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# DISTANCE HEREDITARY GRAPHS *G* OF CONNECTIVITY TWO OR THREE AND $\boldsymbol{diam}(G) = \boldsymbol{diam}(\overline{G}) = 3$ ARE RECONSTRUCTIBLE

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A graph is said to be *reconstructible* if it is determined up to isomorphism from the collection of all its one-vertex deleted unlabeled subgraphs. It is shown that all distance hereditary graphs $G$ of connectivity two or three and $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

**Key words** : Reconstruction; distance; connectivity; distance hereditary graphs.

## 1. INTRODUCTION

The graphs considered in this paper have finite orders and do not have loops or multiple edges. The terms not defined here are taken as in [3]. The *distance* between two vertices $u$ and $v$ in a connected graph $G$, denoted by $d_G(u, v)$ or simply by $d(u, v)$, is the length of a shortest path joining $u$ and $v$. The *eccentricity* $e(u)$ of a vertex $u$ in a connected graph $G$ is the maximum of its distances to other vertices. The *radius* of a connected graph $G$, denoted by $rad(G)$, is the minimum of the vertex eccentricities. A graph $G$ is *self-centered* if $e(v) = rad(G)$ for all the vertices $v$ in $G$.

A *vertex-deleted subgraph* (or *card*) $G - v$ of a graph $G$ is the unlabeled subgraph obtained from $G$ by deleting $v$ and all edges incident with $v$. The collection of all cards of $G$ is called the *deck* of $G$. A graph $H$ is called a *reconstruction* of $G$ if $H$ has the same deck as $G$. A graph is said to be *reconstructible* if it is isomorphic to all its reconstructions. A parameter $p$ defined on graphs is reconstructible if, for any graph $G$, it takes the same value on every reconstruction of $G$. The graph

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reconstruction conjecture (RC), posed by Kelly and Ulam in 1941 (see [2]), says that every graph $G$ on $n \geq 3$ vertices is reconstructible. This conjecture has proved notoriously difficult, and has motivated a large amount of work in graph theory. The manuscripts [1, 2, 4, 6, 7] are surveys of work done on this problem.

Yongzhi [11] settled Problem 3 listed in the survey [1] when he proved that every connected graph is reconstructible if and only if every 2-connected graph is reconstructible. Gupta *et al.* [5] have proved that the RC is true if and only if all connected graphs $G$ such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible. In their paper, Ramachandran and Monikandan [8] have combined these two reductions of the RC and proved that the RC is true if and only if all 2-connected graphs $G$ such that $\text{diam}(G) = 2$ or $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

A graph $G$ is said to be *distance-hereditary* if for all connected induced subgraphs $F$ of $G$, $d_F(u, v) = d_G(u, v)$ for all pairs of vertices $u, v \in F$. In this paper, we prove that all distance hereditary graphs $G$ with connectivity two or three and $\text{diam}(G) = \text{diam}(\overline{G}) = 3$ are reconstructible.

## 2. Reconstruction Using Distance and Connectivity

The proof of Lemma 3 uses the following two well-known results.

*Lemma* 1 — If $diam(G) \geq 3$, then $diam(\overline{G}) \leq 3$.

*Lemma* 2 — If $rad(G) \geq 3$, then $rad(\overline{G}) \leq 2$.

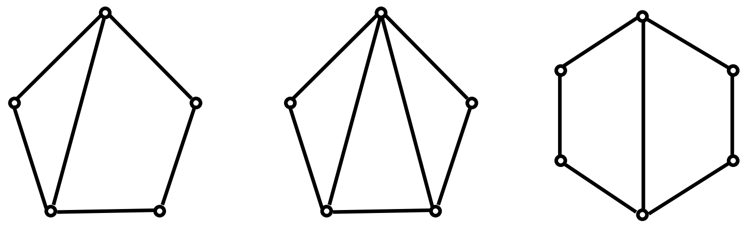
*Lemma* 3 — If $G$ is self-centered with $rad(G) \geq 3$, then $\overline{G}$ is self-centered with $rad(\overline{G}) = 2$.

PROOF : Since $rad(G) \geq 3$, we have $diam(G) \geq 3$. By Lemmas 1 and 2, $diam(\overline{G}) \leq 3$ and $rad(\overline{G}) \leq 2$. Moreover, if $diam(\overline{G})$ or $rad(\overline{G})$ were 1, then $\overline{G}$ would contain a vertex adjacent to all other vertices and that vertex would be an isolated vertex in $G$, giving a contradiction.

Thus, $diam(\overline{G}) = 2$ or 3 and $rad(\overline{G}) = 2$. Also, if $diam(\overline{G})$ were 3, then $\overline{G}$ would contain two vertices $u$ and $v$ such that $d(u, v) = 3$ and $N(u) \cap N(v) = \phi$. Hence, in $\overline{G}$, every vertex in $V - \{u, v\}$ would be adjacent to at most one of the two vertices $u$ and $v$. Therefore, in $G$, vertices $u$ and $v$ are adjacent and every vertex in $V - \{u, v\}$ would be adjacent to $u$ or $v$, which implies $rad(G) \leq 2$, which is a contradiction.

The following characterization of distance-hereditary graphs is given in the book [3] as Theorem 7.22.

***Theorem* 1** — *A graph $G$ is distance-hereditary if and only if it contains no $C_n,\ n \geq 5$, nor any of the graphs in Figure 1 as an induced subgraph.*



$(a)$ $(b)$ $(c)$

Figure 1. Forbidden subgraphs

The next two theorems are useful while proving the recognizability of the graphs stated in Theorem 4.

***Theorem* 2** — *(Tutte [10]). The number of nonseparable spanning subgraphs of $G$ with a given number of edges is reconstructible.*

***Theorem* 3** — *(Gupta et. al [5]). Graphs $G$ with $diam(G) = 2$ and graphs $H$ with $diam(H) = diam(\overline{H}) = 3$ are recognizable.*

Two vertices $u$ and $v$ of a graph $G$ are said to be *bisimilar* if there is an automorphism of $G$ interchanging $u$ and $v$.

***Theorem* 4** — *All distance hereditary graphs $G$ of connectivity $2$ and $diam(G) = diam(\overline{G}) = 3$ are reconstructible.*

PROOF : Recognition: if $H$ is one of the subgraphs in Figure $1$, or a cycle of order strictly less than $n$ (the order of $G$), then, using Kelly's Lemma, we can determine whether $H$ is an induced subgraph of $G$ or not. However, if $C$ is a cycle of order $n$, then $C$ is a nonseparable graph with $n$ edges and hence, using Theorem 2, we can determine whether $C$ is a subgraph of $G$ or not.

Since the connectivity is reconstructible, the recognizability of the class of graphs stated in the theorem follows by Theorem 3.

Since $diam(G) = 3$, we have $rad(G) \leq 3$. If $rad(G) = 1$, then $G$ is reconstructible, since it has an $(n-1)$-vertex. If $rad(G)$ were $3$, then by Lemma 3, the graph $\overline{G}$ would be self-centered with $rad(\overline{G}) = 2$, which would imply $diam(\overline{G}) = 2$, a contradiction. Thus, we assume that $rad(G) = 2$; let $u$ be a vertex in $G$ with $e(u) = 2$.

Let $N_1(u) = \{v \in V(G) : d(u,v) = 1\}$,
$N_2(u) = \{v \in V(G) : d(u,v) = 2\} = Z$,
$X = \{v \in V(G) : d(v,z) \geq 2,$ for any $z \in N_2(u)\}$ and $Y = N_1(u) - X$.

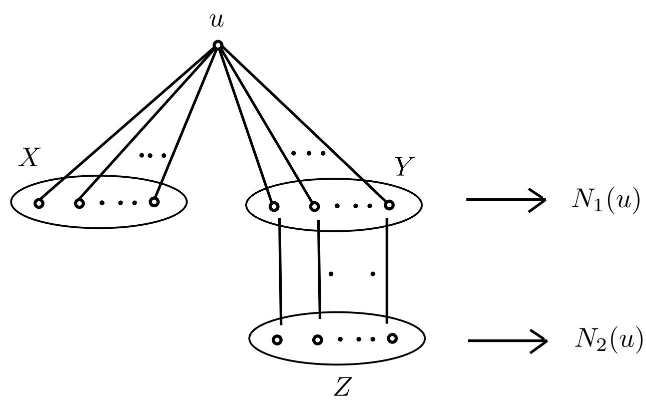



Figure 2. Structure of $G$

Since $\kappa(G) = 2$, there is a 2-vertex-cut in $G$. But the possible 2-vertex-cuts of $G$ are $\{x, z\}$, $\{x_1, x_2\},\ \{x, y\},\ \{u, y\},\ \{u, z\},\ \{y, z\},\ \{z_1, z_2\}, \{u, x\}$ and $\{y_1, y_2\}$, where $x, x_i \in X,\ y, y_i \in Y$ and $z, z_i \in Z$. However, we shall prove that the sets of the form $\{u, x\}$ and $\{y_1, y_2\}$ are the only possible vertex cuts.

We first claim that each vertex in $Z$ has at least two neighbours in $Y$. Suppose this is not the case. Then there exists a vertex $z \in Z$ with exactly one neighbour, say $y \in Y$. Then $d_{G-\{y\}}(u, z) = 3$, which contradicts the facts $e(u) = 2$ and $G$ is distance hereditary. This completes the claim.

Since each vertex $x \in X$ is connected to other vertices by paths passing through $u$, it follows that, any 2-vertex-cut, containing a vertex from $X$, must contain the vertex $u$. Thus the reduced possible collection of 2-vertex-cuts are $\{u, y\},\ \{u, z\},\ \{y, z\},\ \{z_1, z_2\},\ \{u, x\}$ and $\{y_1, y_2\}$.

Next, we prove that $\{u, y\}$ is not a vertex-cut. Assume the contrary. Then all the vertices in $Z$ must lie in a single component of $G - \{u, y\}$ (as otherwise $d_{G-\{y\}}(z_1, z_2)$ would be $4$ for vertices $z_1$ and $z_2$ lying in different components of $G - \{u, y\}$, giving a contradiction since $diam(G) = 3$ and $G$ is distance hereditary (Figure 3)). Consequently, all the vertices in $Y - y$ must lie in a single component containing the vertices from $Z$. Choose a neighbour $z'$ of the vertex $y$ that is in $Z$. Since each vertex in $Z$ is adjacent to at least two vertices in $Y$, there exists $y' \in Y$ such that $y'$ is adjacent to $z'$. Now, since $d(v, w) \leq 2$ for any two vertices $v,\ w$ in $N_1(u)$, and $G - \{v\}$ is connected, it follows that vertex $v$ is adjacent to $y$ for all $v \in N_1(u)$. Choose a vertex $x' \in X$ such that it lies in the component disjoint from $Z$. Now, the subgraph of $G$ induced by the vertices $u,\ x',\ y,\ y'$ and $z'$ is isomorphic to the graph shown in Figure $1$ $(b)$, giving a contradiction to Theorem 1.

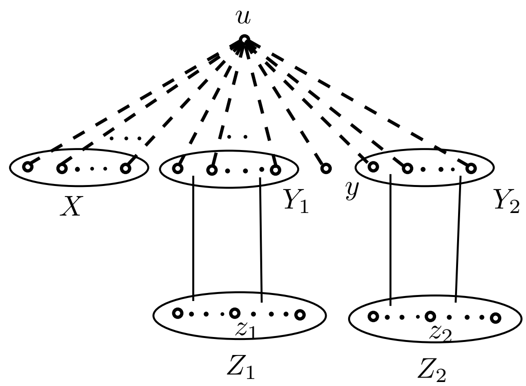



Figure 3. The subgraph $G - \{u, y\}$

We now prove that $\{u, z\}$ is not a 2-vertex-cut of $G$. If possible, suppose $\{u, z\}$ were a 2-vertex-cut of $G$. Then, by similar arguments as given in the above paragraph, the vertices in both $Y$ and $Z - \{z\}$ must lie in a single component. Thus in $G - \{u, z\}$, there exists a component consisting of vertices from $X$ alone so that $G - \{u\}$ becomes disconnected, contradicting $G$.

Suppose, if possible, $\{y, z\}$ were a vertex-cut of $G$. Then, since any two vertices in $N_1(u)$ are connected through $u$ and $G - \{y, z\}$ is disconnected, there exists a component $Z_1$ with vertices only from $Z$. Now $d_{G-\{y\}}(u, z_1) > 2$ for all $z_1 \in Z$, since the vertices in $Z_1$ are connected to the other vertices only through $z$, giving a contradiction to $G$ being distance hereditary.

Also, the set $\{z_1, z_2\}$ can not be a 2-vertex cut of $G$, since each vertex in $Z$ has at least two neighbours in $Y$. Thus, the reduced collection of possible 2-vertex-cuts of $G$ are $\{u, x\}$ and $\{y_1, y_2\}$.

Now, we prove that , if $\{u, x\}$ is a 2-vertex cut of $G$, then $u$ and $x$ are bisimilar vertices in $G$ and that $G$ has at most one 2-vertex-cut of this form. If $v$ and $w$ are any two vertices in $N_1(u)$ lying in different components of $G - \{u, x\}$, then $d(v, w) = 2$ in $G$ and hence in $G - u$. Thus, both $v$ and $w$ are adjacent to $x$. Hence every vertex in $N_1(u) - \{x\}$ is adjacent to $x$, and this $x$ is adjacent to no $y$ in $Y$(by definition of $X$). Hence $u$ and $x$ are bisimilar vertices in $G$. Also, suppose $G$ would contain two such distinct 2-vertex-cuts, say $W_1 = \{u, x_1\}$ and $W_2 = \{u, x_2\}$. Then every vertex in $N_1(u) - \{x_i\}$, for $i = 1, 2$, would be adjacent to both $x_1$ and $x_2$. Therefore, $G - W_1$ would be connected, giving a contradiction.

Similarly, we next prove that if $\{y_1, y_2\}$ is a 2-vertex cut of $G$, then $y_1$ and $y_2$ are bisimilar vertices in $G$. Since $G - \{u, y_1\}$ (respectively, $G - \{u, y_2\}$) is connected and $d(v, w) = 2$ for any two vertices $v, w \in N_1(u)$ lying in different components of $G - \{y_1, y_2\}$, it follows that each vertex in $N_1(u) - \{y_1, y_2\}$ is adjacent to the vertex $y_2$ (respectively, $y_1$). In $G - \{y_1, y_2\}$, the vertices in $X$ are connected to the vertices in $Y - \{y_1, y_2\}$ through $u$. Hence $G - \{y_1, y_2\}$ has a component $Z_1$

consisting of vertices only from $Z$. Clearly, no vertex in $Z_1$ is adjacent to a vertex in $Y-\{y_1,y_2\}$, and every vertex $z\in Z$ has at least two neighbours in $Y$. Therefore, every vertex in $Z_1$ is adjacent to both $y_1$ and $y_2$. Suppose there exists a vertex $z_1\in Z\backslash Z_1$ that is adjacent to one of $y_1$ and $y_2$, say $y_2$. Then $d(z_1,z)=2$ for all $z\in Z_1$. Since $G$ is distance hereditary, the connectedness of $G-y_2$ forces that $z_1$ must be adjacent to $y_1$. Thus, we conclude that if a vertex in $Z$ is adjacent to one of $y_1$ and $y_2$, then it must be adjacent to both $y_1$ and $y_2$. Hence $y_1$ and $y_2$ are bisimilar vertices in $G$.

Finally, we prove that any two 2-vertex-cuts in $Y$ of the form $\{y_1,y_2\}$ are disjoint. Suppose there exist two distinct 2-vertex-cuts $W_1$ and $W_2$ in $Y$ such that $W_1\cap W_2\neq\phi$ ; let $y'\in W_1\cap W_2$. Since $W_1$ and $W_2$ are vertex cuts, $G-W_1$ and $G-W_2$ have components $Z_1'$ and $Z_2'$, respectively, such that $Z_1',\ Z_2'\subseteq Z$ and $N_{G-Z_i'}(Z_i')=W_i$, for $i=1,2$. Choose $z_1\in Z_1,\ z_2\in Z_2,\ y_1\in W_1-y'$ and $y_2\in W_2-y'$. Now $z_1$ is not adjacent to both $z_2$ and $y_2$, and $z_2$ is not adjacent to $y_1$. Also, $y_1,\ y_2$ and $y'$ are mutually adjacent. Now the subgraph of $G$ induced by $\{y_1,y_2,y',z_1,z_2\}$ is isomorphic to the graph in Figure $1$ $(b)$, giving a contradiction to Theorem 1.

We now reconstruct $G$ from a card $G-v$ with a cut vertex. By the above argument, $G-v$ must contain a unique cut vertex, say $w$. Now all the graphs obtained by adding a new vertex to $G-v$ and joining it to all the neighbours of $w$ and also to the vertex $w$ (when $deg(v)\neq|N_{G-v}(w)|$ holds) are isomorphic and they are $G$.

***Theorem* 5** — *Distance hereditary graphs of connectivity* $3$ *and* $diam(G)=diam(\overline{G})=3$ *are reconstructible.*

PROOF : Consider $u,\ X,\ Y$ and $Z$ as in Theorem 4. The possible 3-vertex-cuts of $G$ are $\{x_1,x_2,x_3\}$, $\{x_1,x_2,y\}$, $\{x_1,x_2,z\}$, $\{x,y,z\}$, $\{x,y_1,y_2\}$, $\{x,z_1,z_2\}$, $\{u,x,y\}$, $\{u,y,y_1\}$, $\{u,y,z\}$, $\{y_1,y_2,z_1\}$, $\{y,z_1,z_2\}$, $\{z_1,z_2,z_3\}$, $\{u,x,z\}$, $\{u,z_1,z_2\}$, $\{u,x_1,x_2\}$, and $\{y_1,y_2,y_3\}$ (Figure 3). But, we prove that only the sets of the form $\{u,x_1,x_2\}$ and $\{y_1,y_2,y_3\}$ are the possible 3-vertex-cuts of $G$.

Every vertex in $Z$ has at least three neighbours in $Y$. For, suppose there is a vertex $z\in Z$ with at most two neighbours $y_1,y_2\in Y$. Then $d_{G-\{y_1,y_2\}}(u,z)\geq 3$, which is a contradiction to $e(u)=2$. Any 3-vertex-cut containing a vertex from $X$ must contain the vertex $u$, since each vertex in $X$ is connected to the other vertices by a path passing through $u$. Also, if $W$ is a 3-vertex-cut containing the vertex $u$, then all the vertices of $Z$ would lie in a single component of $G-W$ (as otherwise, $d_{G-(W-\{u\})}(z_1,z_2)$ would be at least 4 for any two vertices $z_1,z_2$ lying in different components of $G-W$, giving a contradiction to $G$ being distance hereditary).

Suppose that $W=\{u,x,y\}$(or $\{u,y,y_1\},\ \{u,y,z\}$) were a 3-vertex cut of $G$. Then, by the above argument, both $Z-W$ and $Y-W$ would lie in a single component of $G-W$. Since $d(v_1,v_2)\leq 2$ for any two vertices in $N_1(u)$, every vertex in $N_1(u)-W$ would be adjacent to all the vertices in $W\cap N_1(u)$. Now, since $W$ is a 3-vertex cut, $X-W$ and $Y-W$ would not lie in a single component of $G-W$. Also it is clear that, for each $z\in Z$, $|N_Y(z)|\geq 3$ and so $N_Z(y)\cap N_Z(y')\neq\phi$; let $z\in N_Z(y)\cap N_Z(y')$. Choose a vertex $x'$ in a component of $X-W$ containing no vertices of $Y$. Now the graph induced by $u,\ x',\ y,\ y'$ and $z$ would be isomorphic to the graph shown in Figure $1$ $(b)$, contradicting Theorem 1.

If $W=\{y_1,y_2,z_1\}$(or $\{y,z_1,z_2\}$) were a 3-vertex-cut of $G$, then $G-W$ would contain a component $Z_1$ consisting vertices only from $Z$ and so $G-(W-\{z_1\})$ would be connected and $d(u,z')>2$ for any vertex $z'\in Z_1$, giving a contradiction to $G$.

Suppose that $W=\{u,x,z\}$(or $\{u,z_1,z_2\}$) were a 3-vertex-cut of $G$. Since $W$ contains the vertex $u$, both $Z-W$ and $Y$ would lie in a single component of $G-W$ and hence the vertices in $X$ may lie in different components of $G-W$. Thus, the set $W-\{z\}$(or $W-\{z_1,z_2\}$) would possibly be a vertex cut of $G$, giving a contradiction to $\kappa(G)=3$.

Also, the set $\{z_1,z_2,z_3\}$ is not a 3-vertex-cut, since each vertex in $Z$ has at least three neighbours in $Y$. Thus, the only possible 3-vertex-cuts of $G$ are $\{u,x_1,x_2\}$ and $\{y_1,y_2,y_3\}$.

Now, if the set $\{u,x_1,x_2\}$ is a 3-vertex-cut of $G$, then, since $d(v_1,v_2)\leq 2$ for any two vertices $v_1$ and $v_2$ in $N_1(u)$, each vertex in $N_1(u)-\{x_1,x_2\}$ is adjacent to both $x_1$ and $x_2$, which implies they are bisimilar. Also, there can be at most one 3-vertex-cut $W$ such that $W\subseteq X\cup\{u\}$. For, if possible, suppose there were two 3-vertex-cuts $W_1$ and $W_2$ such that $W_1,W_2\subseteq X\cup\{u\}$. Then, every vertex in $N_1(u)-W_i$, for $i=1,2$, would be adjacent to the vertices in $W_1$ and $W_2$, which would imply $G-W_1$ is connected, a contradiction to $W_1$.

If $W=\{y_1,y_2,y_3\}$ is a 3-vertex cut of $G$, then, by the above argument, each vertex in $N_1(u)-\{y_1,y_2,y_3\}$ is adjacent to all the three vertices $y_1,y_2$ and $y_3$. Since $W$ is a 3-vertex cut of $G$, there exists a component $Z_1$ consisting vertices of $Z$ alone and no vertex in $Z_1$ is adjacent to a vertex in $Y-\{y_1,y_2,y_3\}$. Therefore, each $z\in Z_1$ is adjacent to $y_1,y_2$ and $y_3$. Suppose there exists $z\in Z-Z_1$ adjacent to some $y_i$, say $y_3$, then $d(z_1,z)=2$ for any $z_1\in Z_1$. Since $G-\{y_1,y_3\}$ is connected and $G$ is distance hereditary, vertex $z_1$ is adjacent to $y_2$. Similarly, vertex $z_1$ is adjacent to both $y_1$ and $y_2$. Thus, $N(y_1)=N(y_2)=N(y_3)$ in $G-\{y_1,y_2,y_3\}$. Moreover, any two 3-vertex cuts in $Y$ are disjoint (as otherwise, we would get Figure $1$ $(b)$ as an induced subgraph of $G$, again contradicting Theorem 1).

Now we reconstruct $G$ as follows. Among the cards with a 2-vertex cut, choose one, say $G-v$, such that $deg_G(v)$ is as maximum as possible. By the above arguments, $G-v$ must contain a unique 2-vertex cut, say $\{w_1, w_2\}$. Now all the graphs obtained by adding a new vertex to $G-v$ and joining it to all the common neighbours of $w_1, w_2$ and also either to one of $w_1$ and $w_2$ (if $d(v) = |N_{G-\{w_1,w_2\}}|+1$ holds) or to both $w_1$ and $w_2$ (otherwise), are isomorphic and they are $G$.

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