DISTANCE HEREDITARY GRAPHS $G$ OF CONNECTIVITY TWO OR THREE AND $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ ARE RECONSTRUCTIBLE

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(Received 9 April 2017; after final revision 7 March 2018; accepted 12 July 2018)

A graph is said to be reconstructible if it is determined up to isomorphism from the collection of all its one-vertex deleted unlabeled subgraphs. It is shown that all distance hereditary graphs $G$ of connectivity two or three and $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

Key words: Reconstruction; distance; connectivity; distance hereditary graphs.

1. INTRODUCTION

The graphs considered in this paper have finite orders and do not have loops or multiple edges. The terms not defined here are taken as in [3]. The distance between two vertices $u$ and $v$ in a connected graph $G$, denoted by $d_G(u, v)$ or simply by $d(u, v)$, is the length of a shortest path joining $u$ and $v$. The eccentricity $e(u)$ of a vertex $u$ in a connected graph $G$ is the maximum of its distances to other vertices. The radius of a connected graph $G$, denoted by $\text{rad}(G)$, is the minimum of the vertex eccentricities. A graph $G$ is self-centered if $e(v) = \text{rad}(G)$ for all the vertices $v$ in $G$.

A vertex-deleted subgraph (or card) $G - v$ of a graph $G$ is the unlabeled subgraph obtained from $G$ by deleting $v$ and all edges incident with $v$. The collection of all cards of $G$ is called the deck of $G$. A graph $H$ is called a reconstruction of $G$ if $H$ has the same deck as $G$. A graph is said to be reconstructible if it is isomorphic to all its reconstructions. A parameter $p$ defined on graphs is reconstructible if, for any graph $G$, it takes the same value on every reconstruction of $G$. The graph

\footnote{Monikandan’s research is supported by the SERB-DST, Govt. of India. Grant No. EMR/2016/000157.}
reconstruction conjecture (RC), posed by Kelly and Ulam in 1941 (see [2]), says that every graph 
G on n ≥ 3 vertices is reconstructible. This conjecture has proved notoriously difficult, and has
motivated a large amount of work in graph theory. The manuscripts [1, 2, 4, 6, 7] are surveys of work
done on this problem.

graph is reconstructible if and only if every 2-connected graph is reconstructible. Gupta et al. [5]
have proved that the RC is true if and only if all connected graphs G such that diam(G) = 2 or
diam(G) = diam(G) = 3 are reconstructible. In their paper, Ramachandran and Monikandan [8]
have combined these two reductions of the RC and proved that the RC is true if and only if all
2-connected graphs G such that diam(G) = 2 or diam(G) = diam(G) = 3 are reconstructible.

A graph G is said to be distance-hereditary if for all connected induced subgraphs F of G, d_F(u, v) =
d_G(u, v) for all pairs of vertices u, v ∈ F. In this paper, we prove that all distance hereditary graphs
G with connectivity two or three and diam(G) = diam(G) = 3 are reconstructible.

2. RECONSTRUCTION USING DISTANCE AND CONNECTIVITY

The proof of Lemma 3 uses the following two well-known results.

Lemma 1 — If diam(G) ≥ 3, then diam(G) ≤ 3.

Lemma 2 — If rad(G) ≥ 3, then rad(G) ≤ 2.

Lemma 3 — If G is self-centered with rad(G) ≥ 3, then G is self-centered with rad(G) = 2.

PROOF : Since rad(G) ≥ 3, we have diam(G) ≥ 3. By Lemmas 1 and 2, diam(G) ≤ 3 and
rad(G) ≤ 2. Moreover, if diam(G) or rad(G) were 1, then G would contain a vertex adjacent to all
other vertices and that vertex would be an isolated vertex in G, giving a contradiction.

Thus, diam(G) = 2 or 3 and rad(G) = 2. Also, if diam(G) were 3, then G would contain two
vertices u and v such that d(u, v) = 3 and N(u) ∩ N(v) = φ. Hence, in G, every vertex in V − {u, v}
would be adjacent to at most one of the two vertices u and v. Therefore, in G, vertices u and v are
adjacent and every vertex in V − {u, v} would be adjacent to u or v, which implies rad(G) ≤ 2,
which is a contradiction.

The following characterization of distance-hereditary graphs is given in the book [3] as Theorem
7.22.

Theorem 1 — A graph G is distance-hereditary if and only if it contains no C_n, n ≥ 5, nor any
of the graphs in Figure 1 as an induced subgraph.
The next two theorems are useful while proving the recognizability of the graphs stated in Theorem 4.

**Theorem 2** — (Tutte [10]). The number of nonseparable spanning subgraphs of $G$ with a given number of edges is reconstructible.

**Theorem 3** — (Gupta et. al [5]). Graphs $G$ with $diam(G) = 2$ and graphs $H$ with $diam(H) = diam(\overline{H}) = 3$ are recognizable.

Two vertices $u$ and $v$ of a graph $G$ are said to be **bisimilar** if there is an automorphism of $G$ interchanging $u$ and $v$.

**Theorem 4** — All distance hereditary graphs $G$ of connectivity 2 and $diam(G) = diam(\overline{G}) = 3$ are reconstructible.

**Proof**: Recognition: if $H$ is one of the subgraphs in Figure 1, or a cycle of order strictly less than $n$ (the order of $G$), then, using Kelly’s Lemma, we can determine whether $H$ is an induced subgraph of $G$ or not. However, if $C$ is a cycle of order $n$, then $C$ is a nonseparable graph with $n$ edges and hence, using Theorem 2, we can determine whether $C$ is a subgraph of $G$ or not.

Since the connectivity is reconstructible, the recognizability of the class of graphs stated in the theorem follows by Theorem 3.

Since $diam(G) = 3$, we have $rad(G) \leq 3$. If $rad(G) = 1$, then $G$ is reconstructible, since it has an $(n-1)$-vertex. If $rad(G)$ were 3, then by Lemma 3, the graph $\overline{G}$ would be self-centered with $rad(\overline{G}) = 2$, which would imply $diam(\overline{G}) = 2$, a contradiction. Thus, we assume that $rad(G) = 2$; let $u$ be a vertex in $G$ with $e(u) = 2$.

Let $N_1(u) = \{v \in V(G) : d(u, v) = 1\}$,

$N_2(u) = \{v \in V(G) : d(u, v) = 2\} = Z$,

$X = \{v \in V(G) : d(v, z) \geq 2, \text{ for any } z \in N_2(u)\}$ and $Y = N_1(u) - X$.
Since \( \kappa(G) = 2 \), there is a 2-vertex-cut in \( G \). But the possible 2-vertex-cuts of \( G \) are \{\( x, z \), \( x_1, x_2 \), \( x, y \), \( u, y \), \( u, z \), \( y, z \), \( z_1, z_2 \), \( u, x \) and \( y_1, y_2 \)\}, where \( x, x_i \in X \), \( y, y_i \in Y \) and \( z, z_i \in Z \). However, we shall prove that the sets of the form \{\( u, x \)\} and \{\( y_1, y_2 \)\} are the only possible vertex cuts.

We first claim that each vertex in \( Z \) has at least two neighbours in \( Y \). Suppose this is not the case. Then there exists a vertex \( z \in Z \) with exactly one neighbour, say \( y \in Y \). Then \( d_{G-\{y\}}(u, z) = 3 \), which contradicts the facts \( e(u) = 2 \) and \( G \) is distance hereditary. This completes the claim.

Since each vertex \( x \in X \) is connected to other vertices by paths passing through \( u \), it follows that, any 2-vertex-cut, containing a vertex from \( X \), must contain the vertex \( u \). Thus the reduced possible collection of 2-vertex-cuts are \{\( u, y \), \( u, z \), \( y, z \), \( z_1, z_2 \), \( u, x \) and \( y_1, y_2 \)\}.

Next, we prove that \{\( u, y \)\} is not a vertex-cut. Assume the contrary. Then all the vertices in \( Z \) must lie in a single component of \( G - \{u, y\} \) (as otherwise \( d_{G-\{y\}}(z_1, z_2) \) would be 4 for vertices \( z_1 \) and \( z_2 \) lying in different components of \( G - \{u, y\} \), giving a contradiction since \( diam(G) = 3 \) and \( G \) is distance hereditary (Figure 3)). Consequently, all the vertices in \( Y - y \) must lie in a single component containing the vertices from \( Z \). Choose a neighbour \( z' \) of the vertex \( y \) that is in \( Z \). Since each vertex in \( Z \) is adjacent to at least two vertices in \( Y \), there exists \( y' \in Y \) such that \( y' \) is adjacent to \( z' \). Now, since \( d(v, w) \leq 2 \) for any two vertices \( v, w \) in \( N_1(u) \), and \( G - \{v\} \) is connected, it follows that vertex \( v \) is adjacent to \( y \) for all \( v \in N_1(u) \). Choose a vertex \( x' \in X \) such that it lies in the component disjoint from \( Z \). Now, the subgraph of \( G \) induced by the vertices \( u, x', y, y' \) and \( z' \) is isomorphic to the graph shown in Figure 1 (b), giving a contradiction to Theorem 1.
are connected to the vertices in \( X \) in \( G \).

We now prove that \( \{u, z\} \) is not a 2-vertex-cut of \( G \). If possible, suppose \( \{u, z\} \) were a 2-vertex-cut of \( G \). Then, by similar arguments as given in the above paragraph, the vertices in both \( Y \) and \( Z - \{z\} \) must lie in a single component. Thus in \( G - \{u\}, \) there exists a component consisting of vertices from \( X \) alone so that \( G - \{u\} \) becomes disconnected, contradicting \( G \).

Suppose, if possible, \( \{y, z\} \) were a vertex-cut of \( G \). Then, since any two vertices in \( N_1(u) \) are connected through \( u \) and \( G - \{y, z\} \) is disconnected, there exists a component \( Z_1 \) with vertices only from \( Z \). Now \( d_{G-\{y\}}(u, z_1) > 2 \) for all \( z_1 \in Z \), since the vertices in \( Z_1 \) are connected to the other vertices only through \( z \), giving a contradiction to \( G \) being distance hereditary.

Also, the set \( \{z_1, z_2\} \) cannot be a 2-vertex cut of \( G \), since each vertex in \( Z \) has at least two neighbours in \( Y \). Thus, the reduced collection of possible 2-vertex-cuts of \( G \) are \( \{u, x\} \) and \( \{y_1, y_2\} \).

Now, we prove that, if \( \{u, x\} \) is a 2-vertex cut of \( G \), then \( u \) and \( x \) are bisimilar vertices in \( G \) and that \( G \) has at most one 2-vertex-cut of this form. If \( v \) and \( w \) are any two vertices in \( N_1(u) \) lying in different components of \( G - \{u, x\} \), then \( d(v, w) = 2 \) in \( G \) and hence in \( G - u \). Thus, both \( v \) and \( w \) are adjacent to \( x \). Hence every vertex in \( N_1(u) - \{x\} \) is adjacent to \( x \), and this \( x \) is adjacent to no \( y \) in \( Y \)(by definition of \( X \)). Hence \( u \) and \( x \) are bisimilar vertices in \( G \). Also, suppose \( G \) would contain two such distinct 2-vertex-cuts, say \( W_1 = \{u, x_1\} \) and \( W_2 = \{u, x_2\} \). Then every vertex in \( N_1(u) - \{x_i\} \), for \( i = 1, 2 \), would be adjacent to both \( x_1 \) and \( x_2 \). Therefore, \( G - W_1 \) would be connected, giving a contradiction.

Similarly, we next prove that if \( \{y_1, y_2\} \) is a 2-vertex cut of \( G \), then \( y_1 \) and \( y_2 \) are bisimilar vertices in \( G \). Since \( G - \{u, y_1\} \) (respectively, \( G - \{u, y_2\} \)) is connected and \( d(v, w) = 2 \) for any two vertices \( v, w \in N_1(u) \) lying in different components of \( G - \{y_1, y_2\} \), it follows that each vertex in \( N_1(u) - \{y_1, y_2\} \) is adjacent to the vertex \( y_2 \) (respectively, \( y_1 \)). In \( G - \{y_1, y_2\} \), the vertices in \( X \) are connected to the vertices in \( Y - \{y_1, y_2\} \) through \( u \). Hence \( G - \{y_1, y_2\} \) has a component \( Z_1 \).
consisting of vertices only from \( Z \). Clearly, no vertex in \( Z_1 \) is adjacent to a vertex in \( Y - \{ y_1, y_2 \} \), and every vertex \( z \in Z \) has at least two neighbours in \( Y \). Therefore, every vertex in \( Z_1 \) is adjacent to both \( y_1 \) and \( y_2 \). Suppose there exists a vertex \( z_1 \in Z \setminus Z_1 \) that is adjacent to one of \( y_1 \) and \( y_2 \), say \( y_2 \). Then \( d(z_1, z) = 2 \) for all \( z \in Z_1 \). Since \( G \) is distance hereditary, the connectedness of \( G - y_2 \) forces that \( z_1 \) must be adjacent to \( y_1 \). Thus, we conclude that if a vertex in \( Z \) is adjacent to one of \( y_1 \) and \( y_2 \), then it must be adjacent to both \( y_1 \) and \( y_2 \). Hence \( y_1 \) and \( y_2 \) are bisimilar vertices in \( G \).

Finally, we prove that any two 2-vertex-cuts in \( Y \) of the form \( \{ y_1, y_2 \} \) are disjoint. Suppose there exist two distinct 2-vertex-cuts \( W_1 \) and \( W_2 \) in \( Y \) such that \( W_1 \cap W_2 \neq \emptyset \); let \( y' \in W_1 \cap W_2 \). Since \( W_1 \) and \( W_2 \) are vertex cuts, \( G - W_1 \) and \( G - W_2 \) have components \( Z'_1 \) and \( Z'_2 \), respectively, such that \( Z'_1, Z'_2 \subseteq Z \) and \( N_{G-Z'_i}(Z'_i) = W_i \), for \( i = 1, 2 \). Choose \( z_1 \in Z_1 \), \( z_2 \in Z_2 \), \( y_1 \in W_1 - y' \) and \( y_2 \in W_2 - y' \). Now \( z_1 \) is not adjacent to both \( z_2 \) and \( y_2 \), and \( z_2 \) is not adjacent to \( y_1 \). Also, \( y_1, y_2 \) and \( y' \) are mutually adjacent. Now the subgraph of \( G \) induced by \( \{ y_1, y_2, y', z_1, z_2 \} \) is isomorphic to the graph in Figure 1 (b), giving a contradiction to Theorem 1.

We now reconstruct \( G \) from a card \( G - v \) with a cut vertex. By the above argument, \( G - v \) must contain a unique cut vertex, say \( w \). Now all the graphs obtained by adding a new vertex to \( G - v \) and joining it to all the neighbours of \( w \) and also to the vertex \( w \) (when \( \text{deg}(v) \neq |N_{G-v}(w)| \) holds) are isomorphic and they are \( G \).

**Theorem 5** — Distance hereditary graphs of connectivity 3 and \( \text{diam}(G) = \text{diam}^*(G) = 3 \) are reconstructible.

**Proof** : Consider \( u, X, Y \) and \( Z \) as in Theorem 4. The possible 3-vertex-cuts of \( G \) are \( \{ x_1, x_2, x_3 \}, \{ x_1, x_2, y \}, \{ x_1, x_2, z \}, \{ x, y, z \}, \{ x, y_1, y_2 \}, \{ x, z_1, z_2 \}, \{ u, x, y \}, \{ u, y, y_1 \}, \{ u, y, z \}, \{ y_1, y_2, z_1 \}, \{ y, z_1, z_2 \}, \{ z_1, z_2, z_3 \}, \{ u, x, z \}, \{ u, z_1, z_2 \}, \{ u, x_1, x_2 \}, \) and \( \{ y_1, y_2, y_3 \} \) (Figure 3). But, we prove that only the sets of the form \( \{ u, x, x_2 \} \) and \( \{ y_1, y_2, y_3 \} \) are the possible 3-vertex-cuts of \( G \).

Every vertex in \( Z \) has at least three neighbours in \( Y \). For, suppose there is a vertex \( z \in Z \) with at most two neighbours \( y_1, y_2 \in Y \). Then \( d_{G-\{y_1,y_2\}}(u,z) \geq 3 \), which is a contradiction to \( e(u) = 2 \). Any 3-vertex-cut containing a vertex from \( X \) must contain the vertex \( u \), since each vertex in \( X \) is connected to the other vertices by a path passing through \( u \). Also, if \( W \) is a 3-vertex-cut containing the vertex \( u \), then all the vertices of \( Z \) would lie in a single component of \( G - W \) (as otherwise, \( d_{G-\{W-\{u\}\}}(z_1, z_2) \) would be at least 4 for any two vertices \( z_1, z_2 \) lying in different components of \( G - W \), giving a contradiction to \( G \) being distance hereditary).

Suppose that \( W = \{ u, x, y \} \) (or \( \{ u, y, y_1 \}, \{ u, y, z \} \) were a 3-vertex cut of \( G \). Then, by the above
argument, both $Z - W$ and $Y - W$ would lie in a single component of $G - W$. Since $d(v_1, v_2) \leq 2$ for any two vertices in $N_1(u)$, every vertex in $N_1(u) - W$ would be adjacent to all the vertices in $W \cap N_1(u)$. Now, since $W$ is a 3-vertex cut, $X - W$ and $Y - W$ would not lie in a single component of $G - W$. Also it is clear that, for each $z \in Z$, $|N_Y(z)| \geq 3$ and so $N_Z(y) \cap N_Z(y') \neq \emptyset$; let $z \in N_Z(y) \cap N_Z(y')$. Choose a vertex $x'$ in a component of $X - W$ containing no vertices of $Y$. Now the graph induced by $u, x', y, y'$ and $z$ would be isomorphic to the graph shown in Figure 1 (b), contradicting Theorem 1.

If $W = \{y_1, y_2, z_1\}$ or $\{y, z_1, z_2\}$ were a 3-vertex-cut of $G$, then $G - W$ would contain a component $Z_1$ consisting vertices only from $Z$ and so $G - (W - z_1)$ would be connected and $d(u, z') > 2$ for any vertex $z' \in Z_1$, giving a contradiction to $G$.

Suppose that $W = \{u, x, z\}$ or $\{u, z_1, z_2\}$ were a 3-vertex-cut of $G$. Since $W$ contains the vertex $u$, both $Z - W$ and $Y$ would lie in a single component of $G - W$ and hence the vertices in $X$ may lie in different components of $G - W$. Thus, the set $W - \{z\}$ or $W - \{z_1, z_2\}$ would possibly be a vertex cut of $G$, giving a contradiction to $\kappa(G) = 3$.

Also, the set $\{z_1, z_2, z_3\}$ is not a 3-vertex-cut, since each vertex in $Z$ has at least three neighbours in $Y$. Thus, the only possible 3-vertex-cuts of $G$ are $\{u, x_1, x_2\}$ and $\{y_1, y_2, y_3\}$.

Now, if the set $\{u, x_1, x_2\}$ is a 3-vertex-cut of $G$, then, since $d(v_1, v_2) \leq 2$ for any two vertices $v_1$ and $v_2$ in $N_1(u)$, each vertex in $N_1(u) - \{x_1, x_2\}$ is adjacent to both $x_1$ and $x_2$, which implies they are bisimilar. Also, there can be at most one 3-vertex-cut $W$ such that $W \subseteq X \cup \{u\}$. For, if possible, suppose there were two 3-vertex-cuts $W_1$ and $W_2$ such that $W_1, W_2 \subseteq X \cup \{u\}$. Then, every vertex in $N_1(u) - W_i$, for $i = 1, 2$, would be adjacent to the vertices in $W_1$ and $W_2$, which would imply $G - W_1$ is connected, a contradiction to $W_1$.

If $W = \{y_1, y_2, y_3\}$ is a 3-vertex cut of $G$, then, by the above argument, each vertex in $N_1(u) - \{y_1, y_2, y_3\}$ is adjacent to all the three vertices $y_1, y_2$ and $y_3$. Since $W$ is a 3-vertex cut of $G$, there exists a component $Z_1$ consisting vertices of $Z$ alone and no vertex in $Z_1$ is adjacent to a vertex in $Y - \{y_1, y_2, y_3\}$. Therefore, each $z \in Z_1$ is adjacent to $y_1, y_2$ and $y_3$. Suppose there exists $z \in Z - Z_1$ adjacent to some $y_i$, say $y_3$, then $d(z_1, z) = 2$ for any $z_1 \in Z_1$. Since $G - \{y_1, y_3\}$ is connected and $G$ is distance hereditary, vertex $z_1$ is adjacent to $y_2$. Similarly, vertex $z_1$ is adjacent to both $y_1$ and $y_2$. Thus, $N(y_1) = N(y_2) = N(y_3)$ in $G - \{y_1, y_2, y_3\}$. Moreover, any two 3-vertex cuts in $Y$ are disjoint (as otherwise, we would get Figure 1 (b) as an induced subgraph of $G$, again contradicting Theorem 1).

Now we reconstruct $G$ as follows. Among the cards with a 2-vertex cut, choose one, say $G - v,$
such that $deg_G(v)$ is as maximum as possible. By the above arguments, $G-v$ must contain a unique 2-vertex cut, say $\{w_1, w_2\}$. Now all the graphs obtained by adding a new vertex to $G-v$ and joining it to all the common neighbours of $w_1, w_2$ and also either to one of $w_1$ and $w_2$ (if $d(v) = |N_{G-\{w_1, w_2\}}|+1$ holds) or to both $w_1$ and $w_2$ (otherwise), are isomorphic and they are $G$.

**ACKNOWLEDGEMENT**

We thank the anonymous referees for their valuable comments and suggestions. The work reported here is supported by the Project EMR/2016/000157 awarded to the second author by the Department of Science and Technology, Government of India, New Delhi.

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