues of the matrix $A-BK^T$ can be obtained by solving the characteristic equation

$$|A-BK^T-\gamma I|=0,$$

which gives $\gamma_1=-1.7993-0.3183k_2$ and $\gamma_2=1.0002$. Generally speaking, we must choose values for k_1 and k_2 such that $\gamma_1\gamma_2<1$, $\gamma_1<1$ and $\gamma_1>-1$. So, we have $-8.7930<k_2<-2.5112$. Figure 15 shows the controled fixed point of system (6.1) for $k_1=4$.

In the same manner, we can control chaos in case ρ is complex which is ommitted here for no repetition.

7. CONCLUSION

In this paper, we have considered a discrete logistic-type equation with complex variables in two different cases of the control parameter ρ in the defining equation: real and complex. Our theoretical analysis and numerical simulations have demonstrated that the map undergoes pitchfork bifurcation, flip bifurcation, Neimark-Sacker bifurcation and chaos. Moreover, we have proven that the map in the real case is chaotic in the sense of Marotto. In addition, it has been shown that the map with complex parameter ρ demonstrated a fractal structure. Finally, we have applied the OGY feedback control technique to control chaos in case ρ is real. It has been investigated that considering the control parameter ρ to be complex led to more richer dynamics.

REFERENCES

1. Sanju and V. S. Varmna, Quadratic map modulated by additive periodic forcing, *Phys. Rev. E*, **48** (1993), 1670-1675.

2. E. A. Jackson and A. Hiiblcr, Periodic entrainment of chaotic logistic map dynamics, *Physica D*, **44** (1990), 404-420.

3. S. J. Linz and M. Lticke, Effect of additive arid multiplicative noise on the first bifurcations of thelogistic map, *Phys. Rev. A*, **33** (1986), 2694-2703.

4. Y. Yamaguchi and K. Sakai, New type of crisis showing hysteresis, *Phys. Rev. A*, **27** (1983), 2755-2758.

5. A. C. Fowler, M. J. McGuinness, and J. D. Gibbon, The complex Lorenz equations, *Physica D*, **4**(2) (1982), 139-163.

6. G. M. Mahmoud, E. E. Mahmoud, and M. E. Ahmed, A hyperchaotic complex Chen system and its dynamics, *International Journal of Applied Mathematics & Statistics*, **12**(D07) (2007), 90-100.

7. G. M. Mahmoud, Periodic solutions of strongly non-linear Mathieu oscillators, *International Journal of Non-Linear Mechanics*, **32**(6) (1997), 1177-1185.

8. G. M. Mahmoud and S. A. Aly, Periodic attractors of complex damped non-linear systems, *International Journal of Non-Linear Mechanics*, **35** (2000), 309-323.

9. G. M. Mahmoud, Stability regions for coupled Hill's equations, *Physica A*, **242** (1997), 139-149.

10. G. M. Mahmoud, On the generalized averaging method of a class of strongly nonlinear forced oscillators, *Physica A*, **199** (1993), 87-95.

11. R. A. Holmgren, *A first course in discrete dynamical systems*, Second Edition, Springer-Vcrlag, (1996).

12. G. M. Mahmoud, Bountis Tassos, G. M. Abd El-Latif, and E. E. Mahmoud, Chaos synchronization of two different chaotic complex Chen and Lü systems, *Nonlinear Dyn*., **55** (2009), 43-53.

13. C. Z. Ning and H. Haken, Detuned lasers and the complex Lorenz equations: Subcritical and supercritical Hopf bifurcations, *Phys. Rev. A*, **41** (1990), 3826-3837.

14. Yong Xu, Wei Xu, and G. M. Mahmoud, On a complex duffing system with random excitation, *Chaos, Solitons and Fractals*, **35** (2008), 126-135.

15. A. A. ElSadany and A. M. A. El-Sayed, On a complex logistic difference equation, *International Journal of Modern Mathematical Sciences*, **4**(1) (2012), 37-47.

16. A. M. A. El-Sayed and S. M. Salman, Dynamic behavior and chaos control in a complex riccati-type map, *Quaestiones Mathematicae* (2015), DOI: 10.2989/16073606.2015.1115441.

17. R. P. Agarwal, *Difference equations and inequalities*, New York, NY, USA, second edition, 2000.

18. S. Elaydi, *An introduction to difference equations*, third edition, Springer Verlag, New York, 2005.

19. S. Elaydi, Is the world evolving discretely?, *Adv. in Appl. Math*., **31**(1)(2003), 19.

20. R. Holmgren, *A first course in discrete dynamical systems*, Second Edition, Springer Verlag, New York, 1996.

21. W. G. Kelley and A. C. Peterson, *Difference equations, an introduction with applications*, Second Edition, Academic Press, New York, 2000.

22. V. L. Kocic and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dord recht, 1993.

23. H. Sedaghat, *Nonlinear difference equations*, in Theory with applications to social science models, Kluwer Academic, Dordrecht, 2003.

24. J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer-Verlag. New York, 1997.

25. E. Ott, *Chaos in dynamical systems*, Cambridge University Press, Cambridge, 1993.

26. Y. A. Kuznetsov, *Elements of applied bifurcation theory, applied mathematical sciences*, V. 112. Springer, New York, third edition, 2004.

27. S. Wiggins, *An introduction to applied nonlinear dynamics and chaos*, New York: Springer-Verlag, 1990.

28. F. E. Udwadia and H. von Bremen, A note on the computation of the largest p-Lyapunov characteristic exponents for nonlinear dynamical systems, *J. Appl. Math. Comput.*, **114** (2000), 205-214.

29. Albers; Alexanderson, *Benoit Mandelbrot: In his own words, Mathematical people : Profiles and interviews*, (2008). Wellesley, Mass: AK Peters, p. 214. ISBN 978-1-56881-340-0.

30. V. I. Oseledec, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.*, **19** (1908), 197-231.

31. J. L. Kaplan and Y. A. York, A regime observed in a fluid flow model of Lorenz, *Comm. Math. Phys.*, **67** (1979), 93-108.

32. J. H. E. Cartwright, Nonlinear stiffness, Lyapunov exponents, and attractor dimension, *Phys. Lett. A*, **264** (1999), 298-304.

33. G. Chen and X. Dong, *From chaos to order: Methodologies, perspectives and applications, world scientific*, (1998).

34. H. G. Schstcr, *Handbook of chaos control*, Wiley-VCH, (1998).

35. F. R. Marotto, On redefining a snap-back repeller, *Chaos Solitons Fractals*, **25** (2005), 25-28.