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ON THE GENUS OF THE k-ANNIHILATING-IDEAL HYPERGRAPH OF COMMUTATIVE RINGS

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Let R be a commutative ring and k an integer greater than 2 and let $\mathcal{A}(R, k)$ be the set of all k-annihilating-ideals of R. The *k-annihilating-ideal hypergraph* of R, denoted by $\mathcal{AG}_k(R)$, is a hypergraph with vertex set $\mathcal{A}(R, k)$, and for distinct elements $I_1, I_2, \ldots, I_k$ in $\mathcal{A}(R, k)$, the set $\{I_1, I_2, \ldots, I_k\}$ is an edge of $\mathcal{AG}_k(R)$ if and only if $\prod_{i=1}^{k} I_i = (0)$ and the product of any $(k-1)$ elements of the set $\{I_1, I_2, \ldots, I_k\}$ is nonzero. In this paper, we characterize all Artinian commutative nonlocal rings R whose $\mathcal{AG}_3(R)$ has genus one.

Key words : k-zero-divisor hypergraph; k-annihilating-ideal hypergraph; incidence graph; toroidal graph.

1. Introduction

The study linking commutative ring theory with graph theory has been started with the concept of the zero-divisor graph of a commutative ring. Let R be a commutative ring and $Z(R)^*$ be the set of all nonzero zero-divisors of R. The *zero-divisor graph* of R, denoted $\Gamma(R)$, is the simple graph with $Z(R)^*$ as the vertex set and two distinct vertices x and y are joined by an edge if and only if $xy = 0$. This definition was introduced by Beck, Anderson and Livingston in [5, 6, 13] and later was studied extensively in [3, 5, 10-12, 16, 29]. In view of this, Eslahchi and Rahimi [19] have introduced and investigated a graph called the *k-zero-divisor hypergraph* of a commutative ring. Recently Behboodi and Rakeei [14, 15] have introduced and investigated the annihilating-ideal hypergraph of a commutative ring. In the literature, there are many papers assigning graphs to ideals of rings, see

[1, 2, 14, 15, 28, 30, 32]. From this view, Selvakumar *et al*. [26] introduced a hypergraph called k-*annihilating-ideal hypergraph* of a commutative ring R. For a commutative ring R and $k \geq 2$ a fixed integer, a nonzero proper ideal I_1 in R is said to be a k-*annihilating-ideal* in R if there exist $(k-1)$ distinct nonzero proper ideals $I_2, I_3, \ldots, I_k$ in R different from I_1 such that $\prod_{i=1}^{k} I_i = (0)$ and the product of any $(k-1)$ elements of $\{I_1, I_2, \ldots, I_k\}$ is nonzero. By $\mathcal{A}(R,k)$ we denote the set of all k-*annihilating-ideals* of R. The k-*annihilating-ideal hypergraph* $\mathcal{AG}_k(R)$ of R is defined as the hypergraph with the vertex set $\mathcal{A}(R,k)$, and for distinct elements $I_1, I_2, \ldots, I_k$ in $\mathcal{A}(R,k)$, the set $\{I_1, I_2, \ldots, I_k\}$ is an *hyperedge* of $\mathcal{AG}_3(R)$ if and only if $\prod_{i=1}^{k} I_i = 0$ and the product of any $(k-1)$ elements of $\{I_1, I_2, \ldots, I_k\}$ is nonzero. Note that the graph constructed by 2-annihilating-ideals is exactly the same as the annihilating ideal graph of a ring. In [26], it is shown that for any Artinian reduced nonlocal ring R with at least three maximal ideals, $\mathcal{AG}_k(R) \cong \mathcal{H}_k(\mathbb{Z}_2^n)$ for $n \geq 3$ and $3 \leq k \leq n$.

Throughout this paper, we assume that R is an Artinian commutative nonlocal ring, $\mathcal{A}(R,k)$, its k-annihilating-ideals for $k \geq 3$. We denote the ring of integers modulo n by $\mathbb{Z}_n$, the nilpotency of the ideal by η and the field with q elements by $\mathbb{F}_q$. For basic definitions on rings, one may refer [8, 22].

2. Preliminaries

In this section, we summarize notation, concepts and results related to the planarity of a graph and hypergraph which will be needed in the subsequent sections.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph G is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained from it by replacing edges with pairwise internally-disjoint paths.

By a surface we mean a compact connected topological space such that each point has a neighborhood homeomorphic to an open disc in $\mathbb{R}^2$. We denote by S_n the surface obtained from the sphere S_0 by adding n handles. It is known that every orientable surface is homeomorphic to precisely one of the surfaces S_n ($n \geq 0$). The number n is called the genus of the surface S_n. The purpose of this paper is to study the question of embeddings of the $k-$*zero-divisor hypergraphs* in torus S_1. An embedding of a graph G into some topological space S is a homeomorphism between the geometric realization of G and a subspace of S. One may think of an embedding of a graph G into S as a drawing of G on S with no edge crossings. An embedding of G into S is *cellular* if each component of $S - G$ (i.e. each face) is homeomorphic to an open disc in $\mathbb{R}^2$. An embedding in which all faces have boundary consisting of exactly three edges is called a triangulation. The genus of a graph G is minimum n such that G can be embedded in S_n and embeddings of G in S_n are called minimum genus embeddings. Note that every minimum genus embedding of G is cellular. This is one of the reasons why in the remaining of this paper when we say that a graph is embedded in a surface, we will assume that it is cellularly embedded. Note that graphs of genus 0 are *planar* graphs and graphs of genus 1 are *toroidal* graphs. One of the most remarkable theorems in topological graph theory, known as Euler's formula, states that if G is a finite connected graph with n vertices, e edges and of genus g, then $n - e + f = 2 - 2g$, where f is the number of faces obtained when G is cellularly embedded in S_g.

Euler's formula can be used in combination with some combinatorial identities and other inequalities to show the nonexistence of certain embeddings. Further note that if H is a subgraph of a graph G, then $g(H) \leq g(G)$. For details on the notion of embedding a graph in a surface, see [7, 16, 20, 21].

A *hypergraph* $\mathcal{H}$ consists of a vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H})$ where each edge is an (unordered) set of vertices. A hypergraph may be used to model relationships with more flexibility than graphs. An edge in a graph joins exactly two elements of the vertex set, but an edge in a hypergraph joins any number of vertices. The hypergraph $\mathcal{H}$ is called a k-uniform whenever every edge e of $\mathcal{H}$ is of size k. The number of edges containing a vertex $v \in V(\mathcal{H})$ is its degree $d_{\mathcal{H}}(v)$. For basic definitions on hypergraphs, one may refer [17]. The *incidence graph* $\mathcal{I}(\mathcal{H})$ of $\mathcal{H}$ is a bipartite graph with vertex $V(\mathcal{H}) \cup E(\mathcal{H})$ and a vertex $v \in V(\mathcal{H})$ is adjacent to a vertex $u \in E(\mathcal{H})$ if the hypervetex v is incident with the hyperedge u in $\mathcal{H}$. The genus $g(\mathcal{H})$, of a hypergraph $\mathcal{H}$ is the genus of its incidence graph; i.e. $g(\mathcal{H}) = g(\mathcal{I}(\mathcal{H}))$ (cf. [33]). For basic definitions on graphs and hypergraphs, one may refer [17, 18].

The following are useful in the sequel of this paper and hence given below:

Proposition 2.1 — [27, Proposition 2.7]. If $(R, \mathfrak{m})$ is a local ring and there is an ideal I of R such that $I \neq \mathfrak{m}^i$ for every i, then R has at least three distinct non-trivial ideals I, J and K such that $I, J, K \neq \mathfrak{m}^i$ for every i.

***Theorem* 2.2** — *[18, Kuratowski]. A graph G is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.*

***Theorem* 2.3** — *[33, Corollary 2]. A hypergraph is planar if and only if its incidence graph is planar.*

Lemma 2.4 — [34]. $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ if $n \geq 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.

Lemma 2.5 — [34]. $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ if $m, n \geq 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,4}) = 2$ if $m = 7, 8, 9, 10$.

Lemma 2.6 — [34]. If G is a graph with n vertices, m edges, girth gr and genus g, then

$$\frac{m(gr-2)}{2gr} - \frac{n}{2} + 1 \leq g.$$

Lemma 2.7 — [34, Euler formula]. If G is a finite connected graph with n vertices, m edges, and genus g, then $n - m + f = 2 - 2g$, where f is the number of faces created when G is minimally embedded on a surface of genus g.

***Theorem* 2.8** — *[33, Corollary 1]. For any hypergraph $\mathcal{H}$, $g(\mathcal{H}) = g(\mathcal{I}(\mathcal{H}))$*

3. Genus of k-annihilating-ideal Hypergraphs

In this section, we focus on genus of 3-annihilating-ideal hypergraphs. We characterize all finite commutative non-local rings R with identity with respect to the nilpotency of ideals in R whose $\mathcal{AG}_3(R)$ has genus one. Using the Euler characteristic formula and a technique of deletion and insertion, we are able to successfully exclude some cases of higher genus. As mentioned earlier, Selvakumar *et al*. [26] determined all finite commutative non-local rings R for which $\mathcal{AG}_3(R)$ is planar.

For $n \geq 2$ a fixed integer, we recall from [4] and [9] that a proper ideal I of R is called a *strongly n-absorbing* if $I_1 I_2 \cdots I_n I_{n+1} \subseteq I$ for ideals $I_1, I_2, \ldots, I_{n+1}$ of R, then there are n of the I_i^s whose product is in I. In view of [4, 9], we first start with some remark.

Remark 3.1 : Let R be a ring. Then $\mathcal{AG}_k(R)$ is completely disconnected (i.e., no edges) if and only if $\{0\}$ is a strongly $(k-1)$-absorbing ideal of R. In particular, if $R = F_1 \times F_2$, where F_1 and F_2 are some fields, then $\mathcal{AG}_3(R)$ is completely disconnected. The proof is clear by definition and note that $\{0\}$ is a strongly 2-absorbing ideal of R.

Example 3.2 : Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then $\mathcal{A}(R, 3) = \{(0)\times\mathbb{Z}_4, \mathbb{Z}_4\times(0), (2)\times(2), (2)\times\mathbb{Z}_4, \mathbb{Z}_4\times(2)\}$. The incidence graph of $\mathcal{AG}_3(\mathbb{Z}_4 \times \mathbb{Z}_4)$ is shown in Figure 1.

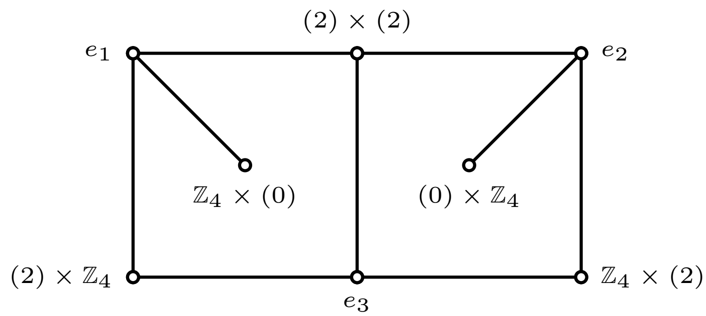

Figure 1: $\mathcal{I}(\mathcal{AG}_3(\mathbb{Z}_4\times\mathbb{Z}_4))$

Recall the following results regarding the planarity of k-annihilating-ideal hypergraph of commutative rings which are essential to examine the genus of k-annihilating-ideal hypergraph.

***Theorem* 3.3** — *[26]. Let $R = F_1\times\cdots\times F_n$, where each F_i is a field and $3\leq k\leq n$. Then the following are true.*

(i) For any k, $3<k<n$, $\mathcal{AG}_k(R)$ is non-planar.

(ii) $\mathcal{AG}_3(R)$ is planar if and only if $R = F_1\times F_2\times F_3$.

(iii) For $n\geq 4$, $\mathcal{AG}_n(R)$ is planar if and only if $R = F_1\times\cdots\times F_n$.

***Theorem* 3.4** — *[26]. Let $R = R_1\times\cdots\times R_n$ be a nonlocal ring, where each $(R_i,\mathfrak{m}_\mathfrak{i})$ is a local ring but not a field. Then $\mathcal{AG}_3(R)$ is planar if and only if $n=2$ and both rings in the decomposition have exactly one nonzero proper ideal.*

***Theorem* 3.5** — *[26]. Let $R = R_1\times\cdots\times R_n\times F_1\times\cdots\times F_n$, $(m,n\geq 1)$ be a finite ring, where each $(R_i,\mathfrak{m}_i)$ is a local ring but not a field and each F_j is a field. Then $\mathcal{AG}_3(R)$ is planar if and only if R is isomorphic to one of the following.*

(i) $R_1\times F_1\times F_2$, where R_1 has exactly only one nonzero ideal.

(ii) $R_1\times F_1$, with the following properties:

(a) $\mathfrak{m}_1^4=(0)$ such that $\mathfrak{m}_1$, $\mathfrak{m}_1^2$ and $\mathfrak{m}_1^3$ are only the nonzero proper ideals of R_1.

(b) $\mathfrak{m}_1^3=(0)$ such that $\mathfrak{m}_1$ and $\mathfrak{m}_1^2$ are only the nonzero proper ideals of R_1.

***Theorem* 3.6** — *Let $R = F_1\times\cdots\times F_n$ be a ring with identity, where each F_i is a field and $n\geq 3$. Then $\mathcal{AG}_3(R)$ is planar or $g(\mathcal{AG}_3(R))\geq 2$.*

PROOF : If $n=3$, then by Theorem 3.3(ii), $\mathcal{AG}_3(R)$ is planar. Suppose that $n\geq 4$. Let G be a subhypergraph of $\mathcal{AG}_3(R)$ induced by $\Omega_1=\{x_1,\ldots,x_{10}\}\subseteq\mathcal{A}(R,3)$, where $x_1=(0)\times(0)\times F_3\times F_4\times(0)\times\cdots\times(0)$, $x_2=F_1\times(0)\times F_3\times(0)\times\cdots\times(0)$, $x_3=F_1\times(0)\times(0)\times F_4\times(0)\times\cdots\times(0)$,

$x_4 = F_1 \times (0) \times F_3 \times F_4 \times (0) \times \cdots \times (0)$, $x_5 = (0) \times F_2 \times F_3 \times (0) \times \cdots \times (0)$, $x_6 = (0) \times F_2 \times (0) \times F_4 \times (0) \times \cdots \times (0)$, $x_7 = (0) \times F_2 \times F_3 \times F_4 \times (0) \times \cdots \times (0)$, $x_8 = F_1 \times F_2 \times (0) \times \cdots \times (0)$, $x_9 = F_1 \times F_2 \times F_3 \times (0) \times \cdots \times (0)$, $x_{10} = F_1 \times F_2 \times (0) \times F_4 \times (0) \times \cdots \times (0)$. Then $\mathcal{I}(G)$ is a subgraph of $\mathcal{I}(\mathcal{AG}_3(R))$, $|V(\mathcal{I}(G))| \geq 32$, $|E(\mathcal{I}(G))| \geq 66$ and $gr(\mathcal{I}(G)) = 4$. By Lemma 2.6, $g(\mathcal{I}(G)) > 1$ and so $g(\mathcal{I}(\mathcal{AG}_3(R))) > 1$, a contradiction. □

Next, we study two lemmas which are needed in the following theorem. We apply insertion and deletion argument for investigating the 3-annihilating-ideal hypergraph $\mathcal{AG}_3(R)$ is either planar or has genus at least two in the following lemma.

Lemma 3.7 — Let $R = R_1 \times R_2$ be a ring, where $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ are local rings such that $\eta(\mathfrak{m}_1) = 2$ and $\eta(\mathfrak{m}_2) = 3$. Then $g(\mathcal{AG}_3(R)) \geq 2$.

PROOF : Let $\mathcal{A}(R, 3) = \{x_1, \ldots, x_8\}$ and $E(\mathcal{AG}_3(R)) = \{e_1, \ldots, e_{10}\}$, where $x_1 = R_1 \times (0)$, $x_2 = (0) \times \mathfrak{m}_2$, $x_3 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $x_4 = R_1 \times \mathfrak{m}_2$, $x_5 = \mathfrak{m}_1 \times \mathfrak{m}_2^2$, $x_6 = R_1 \times \mathfrak{m}_2^2$, $x_7 = (0) \times R_2$, $x_8 = \mathfrak{m}_1 \times R_2 \in \mathcal{A}(R, 3)$, $e_1 = \{x_1, x_3, x_8\}$, $e_2 = \{x_1, x_5, x_8\}$, $e_3 = \{x_2, x_3, x_4\}$, $e_4 = \{x_3, x_6, x_8\}$, $e_5 = \{x_4, x_6, x_7\}$, $e_6 = \{x_4, x_5, x_8\}$, $e_7 = \{x_4, x_5, x_7\}$, $e_8 = \{x_5, x_6, x_8\}$, $e_9 = \{x_5, x_6, x_7\}$, $e_{10} = \{x_3, x_6, x_7\} \in E(\mathcal{AG}_3(R))$. It is observed that $\mathcal{I}(\mathcal{AG}_3(R))$ contains a subdivision of $K_{3,3}$ and so that $g(\mathcal{AG}_3(R)) \geq 1$.

Suppose that $g(\mathcal{AG}_3(R)) = 1$. Then by Theorem 2.8, $g(\mathcal{I}(\mathcal{AG}_3(R))) = 1$. Let $G = \mathcal{I}(\mathcal{AG}_3(R))$, $G' = G - \{x_1, x_2, e_1, e_2, e_3\} - \{e_{10}x_6, e_4x_6, e_5x_7\}$ and $G'' = G' - \{e_9\}$. Then G'' is isomorphic to the subdivision of $K_{3,3}$ and hence $g(G'') = 1$. Since $g(G'') \leq g(G') \leq g(G)$, we obtain that $g(G') = 1$. Note that $|V(G')| = 13$ and $|E(G')| = 18$. Then By Euler's formula 2.7, there are 9 faces when drawing G' on a torus. Fix a representation for G' and let $\{F'_1, \ldots, F'_5\}$ be the faces of G' corresponding to this representation. Note that G'' is isomorphic to the subdivision of $K_{3,3}$. So that the cellular embedding of G'' in S_1 has 3 faces. Let F''_1, F''_2, F''_3 be the faces of G'' obtained by deleting e_9 and all the edges incident with e_9 from the representation of G'. Then $\{F'_1, \ldots, F'_5\}$ can be recovered by inserting e_9 and all the edges incident with e_9 into the representation corresponding to $\{F''_1, F''_2, F''_3\}$. Note that $x_6e_8, x_6e_9 \in E(G')$. Hence x_6, e_8, e_9 should be inserted to the same face, say F''_m of G'' to avoid crossings. Note that $e_8x_i \in E(G')$ for $i = 5, 8$ and $e_9x_i \in E(G')$ for $i = 5, 7$. So that x_5, x_7 and x_8 should be boundary vertices of the face F''_m. Since $x_4e_6, x_4e_7 \in E(G')$. If x_4, e_6, e_7 lie on different faces of cellular embedding of G_1 in S_1, we get an edge crossing. So that x_4, e_6, e_7 should be inserted to the same face F''_m of G'' to avoid crossings. We observe that $e_7x_7, e_6x_8 \in E(G')$. If e_6 and e_7 are not on the same face F''_m, we get an edge crossing. Hence that x_4, e_6, e_7 should be inserted to the same face F''_m of G''.

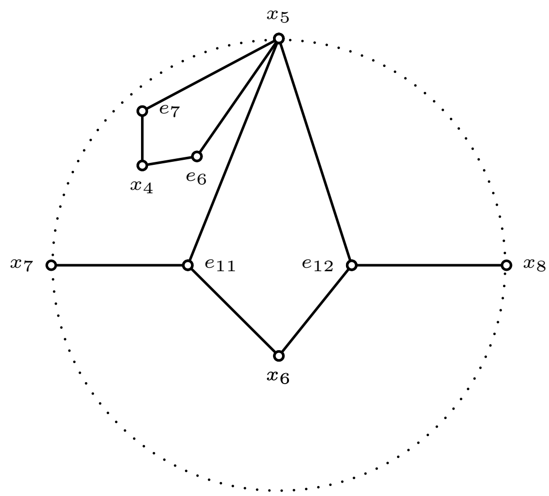

Figure 2: The face F''_m

Consider the following edges of G', $f_1 = e_9x_5,\ f_2 = e_8x_5,\ f_3 = e_9x_6,\ f_4 = e_8x_6,\ f_5 = e_9x_7,\ f_6 = e_8x_8,\ f_7 = e_7x_5,\ f_8 = e_6x_5,\ f_9 = e_7x_4,\ f_{10} = e_6x_4,\ f_{11} = e_7x_7,\ f_{12} = e_6x_8$. Then we obtain the Figure 2 by inserting $x_4, x_6, e_6, e_7, e_8, e_9$ and $f_1, \ldots, f_{10}$. However from the Figure 2, there is no way to insert f_{11} and f_{12} without crossings. Hence we conclude that $g(\mathcal{AG}_3(R)) \geq 2$. □

In the following lemma, we investigate that $\mathcal{AG}_3(R)$ is planar or has genus at least two for non-local commutative ring R having exactly two maximal ideals whose nilpotency is two.

Lemma 3.8 — Let $R = R_1 \times R_2$ be a ring, where $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ are local rings such that $\eta(\mathfrak{m}_1) = 2 = \eta(\mathfrak{m}_2)$. If R_1 or R_2 has nonzero ideals different from their maximal ideals, then $g(\mathcal{AG}_3(R)) \geq 2$.

PROOF : Without loss of generality, assume that R_1 has a nonzero ideal different from $\mathfrak{m}_1^i$ for $1 \leq i \leq 2$. Then by Proposition 2.1, R_1 has at least three distinct non-trivial ideals I, J and K such that $I, J, K \neq \mathfrak{m}_1^i$ for $1 \leq i \leq 2$. Let $\mathcal{A}(R, 3) = \{x_1, \ldots, x_{11}\}$ and $E(\mathcal{AG}_3(R)) = \{e_1, \ldots, e_{26}\}$, where $x_1 = R_1 \times (0)$, $x_2 = \mathfrak{m}_1 \times \mathfrak{m}_2$, $x_3 = I \times \mathfrak{m}_2$, $x_4 = J \times \mathfrak{m}_2$, $x_5 = K \times \mathfrak{m}_2$, $x_6 = R_1 \times \mathfrak{m}_2$, $x_7 = (0) \times R_2$, $x_8 = \mathfrak{m}_1 \times R_2$, $x_9 = I \times R_2$, $x_{10} = J \times R_2$, $x_{11} = K \times R_2 \in \mathcal{A}(R, 3)$ and $e_1 = \{x_1, x_9, x_{10}\}$, $e_2 = \{x_1, x_3, x_{10}\}$, $e_3 = \{x_1, x_4, x_9\}$, $e_4 = \{x_1, x_9, x_{11}\}$, $e_5 = \{x_1, x_3, x_{11}\}$, $e_6 = \{x_1, x_5, x_9\}$, $e_7 = \{x_1, x_8, x_9\}$, $e_8 = \{x_1, x_3, x_8\}$, $e_9 = \{x_1, x_2, x_9\}$, $e_{10} = \{x_1, x_{10}, x_{11}\}$, $e_{11} = \{x_1, x_4, x_{11}\}$, $e_{12} = \{x_1, x_5, x_{10}\}$, $e_{13} = \{x_1, x_8, x_{10}\}$, $e_{14} = \{x_1, x_4, x_8\}$, $e_{15} = \{x_1, x_2, x_{10}\}$, $e_{16} = \{x_1, x_8, x_{11}\}$, $e_{17} = \{x_1, x_5, x_8\}$, $e_{18} = \{x_1, x_2, x_{11}\}$, $e_{19} = \{x_1, x_3, x_9\}$, $e_{20} = \{x_1, x_4, x_{10}\}$, $e_{21} = \{x_1, x_5, x_{11}\}$, $e_{22} = \{x_1, x_2, x_8\}$, $e_{23} = \{x_3, x_6, x_7\}$, $e_{24} = \{x_4, x_6, x_7\}$, $e_{25} = \{x_5, x_6, x_7\}$, $e_{26} = \{x_2, x_6, x_7\} \in E(\mathcal{AG}_3(R))$. Let $G = \mathcal{I}(\mathcal{AG}_3(R))$, $G' = G - \{x_1\}$. It is observed that $deg(x_1) = 22$ in $\mathcal{I}(\mathcal{AG}_3(R))$. Let $\Omega = \{x_3, x_4, x_5, x_6, x_7, x_9, x_{11}, e_1, e_3, e_5, e_{20}, e_{21}, e_{23}, e_{25}\}$. Then the induced subgraph of G' induced by Ω contains a subdivision of $K_{3,3}$ and hence $g(G') \geq 1$. Note that $|V(G')| = 36$ and $|E(G')| = 56$. Then by Euler's formula 2.7, G' has 20 faces in which there is no face of length greater than or equal to 22. Hence there is no way to insert x_1 in G' without crossings. Thus $g(\mathcal{AG}_3(R)) \geq 2$. $\square$

Next, we study that $\mathcal{AG}_3(R)$ is planar or has genus at least two for the nonlocal ring R which is the finite product of local rings.

***Theorem* 3.9** — *Let $R = R_1 \times \cdots \times R_n$, $(n \geq 2)$, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq (0)$. Then $\mathcal{AG}_3(R)$ is either planar or has genus at least two.*

PROOF : Suppose that $n \geq 3$. Since each R_i is a local ring, there exists an ideal I_i in R_i such that $I_i^2 = (0)$ for all i. Let G be a subhypergraph of $\mathcal{AG}_3(R)$ induced by $\Omega = \{x_1, \ldots, x_8\}$, where $x_1 = (0)\times(0)\times R_3\times(0)\times\cdots\times(0)$, $x_2 = R_1\times R_2\times I_3\times(0)\times\cdots\times(0)$, $x_3 = R_1\times(0)\times I_3\times(0)\times\cdots\times(0)$, $x_4 = R_1\times I_2\times I_3\times(0)\times\cdots\times(0)$, $x_5 = (0)\times R_2\times I_3\times(0)\times\cdots\times(0)$, $x_6 = (0)\times I_2\times I_3\times(0)\times\cdots\times(0)$, $x_7 = R_1\times R_2\times I_3\times(0)\times\cdots\times(0)$ and $x_8 = I_1\times R_2\times I_3\times(0)\times\cdots\times(0) \in \mathcal{A}(R,3)$. So that $\mathcal{I}(G)$ is a subgraph of $\mathcal{I}(\mathcal{AG}_3(R))$, $|V(\mathcal{I}(G))| \geq 25$, $|E(\mathcal{I}(G))| \geq 51$ and $gr(\mathcal{I}(G)) = 4$. Then by Lemma 2.6, $g(\mathcal{I}(G)) > 1$ and so $g(\mathcal{AG}_3(R)) \geq 2$. It follows that $n = 2$ and so $R = R_1 \times R_2$. If $\mathfrak{m}_1$ or $\mathfrak{m}_2$ has nilpotency greater than or equal to 3, then by Lemma 3.7, we get that $g(\mathcal{AG}_3(R)) \geq 2$. So we conclude that $\eta(\mathfrak{m}_1) = \eta(\mathfrak{m}_2) = 2$. By Lemma 3.8 and Theorem 3.4, $\mathcal{AG}_3(R)$ is planar or $g(\mathcal{AG}_3(R)) \geq 2$. $\square$

In the following theorem, we deal with 3-annihilating-ideal hypergraph for an Artinian ring which is a finite product of local rings and fields having at least three maximal ideals.

***Theorem* 3.10** — *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a ring with at least three maximal ideals, $m \geq 1, n \geq 1$ where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq (0)$ and each F_j is a field. Then $\mathcal{AG}_3(R)$ is planar or has genus at least two.*

PROOF : *Case* 1 : $n + m \geq 4$.

It shows that $\mathcal{AG}_3(F_1 \times \cdots \times F_n \times \cdots \times F_{n+m})$ is a subhypergraph of $\mathcal{AG}_3(R)$ and as in the Proof of Theorem 3.6, we observe that $g(\mathcal{AG}_3(F_1 \times \cdots \times F_n \times \cdots \times F_{n+m})) \geq 2$. It follows that $g(\mathcal{AG}_3(R)) \geq 2$.

Case 2 : $n + m = 3$.

Suppose $n = 2$ and $m = 1$. It shows that $R = R_1 \times R_2 \times F_1$. Since R_1 and R_2 are local rings, there exists a non-trivial ideal I_1 in R_1 and I_2 in R_2 such that $I_1^2 = (0) = I_2^2$. Let H be the subhypergraph of $\mathcal{AG}_3(R)$. Let $x_1 = R_1 \times I_2 \times F_1$, $x_2 = R_1 \times (0) \times F_1$, $x_3 = R_1 \times I_2 \times (0)$, $x_4 = I_1 \times I_2 \times (0)$, $x_5 = I_1 \times R_2 \times F_1$, $x_6 = (0) \times R_2 \times F_1$, $x_7 = I_1 \times R_2 \times (0) \in \mathcal{A}(R, 3)$ and $e_1 = \{x_1, x_4, x_5\}$, $e_2 = \{x_1, x_4, x_6\}$, $e_3 = \{x_1, x_4, x_7\}$, $e_4 = \{x_2, x_4, x_5\}$, $e_5 = \{x_2, x_4, x_6\}$,

$e_6 = \{x_2, x_4, x_7\}$, $e_7 = \{x_3, x_4, x_5\}$, $e_8 = \{x_3, x_4, x_6\}$, $e_9 = \{x_3, x_4, x_7\} \in E(H)$. It is observed that $\mathcal{I}(H)$ contains a subdivision of $K_{3,3}$. It shows that $g(\mathcal{AG}_3(R)) \geq 1$.

Consider $H_1 = \mathcal{I}(H)$ and $H' = H - \{x_4\}$. Then H' is a subdivision of $K_{3,3}$ and the cellular embedding of H' in S_1 has 3 faces. Next, we proceed to prove that $g(H_1) \geq 2$ by a deletion and insertion argument. Note that H' is a subdivision of $K_{3,3}$ in which each 9 edges $x_1x_5,\ x_1x_6,\ x_1x_7,\ x_2x_5,\ x_2x_6,\ x_2x_7,\ x_3x_5,\ x_3x_6,\ x_3x_7$ of H' is subdivided by a single edge $e_1, \ldots, e_9$ respectively.

Now H_1 can be obtained from H' by inserting x_4 and all edges incident with x_4. Since $x_4e_i \in E(H_1)$ for all $i = 1, \ldots, 9$, there is no way to insert x_4 in the cellular embedding of H' in S_1 without crossings. Therefore, we conclude that $g(H_1) \geq 2$ and so that $g(\mathcal{AG}_3(R)) \geq 2$.

Suppose $m = 1$ and $n = 2$. Then $R = R_1 \times F_1 \times F_2$. Assume that $\eta(\mathfrak{m}_1) = t \geq 3$. Let G be the subhypergraph of $\mathcal{AG}_3(R)$ induced by $\Omega_1 = \{x_1, \ldots, x_{10}\} \subseteq \mathcal{A}(R, 3)$, where $x_1 = (0) \times F_1 \times F_2$, $x_2 = \mathfrak{m}_1 \times (0) \times (0)$, $x_3 = \mathfrak{m}_1 \times (0) \times F_2$, $x_4 = \mathfrak{m}_1 \times F_1 \times (0)$, $x_5 = \mathfrak{m}_1 \times F_1 \times F_2$, $x_6 = \mathfrak{m}_1^{n-1} \times (0) \times F_2$, $x_7 = \mathfrak{m}_1^{n-1} \times F_1 \times (0)$, $x_8 = \mathfrak{m}_1^{n-1} \times F_1 \times F_2$, $x_9 = R_1 \times (0) \times (0)$, $x_{10} = R_1 \times (0) \times F_2$, $x_{11} = R_1 \times F_1 \times (0) \in \mathcal{A}(R, 3)$. Then $\mathcal{I}(G)$ is a subgraph of $\mathcal{I}(\mathcal{AG}_3(R))$, $|V(\mathcal{I}(\mathcal{AG}_3(R)))| \geq 35$, $|E(\mathcal{I}(\mathcal{AG}_3(R)))| \geq 72$ and $gr(\mathcal{I}(\mathcal{AG}_3(R))) = 4$. By Lemma 2.6, $g(G) \geq 2$ and so that $g(\mathcal{AG}_3(R)) \geq 2$.

Next, let us assume that $\eta(\mathfrak{m}_1) = 2$. If R_1 has no nonzero proper ideals different from $\mathfrak{m}_1$, then by Theorem 3.5(i), $\mathcal{AG}_3(R)$ is planar. If R_1 has a nonzero proper ideal different from $\mathfrak{m}_1$, then by Proposition 2.1, R_1 has at least three ideals different from $\mathfrak{m}_1$, say I, J, K. Let H be the subhypergraph of $\mathcal{AG}_3(R)$ induced by $\Omega_2 = \{x_1, \ldots, x_7\} \subseteq \mathcal{A}(R, 3)$, where $x_1 = R_1 \times (0) \times F_2$, $x_2 = I \times F_1 \times F_2$, $x_3 = J \times F_1 \times F_2$, $x_4 = K \times F_1 \times F_2$, $x_5 = I \times F_1 \times (0)$, $x_6 = J \times F_1 \times (0)$, $x_7 = K \times F_1 \times (0) \in \mathcal{A}(R, 3)$. Then it is easy to observe that $\mathcal{I}(H)$ contains a subdivision of $K_{3,3}$ and x_1 is adjacent to all edges in subhypergraph H of $\mathcal{AG}_3(R)$. Hence as before, we prove that $g(\mathcal{AG}_3(R)) \geq 2$. □

***Theorem* 3.11** — *Let $R = R_1 \times F_1$, where $(R_1, \mathfrak{m}_1)$ is a local ring with $\mathfrak{m}_1$ is principal and F_1 is a field. Then $\mathcal{AG}_3(R)$ is toroidal if and only if the nilpotency of $\mathfrak{m}_1$ is five.*

PROOF : Assume that $\mathcal{AG}_3(R)$ is toroidal. Suppose that $\eta(\mathfrak{m}_1) = n \geq 6$. Since $\mathfrak{m}_1$ is principle, $\mathfrak{m}_1 = \langle a \rangle$ for some $a \in R_1^*$ and so $\mathfrak{m}_1$, $\mathfrak{m}_1^2$, $\ldots$, $\mathfrak{m}_1^{n-1}$ are only non-trivial ideals in R_1. Let G be a subhypergraph of $\mathcal{AG}_3(R)$ induced by $\Omega_1 = \{x_1, \ldots, x_{10}\} \subseteq \mathcal{A}(R, 3)$, where $x_1 = \mathfrak{m}_1 \times (0)$, $x_2 = \mathfrak{m}_1^{n-4} \times (0)$, $x_3 = \mathfrak{m}_1^{n-3} \times (0)$, $x_4 = \mathfrak{m}_1^{n-2} \times (0)$, $x_5 = R_1 \times (0)$, $x_6 = \mathfrak{m}_1 \times F_1$, $x_7 = \mathfrak{m}_1^{n-4} \times F_1$, $x_8 = \mathfrak{m}_1^{n-3} \times F_1$, $x_9 = \mathfrak{m}_1^{n-2} \times F_1$, $x_{10} = \mathfrak{m}_1^{n-1} \times F_1 \in \mathcal{A}(R, 3)$ and let $e_1 = \{x_1, x_6, x_9\}$, $e_2 = \{x_1, x_7, x_8\}$, $e_3 = \{x_1, x_7, x_9\}$, $e_4 = \{x_1, x_8, x_9\}$, $e_5 = \{x_1, x_4, x_6\}$, $e_6 = \{x_1, x_3, x_7\}$, $e_7 =$

$\{x_1, x_2, x_3\}$, $e_8 = \{x_1, x_2, x_8\}$, $e_9 = \{x_2, x_6, x_8\}$, $e_{10} = \{x_2, x_7, x_8\}$, $e_{11} = \{x_2, x_3, x_7\}$, $e_{12} = \{x_3, x_6, x_7\}$, $e_{13} = \{x_2, x_3, x_6\}$, $e_{14} = \{x_5, x_6, x_{10}\}$, $e_{15} = \{x_5, x_7, x_9\}$, $e_{16} = \{x_5, x_7, x_{10}\}$, $e_{17} = \{x_5, x_8, x_9\}$, $e_{18} = \{x_5, x_8, x_{10}\}$ and $e_{19} = \{x_5, x_9, x_{10}\} \in E(\mathcal{AG}_3(R))$. It follows that $\mathcal{I}(G)$ contains a subdivision of $K_{3,3}$ and hence $g(G) \geq 1$.

Suppose that $g(G) = 1$. Then by Theorem 2.8, $g(\mathcal{I}(G)) = 1$. Let $H = \mathcal{I}(G)$, $H' = H - \{x_4, x_5, x_{10}, e_4, e_5, e_8, e_9, e_{11}, e_{14}, e_{16}, e_{18}, e_{19}\} - \{x_5e_{15}, x_2e_{10}, x_3e_{12}, x_2e_{13}, x_1e_7, x_5e_{17}\}$ and H'' $= H' - \{e_2, e_{15}\} - \{e_{10}x_7\}$. Hence H'' is isomorphic to the subdivision of $K_{3,3}$ and so $g(H'') = 1$. Since $g(\mathcal{I}(G)) = 1$, $g(H) = 1$. Since $g\,(H'') \leq g\,(H') \leq g\,(H)$, we obtain that $g\,(H') = 1$. Note that $|V\,(H')\,| = 17$ and $|E\,(H')\,| = 32$. Then by Euler's formula 2.7, there are 15 faces when drawing H' on a torus. Fix a representation of H' and let $\{F'_1, \ldots, F'_{15}\}$ be the set of all faces of H' corresponding to this representation. We observe that H'' is the subdivision of $K_{3,3}$ and this graph has 3 faces. Write the faces of H'' as F''_1, F''_2 and F''_3 obtained by deleting e_2 and e_{15} and all the edges incident with e_2, e_{15} and the edge $e_{10}x_7$ from the representation of H''. Then $\{F'_1, \ldots, F'_{15}\}$ can be recovered by inserting e_2 and e_{15} and all the edges incident with e_2, e_{15} and the edge $e_{10}x_7$ into the representation corresponding to $\{F''_1, F''_2, F''_3\}$. Note that $x_8e_2, x_8e_{10} \in E(H')$. Hence x_8, e_2 and e_{10} should be inserted to the same face F''_m of H'' to avoid crossings. Since $e_2x_i \in E(H')$ for $i = 1, 7$ and $e_{10}x_i \in E(H')$ for $i = 1, 7$. Hence x_1, x_2 and x_7 should be boundary vertices of the face F''_m. We observe that $x_9e_3, x_9e_{15} \in E(H')$. If x_7, e_3 and e_{15} lie on different faces of cellular embedding of H' in S_1, we get an edge crossing. Hence x_7, e_3 and e_{15} should be inserted to the same face of cellular embedding of H' in S_1. Moreover $e_1x_i \in E(H')$ for $i = 1, 9$ and $e_{17}x_i \in E(H')$ for $i = 8, 9$. If e_1, e_{17}, x_8 and x_9 lie on the face different from F''_m, we get an edge crossing. So that e_1, e_{17}, x_8 and x_9 should be inserted to the same face F''_m of H''.

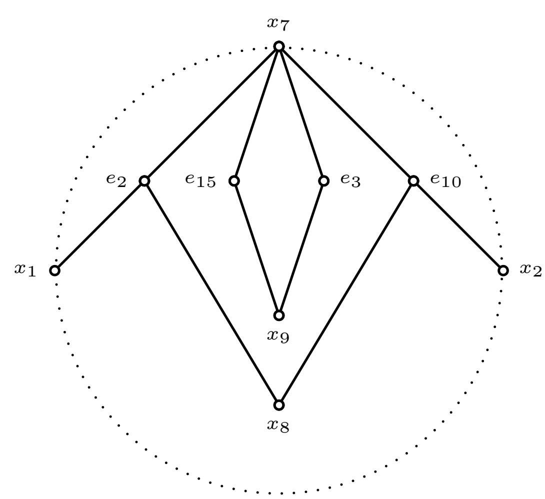

Figure 3: The face F''_m

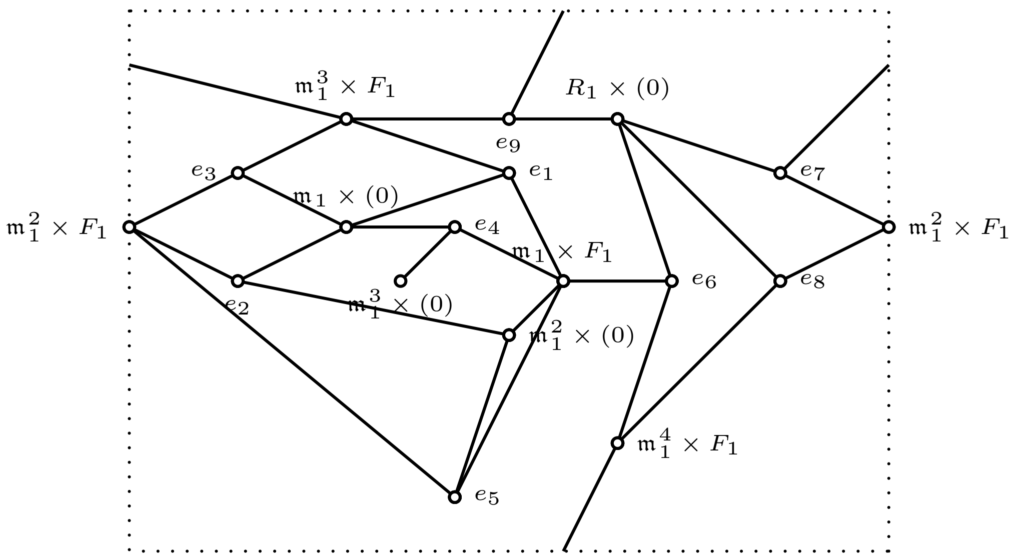

Figure 4: The embedding of $\mathcal{AG}_3(R)$ in S_1

Consider the edges $f_1 = e_2x_8$, $f_2 = e_{10}x_8$, $f_3 = e_2x_1$, $f_4 = e_2x_7$, $f_5 = e_{10}x_2$, $f_6 = e_{10}x_7$, $f_7 = e_{15}x_9$, $f_8 = e_3x_9$, $f_9 = e_{15}x_7$, $f_{10} = e_3x_7$, $f_{11} = e_1x_1$ and $f_{12} = e_1x_9 \in E(G)$. After inserting $f_1, \ldots, f_{10}$ into F''_m, we obtain the Figure 3. However, it is easy to see from Figure 3, we cannot insert x_1, e_1 and f_{11}, f_{12} into F''_m without crossings. Hence we conclude that $g(H) \geq 2$ and hence $g(\mathcal{AG}_3(R)) \geq 2$ which is a contradiction. Thus we conclude that $\eta(\mathfrak{m}_1) = 5$. The converse follows from Figure 4. □

***Theorem* 3.12** — *Let $R = R_1 \times F_1$, where R_1 is a local ring with maximal ideal $\mathfrak{m}_1$ of nilpotency two and F_1 is a field. Then $\mathcal{AG}_3(R)$ is toroidal if and only if R_1 has exactly three nonzero proper ideals different from $\mathfrak{m}_1$.*

PROOF : Assume that $\mathcal{AG}_3(R)$ is toroidal. Suppose R_1 has no ideals different from $\mathfrak{m}_1$. Then it is clear that $\mathcal{A}(R, 3) = \emptyset$. Hence let us assume that R_1 has at least four nonzero proper ideals I_1, I_2, I_3 and I_4 different from $\mathfrak{m}_1$. Let G be a subhypergraph of $\mathcal{AG}_3(R)$ induced by $\Omega_1 = \{x_1, \ldots, x_6\} \subseteq \mathcal{A}(R, 3)$ where $x_1 = R_1 \times (0)$, $x_2 = \mathfrak{m}_1 \times F_1$, $x_3 = I_1 \times F_1$, $x_4 = I_2 \times F_1$, $x_5 = I_3 \times F_1$, $x_6 = I_4 \times F_1 \in \mathcal{A}(R, 3)$ and let $e_1 = \{x_1, x_2, x_3\}$, $e_2 = \{x_1, x_2, x_4\}$, $e_3 = \{x_1, x_2, x_5\}$, $e_4 = \{x_1, x_2, x_6\}$, $e_5 = \{x_1, x_3, x_4\}$, $e_6 = \{x_1, x_3, x_5\}$, $e_7 = \{x_1, x_3, x_6\}$, $e_8 = \{x_1, x_4, x_5\}$, $e_9 = \{x_1, x_4, x_6\}$, and $e_{10} = \{x_1, x_5, x_6\} \in E(\mathcal{AG}_3(R))$. Note that $\mathcal{I}(G)$ contains a subdivision of $K_{3,3}$ and hence $g(G) \geq 1$.

Suppose that $g(G) = 1$. Then by Theorem 2.8, $g(\mathcal{I}(G)) = 1$. Let $H = I(G)$, $H' = H - \{e_8\} - \{e_4x_2, e_5x_1, e_6x_1\}$ and $H'' = H' - \{x_6, e_4, e_7, e_9, e_{10}\}$. Hence H'' is isomorphic to the subdivision of $K_{3,3}$ and hence $g(H'') = 1$. Suppose that $g(H) = 1$. Hence we conclude that $g(H') = 1$ and by

Euler's formula 2.7, there are 9 faces when drawing H' on torus. Fix a representation for H' and let $\{F_1', \ldots, F_9'\}$ be the set of faces of H' corresponding to this representation. Since $K_{3,3}$ has three faces, H'' has three faces. Let F_1'', F_2'', F_3'' be the faces of H'' obtained by deleting $x_6, e_4, e_7, e_9, e_{10}$ and all edges incident with $x_6, e_4, e_7, e_9, e_{10}$ from the representation of H'. Again $\{F_1', \ldots, F_9'\}$ be recovered by inserting $x_6, e_4, e_7, e_9, e_{10}$ and all the edges incident with $x_6, e_4, e_7, e_9, e_{10}$ into the representation corresponding to $\{F_1'', F_2'', F_3''\}$. Let F_n'' be the face of H'' into which $x_6, e_4, e_7, e_9, e_{10}$ are inserted during the recovering process from H'' to H'.

Since $x_6e_i \in E(H')$ for $i = 4, 7, 9, 10$. So that $x_6, e_4, e_7, e_9, e_{10}$ should be inserted to the same face F_n'' of H'' to avoid crossings. Note that $x_1e_4, x_1e_7, x_1e_9, x_1e_{10}, x_3e_7, x_4e_9, x_5e_{10}, x_6e_4, x_6e_7,$ $x_6e_9, x_6e_{10} \in E(H')$. If x_1, x_3, x_4 and x_5 are not in the same face of cellular embedding of H' in S_1, then we get an edge crossing. Hence x_1, x_3, x_4, x_5 lie on the boundary vertices of the face F_n'' of H''. Consider the edges $f_1 = x_1e_4, f_2 = x_6e_4, f_3 = x_1e_7, f_4 = x_3e_7, f_5 = x_6e_7, f_6 = x_4e_9, f_7 = x_6e_9, f_8 = x_1e_{10}, f_9 = x_6e_{10}, f_{10} = x_1e_9, f_{11} = x_5e_{10} \in E(G)$. Then we obtain the Figure 5 by inserting $x_6, e_4, e_7, e_9, e_{10}$ and $f_1, \ldots, f_9$ into F_n''. However from the Figure 5, we see that there is no way to insert f_{10} and f_{11} into F_n'' without crossings. We get a contradiction. So that $g(H) \geq 2$ and hence $g(\mathcal{AG}_3(R)) \geq 2$.

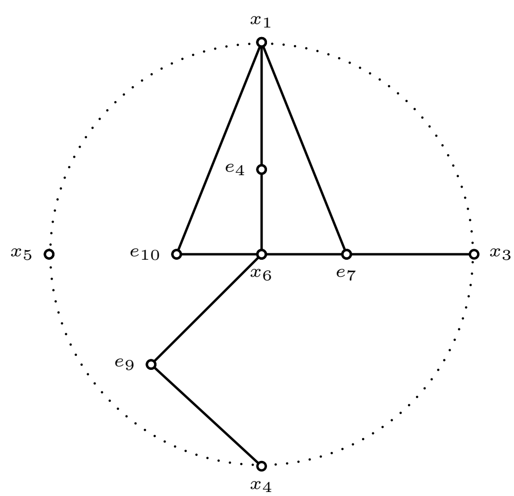

Figure 5: The face F_n''

Thus we observe from Proposition 2.1, R_1 has exactly three nonzero proper ideals I_1, I_2 and I_3 different from the maximal ideal $\mathfrak{m}_1$.

The converse follows from Figure 6. □

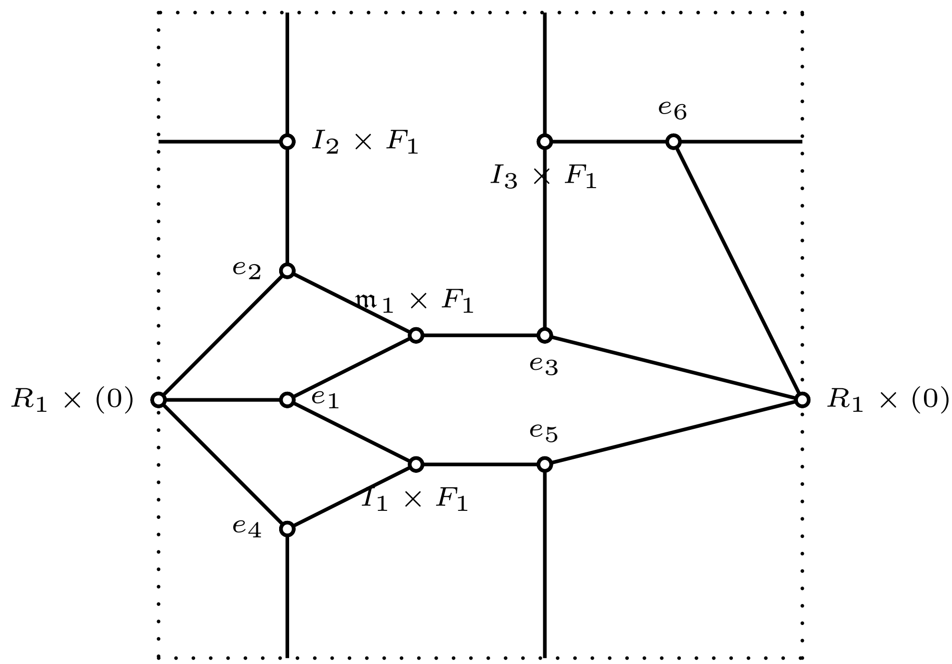

Figure 6: The embedding of $\mathcal{AG}_3(R)$ in S_1

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References

1. G. Aalipour, S. Akbari, R. Nikandish, M. J. Nikmehr, and F. Shaveisi, On the coloring of the annihilating-ideal graph of a commutative ring, *Discrete Math.*, **312** (2012), 2620-2626.
2. G. Aalipour, S. Akbari, R. Nikandish, M. J. Nikmehr, and F. Shaveisi, Minimal prime ideals and cycles in annihilating-ideal graphs, *Rocky Mountain J. Math*., **43**(5) (2013), 1415-1425.
3. M. Afkhami and K. Khashyarmanesh, The cozero-divisor graph of a commutative ring, *Southeast Asian Bull. Math.*, **35** (2011), 753-762.
4. D. F. Anderson and A. Badawi, On n-absorbing ideals of commutative rings, *Comm. Algebra*, **39** (2011), 1646-1672.
5. D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston, *The zero-divisor graph of a commutative ring, II*, in: Ideal Theoretic Methods in Commutative Algebra, Columbia, MO, 1999, Dekker, New York, 2001, 61-72.

6. D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217** (1999), 434-447.

7. D. Archdeacon, Topological graph theory: A survey, *Congr. Numer.,* **115** (1996), 5-54.

8. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company, 1969.

9. A. Badawi, On 2-absorbing ideals of Commutative rings, *Bull. Austral. Math. Soc.*,, **75** (2007), 417-429.

10. A. Badawi, On the total graph of a ring and its related graphs: A survey, Recent Advances in Commutative Rings, Integer-Valued Polynomial and Polynomial Functions, Fontana, Macro, Sophie, Glaz, Sarah (Eds.), 2014, Springer.

11. A. Badawi, On the annihilator graph of a commutative ring, *Comm. Algebra*, **42**(1) (2014), 108-121, doi: 10.1080/00927872.212.707262.

12. A. Badawi, On the dot product graph of a commutative ring, *Comm. Algebra,* **43** (2015), 43-50, doi: 10.1080/00927872.2014.897188.

13. I. Beck, Coloring of commutative rings, *J. Algebra* **116** (1988), 208-226.

14. M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings I, *J. Algebra Appl.,* **10**(4) (2011), 727-739.

15. M. Behboodi and Z. Rakeei, The annihilating-ideal graph of commutative rings II, *J. Algebra Appl.*, **10**(4) (2011), 741-753.

16. R. Belshoff and J. Chapman, Planar zero-divisor graphs, *J. Algebra*, **316**(1) (2007), 471-480.

17. C. Berge, *Graphs and hypergraphs*, North-Holland Publishing Company, Londan, 2003.

18. G. Chartrand and L. Lesniak, *Graphs and digraphs*, Wadsworth and Brooks/ Cole, Monterey, CA, 1986.

19. Ch. Eslahchi and A. M. Rahimi, The k-zero divisor Hypergraph of a commutative ring, *Int. J. Math. and Math. Sciences*, **2007** (2007), Article ID 50875, (15 pages), http://dx.doi.org/10.1155/2007/50875.

20. A. Gagarin and W. Kocay, Embedding graphs containing K_5-subdivisions, *Ars Combinatoria*, **64** (2002), 33-50.

21. A. Gagarin, W. Myrvold, and J. Chambers, Forbidden minors and subdivisions for toroidal graphs with no $K_{3,3}$'s, *Electron. Notes in Discrete Math*., **22** (2005), 151-156.

22. I. Kaplansky, *Commutative rings*, rev. ed., University of Chicago Press, Chicago, 1974.

23. H. R. Maimani, M. Salimi, A. Sattari, and S. Yassemi, Comaximal graph of commutative rings, *J. Algebra*, **319** (2008), 1801-1808.

24. K. Selvakumar and V. Ramanathan, Classification of nonlocal rings with genus one 3-zero-divisor hypergraphs, *Comm. Algebra,* **45**(1) (2017), 275-284.

25. K. Selvakumar and V. Ramanathan, Classification of nonlocal rings with projective 3-zero-divisor hypergraph, *Ricerche Math.*, **66**(2) (2017), 457-468.

26. K. Selvakumar and V. Ramanathan, *The k-annihilating-ideal hypergraph of a commutative ring*, AKCE Int. J. Graphs Combin., https://doi.org/10.1016/j.akcej.2019.02.08.

27. K. Selvakumar and P. Subbulakshmi, On the genus of the graph associated to a commutative ring, *Discrete Math. Algorithm Appl.*, **9**(4) (2017), 1750058:11pp, doi:10.1142/S1793830917500586.

28. F. Shaveisi and R. Nikandish, The Nil-graph of Ideals of a commutative ring, *Bull. Malays. Math. Sci. Soc.*, **39** (2016), 3-11.

29. N. O. Smith, Planar zero-divisor graphs, *Int. J. Comm. Rings*, **2**(4) (2002), 177-188.

30. T. Tamizh Chelvam and K. Selvakumar, Central sets in the annihilating graph of a commutative ring, *J. Combin. Math. Combin. Comput.*, **88** (2014), 277-288.

31. T. Tamizh Chelvam, K. Selvakumar, and V. Ramanathan, On the planarity of the k-zero-divisor hypergraphs, *AKCE Inter. J. Graphs and Combin.*, **12** (2015), 169-176.

32. T. Tamizh Chelvam and K. Selvakumar, On the connectivity of the annihilating-ideal graphs, *Discuss. Math. Gen. Algebra Appl.*, **35**(2) (2015), 195-204, doi:10.7151/dmgaa.1241.

33. T. R. S. Walsh, Hypermaps versus bipartite maps, *J. Combin. Theory B*, **18** (1975), 155-163.

34. A. T. White, *Graphs, groups and surfaces*, North-Holland, Amsterdam, (1973).