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CLASSIFICATION OF THE PENTAVALENT SYMMETRIC GRAPHS OF ORDER 18*P*

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A graph is symmetric if its automorphism group is transitive on the arc set of the graph. In this paper, we give a complete classification of connected pentavalent symmetric graphs of order $18p$, for each prime p. It is shown that, such graphs there exist if and only if $p = 2, 7$ or 19, and up to isomorphism, there are only four such graphs.

Key words : Pentavalent symmetric graph; connected graph; arc-transitive graph; automorphism group.

1. INTRODUCTION

Throughout this article graphs are assumed to be finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group of Γ, respectively. For u in $V(\Gamma)$, we let $N(u)$ is the set of vertices adjacent to u in Γ. Let G be a permutation group on a set Ω and let $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is semiregular on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$, and regular if G is transitive and semiregular.

An s-arc in a graph Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of Γ such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. A 1-arc just called an arc. A graph Γ is said to be (G, s)-arc-transitive or (G, s)-regular if G acts transitively or regularly on the set of s-arcs of Γ, and (G, s)-transitive if G acts transitively on the s-arcs but not on the $(s+1)$-arcs of Γ , where G is a subgroup of $\mathrm{Aut}(\Gamma)$. A graph Γ is said to be s-arc-transitive, s-regular or s-transitive if it is $(\mathrm{Aut}(\Gamma), s)$-arc transitive, $(\mathrm{Aut}(\Gamma), s)$-regular or $(\mathrm{Aut}(\Gamma), s)$-transitive, respectively. In particular, 0-arc-transitive means vertex-transitive and 1-arc-transitive means arc-transitive or symmetric. A graph Γ is edge-transitive if Aut(Γ) is transitive on the edge set $E(\Gamma)$.

Classification of various symmetric graphs with different orders have been studied extensively. For example, Alaeiyan and Hosseinipoor [1], have recently classified cubic symmetric graphs of orders $18p$. Moreover, there are many investigating the researches classifications of pentavalent symmetric graphs of orders $2pq$, $12p$, $4pq$, $30p$ and $2pqr$, where p, q and r are distinct primes [6-8, 11, 14, 20].

In this paper we shall classify pentavalent symmetric graphs of order $18p$ for a prime p.

2. Preliminaries

In this section, we introduce some notational conventions and preliminary results which will be used later. Throughout this paper, we will denote by $\mathbb{Z}_n$, F_n, D_{2n}, A_n and S_n the cyclic group of order n, the Frobenius group of order n, the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, respectively.

For a subgroup H of a group G, denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. Obviously $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.1 — [9, Chapter I, Theorem 4.5]. The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group Aut(H) of H.

Let Γ be a graph and let N be a subgroup of Aut(Γ). The quotient Γ_N of Γ relative to the orbits of N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there is an edge in Γ between those two orbits. In view of [12, Theorem 9], we have the following.

Proposition 2.2 — Let Γ be a connected pentavalent (G, s)-arc-transitive graph for some $s \geq 1$, and let N be a normal subgroup of G with more than two orbits on $V(\Gamma)$. Then Γ_N is also a pentavalent symmetric graph and N is the kernel of the action of G on $V(\Gamma_N)$. Moreover, N is semiregular on $V(\Gamma)$ and G/N is an s-arc-transitive subgroup of Aut(Γ_N).

Proposition 2.3 — [15, Theorem 8.5.3]. Let p and q be primes, and let m and n be non-negative integers. Then every group of order $p^m q^n$ is solvable.

By [8, Proposition 2.3] and [10], we can list all non-abelian simple groups of order $2^i \cdot 3^j \cdot 5 \cdot p$, where $1 \leq i \leq 19$ and $1 \leq j \leq 4$ in the Table 1.

In the next proposition, we can describe vertex stabilizers of connected pentavalent symmetric graphs.

Table 1: Non-abelian simple $\{2, 3, 5, p\}$-groups

3-prime factor		4-prime factor	
G	Order	G	Order
A_5	$2^2 \cdot 3 \cdot 5$	A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
A_6	$2^3 \cdot 3^2 \cdot 5$	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
PSL$(2, 7)$	$2^3 \cdot 3 \cdot 7$	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$
PSL$(2, 2^3)$	$2^3 \cdot 3^2 \cdot 7$	PSL$(2, 11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$
PSL$(2, 17)$	$2^4 \cdot 3^2 \cdot 17$	PSL$(2, 2^4)$	$2^4 \cdot 3 \cdot 5 \cdot 17$
PSL$(3, 3)$	$2^4 \cdot 3^3 \cdot 13$	PSL$(2, 19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$
PSU$(3, 3)$	$2^5 \cdot 3^3 \cdot 7$	PSL$(2, 31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$
PSU$(4, 2)$	$2^6 \cdot 3^4 \cdot 5$	PSL$(3, 4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
		$PSL(2, 3^4)$	$2^4 \cdot 3^4 \cdot 5 \cdot 41$
		M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
		M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
		$Sp_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$

Proposition 2.4 — [5, Theorem 1.1]. Let Γ be a connected pentavalent (G, s)-transitive graph for some $G \leqslant$Aut(Γ) and $s \geq 1$. Let $v \in V(\Gamma)$. Then $s \leq 5$ and one of the following holds:

(1) For $s = 1$, $G_v \cong \mathbb{Z}_5$, D_{10} or D_{20};

(2) For $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5;

(3) For $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$;

(4) For $s = 4$, $G_v \cong ASL(2, 4)$, $AGL(2, 4)$, $A\Sigma L(2, 4)$ or $A\Gamma L(2, 4)$;

(5) For $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma L(2, 4)$.

Let p be a prime integer. Let $V = \{0, 1, \ldots, p-1\}$ and $V' = \{0', 1', \ldots, (p-1)'\}$ be two disjoint copies of $\mathbb{Z}_p$. Let r be a positive integer dividing $p-1$ and $H(p, r)$ the unique subgroup of $\mathbb{Z}_p^*$ of order r. The graph $G(2p, r)$ has vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, r)\}$. By Cheng and Oxley [3], we have the following proposition.

Proposition 2.5 — Let Γ be a connected pentavalent symmetric graph of order $2p$ for a prime p. Then one of the following occurs:

(1) $\Gamma \cong K_6$, the complete graph of order 6, and Aut$(K_6) = S_6$;

(2) $\Gamma \cong K_{5,5}$, the complete bipartite graph of order 10, and Aut$(K_{5,5}) = (S_5 \times S_5) \rtimes \mathbb{Z}_2$;

(3) $\Gamma \cong G(2p, 5)$ with $p \equiv 1$ (mod 5). For $p > 11$, Aut$(G(2p, 5)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$, and for $p = 11$, Aut$(G(2p, 5)) =$ PSL$(2, 11) \rtimes \mathbb{Z}_2$.

In order to construct pentavalent symmetric graphs, by using of finite groups, we need to introduce the so called coset graph and Cayley graph.

Let G be a finite group and $H \leq G$. Suppose D is union of some double cosets of H in G such that $D^{-1} = D$. The coset graph $Cos(G, H, D)$ of G with respect to H and D is defined to have vertex set $[G : H]$, the set of right cosets of H in G, and edge set $\{\{Hg, Hdg\} \mid g \in G, d \in D\}$. The graph $Cos(G, H, D)$ has valency $|D|/|H|$ and is connected if and only if D generates the group G. Clearly, $Cos(G, H, D) \cong Cos(G, H^\alpha, D^\alpha)$ for each $\alpha \in$ Aut(G). Let S be a generator subset of G with $1 \notin S$ and $S = S^{-1}$. Clearly, the coset graph $\Gamma = Cos(G, 1, S)$ is a connected undirected simple graph, which is called a Cayley graph on G with respect to S and denoted by $Cay(G, S)$.

Proposition 2.6 — [8, Proposition 2.9]. Let Γ be a graph and let G be a vertex-transitive subgroup of Aut(Γ). Then Γ is isomorphic to a coset graph $Cos(G, H, D)$, where $H = G_u$ is the stabilizer of $u \in V(\Gamma)$ in G and D consists of all elements of G which map u to one of its neighbors. Further,

(1) Γ is connected if and only if D generates the group G;

(2) Γ is G-arc-transitive if and only if D is a single double coset. In particular, if $g \in G$ interchanges u and one of its neighbors, then $g^2 \in H$ and $D = HgH$;

(3) The valency of Γ equal to $|D|/|H| = |H : H \cap H^g|$.

Denote by $\mathbf{I}_{12}$ the Icosahedron graph, and by $K_{6,6} - 6K_2$ the complete bipartite graph of order 12 minus a one-factor.

Proposition 2.7 — [6, Proposition 3.2]. Let Γ be a connected pentavalent symmetric graph of order $6p$ for a prime p. Then one of the following occurs:

(1) $\Gamma \cong \mathbf{I}_{12}$ and Aut$(\Gamma) \cong A_5 \times \mathbb{Z}_2$ with $p = 2$;

(2) $\Gamma \cong K_{6,6} - 6K_2$ and Aut$(\Gamma) \cong S_5 \times \mathbb{Z}_2$ with $p = 2$;

(3) $\Gamma \cong \mathcal{G}_{42}$ and Aut$(\Gamma) \cong$Aut(PSL$(3, 4)$) with $p = 7$;

(4) $\Gamma \cong \mathcal{G}_{66}$ and Aut$(\Gamma) \cong$ PGL$(2, 11)$ with $p = 11$;

(5) $\Gamma \cong \mathcal{G}_{114}$ and $\mathrm{Aut}(\Gamma) \cong \mathrm{PGL}(2, 19)$ with $p = 19$;

Let G be a simple group and Z an abelian group. We call an extension E of Z by G a central extension of G if $Z \leq Z(E)$. If E is perfect, that is, the derived group $E^{'} = E$, we call E a covering group of G. Schur [16] proved that for every non-abelian simple group G there is a unique maximal covering group M such that every covering group of G is a factor group of M. This group M is called the full covering group of G, and the center of M is called the Schur multiplier of G, denoted by $\mathrm{Mult}(G)$.

Proposition 2.8 — [20, Lemma 2.8]. Let G be a group, and let N be an abelian normal subgroup of G such that $\mathrm{Aut}(N)$ is solvable. If G/N is a non-abelian simple group, then $G = G^{'}N$ and $G^{'} \cap N \lesssim \mathrm{Mult}(G/N)$.

3. Constructions

First, we introduce a pentavalent symmetric graph of order 36 which were constructed in [6].

Example 3.1 : Let $G = A_6$. By [6, Construction I], G has a subgroup $H \cong D_{10}$ and an element g of order 4 such that $|H : H \cap H^g| = 5$ and $< H, g > \;\; = G$. Denote by $\mathcal{G}_{36}$ the coset graph $Cos(G, H, HgH)$. By [6, Lemma 3.3], $\mathcal{G}_{36}$ is a pentavalent symmetric 2-transitive graph of order 36 and $\mathrm{Aut}(\mathcal{G}_{36}) \cong \mathrm{Aut}(A_6)$. Furthermore, every connected pentavalent symmetric graph of order 36 admitting A_6 as an arc-transitive automorphism group is isomorphic to $\mathcal{G}_{36}$.

Now, we construct several pentavalent symmetric graphs of order $18p$ for some prime p.

Example 3.2 : Let $G = A_9$. By Atlas [4, PP. 37], G has a maximal subgroup $H \cong (A_5 \times A_4) \rtimes \mathbb{Z}_2$ and an involution $g \in G$ such that $|HgH|/|H| = 5$ and $< H, g > \;\; = G$. Denote by $\mathcal{G}_{126}$ the coset graph $Cos(G, H, HgH)$.

Lemma 3.3 — Each connected pentavalent symmetric graph Γ of order 126 admitting A_9 as an arc-transitive automorphism group is isomorphic to $\mathcal{G}_{126}$. Moreover, Γ is 3-transitive and $Aut(\Gamma) \cong S_9$.

PROOF : Let $G = A_9$. As Γ is a G-arc-transitive graph of order 126, $G_u = 1440$ for any vertex $u \in V(\Gamma)$. So by Proposition 2.4, we have $G_u \cong (A_5 \times A_4) \rtimes \mathbb{Z}_2$. By Proposition 2.6, $\Gamma \cong Cos(G, G_u, G_u f G_u)$ for some 2-element $f \in G$ such that $f^2 \in G_u$, $< G_u, f >= G$ and $|G_u : G_u \cap G_u^f| = 5$. Since G has one conjugacy class of subgroup isomorphic to $(A_5 \times A_4) \rtimes \mathbb{Z}_2$, we may assume that $G_u = H$. Set $L = H \cap H^f$. So $f \in N_G(L)$. By Atlas [4], it is easy to see that $N_G(L) \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2^4$. As $N_G(L)/L \cong \mathbb{Z}_2$, we have $N_G(L) = L \cup Lg$, so $f \in Lg$. Thus

$HfH = HgH$, and hence $\Gamma \cong Cos(G, H, HfH) = Cos(G, H, HgH) = \mathcal{G}_{126}$. Moreover, by [2], $|Aut(\Gamma)| = 362880$. Thus $Aut(\Gamma) \cong S_9$. □

Example 3.4 : Let $G \leq T$ be two subgroups of S_{20} such that $G \cong \mathrm{PSL}(2,19)$ and $T \cong \mathrm{PGL}(2,19)$ and T contains the following elements:

$a_1 = (1\ 7)(2\ 9)(3\ 20)(4\ 5)(6\ 13)(8\ 14)(10\ 11)(12\ 15)(16\ 18)(17\ 19)$,

$a_2 = (1\ 9\ 5\ 18\ 3)(2\ 7\ 20\ 16\ 4)(6\ 10\ 17\ 12\ 14)(8\ 15\ 19\ 11\ 13)$,

$g = (1\ 17)(2\ 9)(3\ 16)(4\ 8)(5\ 19)(6\ 20)(7\ 15)(10\ 18)(11\ 12)(13\ 14)$,

$b_1 = (1\ 13)(2\ 18)(3\ 14)(4\ 5)(6\ 17)(7\ 19)(8\ 11)(9\ 12)(10\ 20)(15\ 16)$,

$b_2 = (1\ 11)(2\ 15)(3\ 12)(4\ 6)(5\ 17)(7\ 20)(8\ 13)(9\ 14)(10\ 19)(16\ 18)$,

$b_3 = (1\ 4\ 3\ 19\ 18)(2\ 7\ 14\ 5\ 13)(6\ 12\ 10\ 16\ 11)(8\ 15\ 20\ 9\ 17)$,

$g_1 = (1\ 6\ 5\ 9)(2\ 7\ 19\ 3)(4\ 11\ 14\ 17)(8\ 18\ 16\ 13)(10\ 20\ 15\ 12)$,

$g_2 = (1\ 14)(2\ 12)(3\ 10)(4\ 5)(6\ 11)(7\ 15)(9\ 17)(13\ 18)(19\ 20)$.

It is easy to see that $H_1 := < a_1, a_2 > \cong D_{10}$, $H_2 := < b_1, b_2, b_3 > \cong D_{20}$, $G = < a_1, a_2, g >$ and $T = < b_1, b_2, b_3, g_1 > = < b_1, b_2, b_3, g_2 >$. Define the following coset graphs:

$\mathcal{G}_{PSL(2,19)} = Cos(G, H_1, H_1gH_1)$,

$\mathcal{G}_{PGL(2,19)} = Cos(T, H_2, H_2g_1H_2) \cong Cos(T, H_2, H_2g_2H_2)$.

By Magma [2], the graphs $\mathcal{G}_{PSL(2,19)}$ and $\mathcal{G}_{PGL(2,19)}$ are non-isomorphic connected 1-transitive graphs of order 342 such that $\mathrm{Aut}(\mathcal{G}_{PSL(2,19)}) = \mathrm{PSL}(2,19)$ and $\mathrm{Aut}(\mathcal{G}_{PGL(2,19)}) = \mathrm{PGL}(2,19)$.

Lemma 3.5 — Each connected pentavalent symmetric graph Γ of order 342 admitting $\mathrm{PSL}(2,19)$ as an arc-transitive automorphism group with vertex stabilizer isomorphic to D_{10} is isomorphic to $\mathcal{G}_{PSL(2,19)}$.

PROOF : Let Γ be a connected pentavalent symmetric graph of order 342 admitting $G = \mathrm{PSL}(2,19)$ as an arc-transitive automorphism group. Then $|G_v| = 10$, for any $v \in V(\Gamma)$. By Proposition 2.4, we have $G_v \cong D_{10}$, so from Proposition 2.6, we can deduce that $\Gamma \cong Cos(G, G_v, G_vfG_v)$ for some 2-element $f \in G$ such that $f^2 \in G_v$, $< f, G_v > = G$ and $|G_v : G_v \cap G_v^f| = 5$. Since all subgroups of G isomorphic to D_{10} are conjugate in $\mathrm{Aut}(G)$, $G_v \cong H_1$, so $\Gamma \cong Cos(G, H_1, H_1fH_1)$.

As $|H_1 \cap H_1^f| = 2$ and G has one conjugacy class of involutions, we can take an involution $x \in H_1$ and set $H_1 \cap H_1^f = < x >$. Therefore $C_G(x) \cong D_{20}$. By Magma [2], there are four involution $f \in D_{20}$, such that $f^2 \in H_1$ and $< H_1, f > = G$. Since these involutions are conjugate in G, the coset graphs $Cos(G, H_1, H_1 f H_1)$ corresponding to the four involutions are isomorphic to each other. It follows that $\Gamma = Cos(G, H_1, H_1 f H_1) \cong \mathcal{G}_{PSL(2,19)}$. □

Lemma 3.6 — Each connected pentavalent symmetric graph Γ of order 342 admitting $\mathrm{PGL}(2, 19)$ as an arc-transitive automorphism group with vertex stabilizer isomorphic to D_{20} is isomorphic to $\mathcal{G}_{PGL(2,19)}$.

PROOF : Let Γ be a connected pentavalent symmetric graph of order 342 admitting $T = \mathrm{PGL}(2, 19)$ as an arc-transitive automorphism group. Then $|T_v| = 20$, for any $v \in V(\Gamma)$. By Proposition 2.4, we have $T_v \cong D_{20}$. From Proposition 2.6, we have $\Gamma \cong Cos(T, T_v, T_v f T_v)$ for some 2-element $f \in T$ such that $f^2 \in T_v$, $< f, T_v > = T$ and $|T_v : T_v \cap T_v^f| = 5$. Since all subgroups of T isomorphic to D_{20}, are conjugate in T we have $T_v \cong H_2$. Thus $\Gamma \cong Cos(T, H_2, H_2 f H_2)$. Note that $|T_v \cap T_v^f| = 4$, so $T_v \cap T_v^f = P$ is the Sylow 2-subgroup of D_{20}. Therefore $f \in N_T(P) \cong S_4$ and $|H_2 f H_2| = 100$. By Magma [2], there are eight choices for f such that some of them are conjugate in T. By Magma [2], there are two representatives g_1 and g_2 of these f, such that $Cos(T, H_2, H_2 g_1 H_2) \cong Cos(T, H_2, H_2 g_2 H_2)$. This implies that $\Gamma \cong \mathcal{G}_{PGL(2,19)}$. □

4. THE MAIN RESULT

In this section, we classify pentavalent symmetric graphs of order $18p$ for p a prime. The following theorem is the main result of this paper.

***Theorem* 4.1** — *Let Γ be a connected pentavalent symmetric graph of order $18p$ for a prime p. Then one of the following occurs:*

(1) Γ is 2-transitive, and $\Gamma \cong \mathcal{G}_{36}$ with Aut$(\Gamma) \cong$ Aut(A_6);

(2) Γ is 3-transitive, and $\Gamma \cong \mathcal{G}_{126}$ with Aut$(\Gamma) \cong S_9$.

(3) Γ is 1-transitive, and $\Gamma \cong \mathcal{G}_{PSL(2,19)}$ with Aut$(\Gamma) \cong$ PSL$(2, 19)$;

(4) Γ is 1-transitive, and $\Gamma \cong \mathcal{G}_{PGL(2,19)}$ with Aut$(\Gamma) \cong$ PGL$(2, 19)$.

PROOF : Let Γ be a connected pentavalent symmetric graph of order $18p$ and $A :=\mathrm{Aut}(\Gamma)$. By Weiss [17, 18], $|A_v| \big| 5 \cdot 3^2 \cdot 2^{17}$ for each $v \in V(\Gamma)$. Hence $|A| = 2^s \cdot 3^t \cdot 5 \cdot p$ with $1 \leq s \leq 18$ and $2 \leq t \leq 4$. From [19], there exists no connected pentavalent symmetric graph of order 54. So $p \neq 3$. We divide the proof into the following two cases.

Case I : A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then N is an elementary abelian q-group with $q = 2$ or p. It is easy to see that N has more than two orbits on $V(\Gamma)$, beacause otherwise $9p \big| |N|$, a contradiction. Therefore by Proposition 2.2, N is semiregular and the quotient graph Γ_N of Γ relative to N is a pentavalent symmetric graph with A/N as an arc-transitive group of automorphisms of Γ_N. As N is semiregular, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or $\mathbb{Z}_p$, when $p > 3$. If $N \cong \mathbb{Z}_2$, then Γ_N is a pentavalent graph of odd order $9p$, which is impossible. By [13], we know that there is no pentavalent graph of order 18, so N cannot be isomorphic to $\mathbb{Z}_p$. Assume that $N \cong \mathbb{Z}_3$. Then Γ_N is a pentavalent symmetric graph of order $6p$. By Proposition 2.7, $\Gamma_N \cong I_{12}$, $K_{6,6} - 6K_2$, $\mathcal{G}_{42}$, $\mathcal{G}_{66}$ or $\mathcal{G}_{114}$.

Let $\Gamma_N \cong I_{12}$. Then $A/N \lesssim \mathrm{Aut}(I_{12}) \cong A_5 \times \mathbb{Z}_2$. Since A/N is arc-transitive on Γ_N, we have $60 \big| |A/N|$. Thus, A/N contains an arc-transitive subgroup $H/N \cong A_5$. Set $C := C_H(N)$, the centralizer of N in H. Then $N \leq C$, and by Proposition 2.1, $H/C \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_2$. Since $C/N \leq H/N$ and $H/N \cong A_5$ is a non-abelian simple group, $C = H$, and hence N is in the center $Z(H)$ of H. As H is non-solvable, its derived subgroup $H^{'}$, is also non-solvable. Then $1 < H^{'}N/N \trianglelefteq H/N$, and hence $H^{'}N/N = H/N$. If $N \leq H^{'}$ then $H^{'} = H$, and hence H is a covering group of A_5. However, by [9, Chapter V, Theorem 25.7], the Schur multiplier of A_5 is $\mathbb{Z}_2$, a contradiction. Thus, $N \nleq H^{'}$. Then $N \cap H^{'} = 1$, and so $H = N \times H^{'}$ with $H^{'} \cong A_5$. If $H^{'}$ has more than two orbits on $V(\Gamma)$, then $H^{'}$ is semiregular and so $|H^{'}| = |v^{H^{'}}| \big| |V(\Gamma)|$, for each $v \in V(\Gamma)$. Thus by Proposition 2.3, $H^{'}$ is solvable, which is a contradiction. Hence $H^{'}$ has at most two orbits on $V(\Gamma)$. This implying that $18 \big| |H^{'}| = 60$, which is impossible.

Let $\Gamma_N \cong K_{6,6} - 6K_2$. Then $A/N \lesssim \mathrm{Aut}(\Gamma_N) = S_6 \times \mathbb{Z}_2$. Thus by Magma [2], A/N contains an arc-transitive subgroup $H/N \cong A_5 \times \mathbb{Z}_2$ or S_5. However H/N has a normal subgroup $M/N \cong A_5$. With a similar discussion as in the above paragraph, obtained $M = M^{'} \times N$ with $M^{'} = A_5$. By Proposition 2.2, M/N has at most two orbits on $V(\Gamma_N)$. If M/N be transitive on $V(\Gamma_N)$ then M is transitive on $V(\Gamma)$. So $|M_v| = 5$ and $M_v \cong \mathbb{Z}_5$. From Proposition 2.6, $\Gamma = Cos(M, M_v, M_v g M_v)$ where g is a 2-element in M such that $g^2 \in M_v$ and $< M_v, g > = M$. By Magma [2] there is no such $g \in M$, a contradiction. Therefore M/N has two orbits on $V(\Gamma_N)$ and this implies that $M^{'}$ has two orbits on $V(\Gamma)$. It means that $60 = |M^{'}| = |M_v^{'}| \times 18$, which is impossible.

Let $\Gamma_N \cong \mathcal{G}_{66}$. Then $A/N \lesssim \mathrm{Aut}(\Gamma_N) \cong \mathrm{PGL}(2, 11)$. Since A/N is arc-transitive on Γ_N, $66 \cdot 5 \big| |A/N|$. By Atlas [4, pp. 7], A/N has a normal subgroup $H/N \cong \mathrm{PSL}(2, 11)$. Since $\mathrm{Mult}(\mathrm{PSL}(2, 11)) = \mathbb{Z}_2$, we have $H \cong \mathbb{Z}_3 \times \mathrm{PSL}(2, 11)$ and $H^{'} \cong \mathrm{PSL}(2, 11)$. Let M be a subgroup of H such that $M \cong \mathrm{PSL}(2, 11)$. As $H^{'}$ is characteristic in H, M is a normal subgroup of A. Clearly M has at most two orbits on $V(\Gamma)$. It follows that $9 \big| |M| = 2^2 \cdot 3 \cdot 5 \cdot 11$, a contradiction.

Let $\Gamma_N \cong \mathcal{G}_{42}$. Then $A/N \lesssim$ Aut(Γ_N) $\cong$ Aut(PSL$(3,4)$) $\cong$ PSL$(3,4) \cdot D_{12}$. Since A/N is arc-transitive on Γ_N, $42 \cdot 5 \big| |A/N|$, A/N has a normal subgroup $M/N \cong$ PSL$(3,4)$, so $M \cong$ PSL$(3,4) \rtimes \mathbb{Z}_3$. Since $M/N \trianglelefteq A/N$ and $M^{'} =$ PSL$(3,4)$ is characteristic in M, we have PS$L(3,4) \trianglelefteq A$. By Proposition 2.2, $M^{'}$ has at most two orbits on $V(\Gamma)$. Therefore $|M^{'}_v| = 320$ or 160. Since PSL$(3,4)$ has no subgroups of order 320 by Magma [2], $|M^{'}_v| = 160$, and thus $M^{'}$ is transitive on $V(\Gamma)$. From Proposition 2.6, $\Gamma = Cos(M^{'}, M^{'}_v, M^{'}_v g M^{'}_v)$ where g is a 2-element in $M^{'}$ such that $g^2 \in M^{'}_v$ and $< M^{'}_v, g > = M^{'}$, for $v \in V(\Gamma)$. By Magma [2], there is no such $g \in M^{'}$, a contradiction.

Let $\Gamma_N \cong \mathcal{G}_{114}$. Then $A/N \lesssim$ Aut(Γ_N) $\cong$ PGL$(2,19)$. Since A/N is arc-transitive on Γ_N, $114 \cdot 5 \big| |A/N|$. By Atlas [4, pp. 11], A/N has a normal subgroup $H/N \cong$ PSL$(2,19)$. As Mult(PSL$(2,19)$) $= \mathbb{Z}_2$ and derived subgroup $H^{'} \cong$ PSL$(2,19)$, so $H = N \times$ PSL$(2,19)$. Since $H \trianglelefteq A$ and $H^{'} =$ PSL$(2,19)$ is characteristic in H, we have $H^{'} \trianglelefteq A$. Obviously that $H^{'}$ has at most two orbits on $V(\Gamma)$. If $H^{'}$ has two orbits, then $|H^{'}_v| = 30$. By Atlass [4], PSL$(2,19)$ has no subgroups of order 30. Therefore $H^{'}$ is transitive on $V(\Gamma)$ and $|H^{'}_v| = 10$. By Proposition 2.4, $H^{'}_v \cong D_{10}$. Since $H^{'}_v \trianglelefteq A_v$ for any $v \in V(\Gamma)$, by primitivity of A_v on $N(v)$ [15, Corollary 9.14], we have $H^{'}_v$ is transitive on $N(v)$. This implies that $H^{'}$ is transitive on $E(\Gamma)$. From [3, Proposition 1.2], $H^{'}$ is transitive on $A(\Gamma)$. Hence by Example 3.4 and Lemma 3.5, $\Gamma \cong \mathcal{G}_{PSL(2,19)}$.

Suppose now that $N \cong \mathbb{Z}_3^2$. Then Γ_N is a pentavalent symmetric graph of order $2p$. By Proposition 2.5, we have $\Gamma_N \cong K_{5,5}$, K_6 or $G(2p, 5)$ with $5 \big| p-1$. Concider M/N, as a normal minimal subgroup of A/N.

Let $\Gamma_N \cong K_{5,5}$. Then $p = 5$ and Aut$(K_{5,5}) = (S_5 \times S_5) \rtimes \mathbb{Z}_2$. Suppose that M/N is solvable. Then $M/N \cong \mathbb{Z}_2$, $\mathbb{Z}_5$ or $\mathbb{Z}_5^2$. If $M/N \cong \mathbb{Z}_2$ or $\mathbb{Z}_5$, then M has more than two orbits on $V(\Gamma)$ and so Γ_M is a pentavalent symmetric graph of odd order p and 2 respectively, which is impossible. If $M/N \cong \mathbb{Z}_5^2$, then $M \cong \mathbb{Z}_3^2 \times \mathbb{Z}_5^2$. It is easy to see that M has two orbits on $V(\Gamma)$ and $M_v \cong \mathbb{Z}_5$ for $v \in V(\Gamma)$. Since M is abelian, we have $\Gamma \cong 9K_{5,5}$ a union of nine copies of $K_{5,5}$, which contradicts the connectivity of Γ. Now assume that M/N is unsolvable. So $M/N \cong A_5$ or $A_5 \times A_5$. Obviously, M/N has at most two orbits on $V(\Gamma_N)$. we have $(M/N)_u \trianglelefteq (A/N)_u$ for any $u \in V(\Gamma_N)$ and $(A/N)_u$ is primitive on $N(u)$. So $5 \big| |(M/N)_u|$, and hence $25 \big| |M/N|$. Therefore $M/N \cong A_5 \times A_5$. If M/N is transitive on $V(\Gamma_N)$, then M/N has two subgroups T, L such that $T \cong L \cong A_5$ and $(M/N)_u = T_u \times L_u$. Thus $5^2 \big| |(M/N)_u|$ and $5^3 \big| |N|$, a contradiction. It follows that M/N has two orbits on $V(\Gamma_N)$. Let $B/N \cong A_5$ and $B/N \trianglelefteq M/N$. Similarly as above, B/N has two orbits on $V(\Gamma_N)$ and $5 \big| |(B/N)_u|$. Thus $25 \big| |B/N|$, a contradiction.

Let $\Gamma_N \cong K_6$. Then $A/N \lesssim$ Aut$(K_6) \cong S_6$. Since A/N is arc-transitive on Γ_N, $6 \cdot 5 \big| |A/N|$, so A/N is isomorphic to A_5, S_5, A_6 or S_6. Suppose that $A/N \cong A_5$ or A_6. By Proposition 2.8, $A = N \cdot T$ is a central extension of N by T, where $T = A_5$ or A_6. Since Mult$(T) = \mathbb{Z}_2$ or $\mathbb{Z}_6$, $A^{'} = T \trianglelefteq A$. If $A^{'}$ has at least three orbits on $V(\Gamma)$, then $|A^{'}|\big|18p$. It follows that $4\big|18p$ or $5\big|18p$ where $p \neq 2$ and $p \neq 5$ respectively. However, it is impossible. Thus $A^{'}$ has at most two orbits on $V(\Gamma)$, so $9\big||A^{'}|$. It means that $A^{'} \cong A_6$ and $A^{'}$ is not semiregular on $V(\Gamma)$. Since $A^{'} \trianglelefteq A$, $(A_6)_v \trianglelefteq A_v$ for any $v \in V(\Gamma)$. By the primitivity of A_v on $N(v)$, one has $(A_6)_v$ is transitive on $N(v)$, so $5\big||(A_6)_v|$. Therefore we have $|(A_6)_v| = 10$ or 20. From Atlas [4], we know that A_6 has no subgroup of order 20, so $|(A_6)_v| = 10$. From Example 3.1, $(A_6)_v \cong D_{10}$ and $\Gamma \cong \mathcal{G}_{36}$. Now assume that $A/N \cong S_5$ or S_6. In this case A/N has a normal subgroup $M/N \cong A_5$ or A_6. Similarly, M is a central extension of N by T, where $T = A_5$ or A_6, and $M^{'} = T$ which is normal in A. By the above argument, we obtain $\Gamma \cong \mathcal{G}_{36}$.

Let $\Gamma_N \cong G(2p, 5)$, $p > 11$. Then by Proposition 2.5, Aut$(G(2p,5)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_5) \rtimes \mathbb{Z}_2$. Thus Aut$(\Gamma_N)$ has a normal Sylow p-subgroup PN/N. It follows that $P \trianglelefteq A$ because P is characteristic in PN, which is impossible because A has no normal subgroup of order p. Now, let $p = 11$, then Aut$(\Gamma_N) \cong$ PSL$(2, 11) \rtimes \mathbb{Z}_2$. If M/N be solvable, then $M/N \cong \mathbb{Z}_2$ or $\mathbb{Z}_p$ beacause $|V(\Gamma_N)| = 2p$. This means that Γ_M is a pentavalent graph of order p or 2, a contradiction. If M/N be nonsolvable, then $M/N \cong$ PSL$(2, 11)$. Denote by $M^{'}$ the derived group of M. By Atlas [4], Mult(PSL$(2, 11)) = \mathbb{Z}_2$, so $M = M^{'} \times N$ with $M^{'} =$ PSL$(2, 11)$. Since $M^{'}$ is characteristic in M, we have $M^{'} \trianglelefteq A$. By Proposition 2.2, $M^{'}$ has at most two orbits on $V(\Gamma)$. So $9\big||M^{'}|$, which is impossible.

Case II : A has no solvable minimal normal subgroups.

For convenience, we still use N to denote a minimal normal subgroup of A. Then N is unsolvable. In this case, there is a non-abelian simple group T and a positive integer m such that $N \cong T^m$. By Proposition 2.2, N has at most two orbits on $V(\Gamma)$. Hence $|N| = 18p|N_v|$ or $9p|N_v|$ and $|N|$ has at least three distinct prime factors.

If $p = 2$, then $|V(\Gamma)| = 36$ and $|N| = 2^m \cdot 3^m \cdot 5^m$. But $|A|$ is not divisible by 5^2, so $m = 1$ and N is simple. By Table 1, $N \cong A_6$ or $PSU(4, 2)$. Suppose $N \cong PSU(4, 2)$. Since N has at most two orbits, we have $|N_v| = 2^4 \cdot 3^2 \cdot 5$ or $2^5 \cdot 3^2 \cdot 5$. By Atlas [4, pp. 26], $N_v \cong S_6$. This is impossible, because S_6 cannot have a permutation representation of degree 5. Thus $N \cong A_6$. If A_6 has two orbits on $V(\Gamma)$, then $|N_v| = 20$. By Atlas [4], we know that A_6 has no subgroup of order 20. Thus A_6 is transitive on $V(\Gamma)$ and so $(A_6)_v \cong D_{10}$. By primitivity of $(A_6)_v$ on $N(v)$ we can conclude that N is arc-transitive on Γ. Therefore by Example 3.1, $\Gamma \cong G_{36}$.

Let $p = 5$. Note that $5\big||N_v|$, beacause $N_v \trianglelefteq A_v$ and A_v is primitve on $N(v)$. Therefore $25\big||N|$.

From Table 1, there are not any simple group with this property.

Assume now that $p > 5$. Set $C := C_A(N)$. We claim that $C = 1$. If not, N is a non-abelian and simple group, so $C \cap N = 1$. We know that A has only one element of order p and $p \mid |N|$, thus $p \nmid |C|$. It follows that C has at least p orbits on $V(\Gamma)$, hence C is semiregular on $V(\Gamma)$. By Proposition 2.3, C is solvable which is contrary to our assumption. Hence $C = 1$. From Proposition 2.1, we have $A \cong A/C \lesssim \mathrm{Aut}(N)$, that is, A is an almost simple group. As N has at most two orbits on $V(\Gamma)$, we have $p \big| |N|$. Since N is unsolvable, N is not semiregular on $V(\Gamma)$ and $N_v \neq 1$ for each $v \in V(\Gamma)$. By primitivity of A_v on $N(v)$, we have $5 \big| |N_v|$. Therefore, N is a non-abelian simple $\{2, 3, 5, p\}$-group. By Table 1, N is isomorphic to one of the following groups:

A_7, A_8, A_9, $\mathrm{PSL}(2, 19)$, $\mathrm{PSL}(3, 4)$, $\mathrm{PSL}(2, 3^4)$, M_{11}, M_{12}, $\mathrm{Sp}_6(2)$.

Recall that N has at most two orbits on $V(\Gamma)$ and $|N_v| = 9p$ or $18p$.

Suppose $N \cong A_7$. If N has two orbits on $V(\Gamma)$, then $|N_v| = 40$. By Magma [2], A_7 has no subgroups of order 40. Hence N is transitive on $V(\Gamma)$ and $|N_v| = 20$. From Proposition 2.6, $\Gamma \cong Cos(N, N_v, D)$, where $g \in N$ such that $g^2 \in N_v$, $D = N_v g N_v$ and $|N_v : N_v \cap N_v^g| = 5$. By Magma [2], there are four choices for g. But $< N_v, g >$ is a proper subgroup of N which is contrary to the fact that Γ is connected.

Suppose that $N \cong A_8$. Then $p = 7$ and $|N_v| = 2^5 \cdot 5$ or $2^6 \cdot 5$. However, by Magma [2], A_8 has no subgroups of order $2^5 \cdot 5$ or $2^6 \cdot 5$. By checking the subgroups of M_{11} and M_{12} by Magma [2], we can conclude that $N \neq M_{11}$ and M_{12}.

Suppose $N \cong \mathrm{PSL}(3, 4)$. If N has two orbits on $V(\Gamma)$, then $|N_v| = 2^6 \cdot 5$. By Magma [2], $\mathrm{PSL}(3, 4)$ has no subgroups of order $2^6 \cdot 5$. Hence N is transitive on $V(\Gamma)$ and $|N_v| = 2^5 \cdot 5$. From Proposition 2.6, $\Gamma \cong Cos(N, N_v, D)$, where g is a 2-element in N such that $g^2 \in N_v$ and $N_v \cap N_v^g = P$ where P is the Sylow 2-subgroup of N_v. Thus $g \in N_N(P)$ and $< N_v, g >= N$. By Magma [2], there exists no 2-element $g \in N_N(P)$ such that $< N_v, g >= N$, a contradiction.

Let $N \cong Sp_6(2)$. As N is almost simple, $A = Aut(N) = N$ and N is transitive on $V(\Gamma)$. So $|N_v| = 2^8 \cdot 3^2 \cdot 5$. By Atlas [4], $N_v \cong S_6 \rtimes \mathbb{Z}_{2^5}$. From Proposition 2.6, $\Gamma \cong Cos(N, N_v, D)$, where g is a 2-element in N such that $g^2 \in N_v$ and $|N : N_v \cap N_v^g| = |D|/|N_v| = 5$. Thus $N_v \cap N_v^g \leqslant N$ and $|N_v \cap N_v^g| = 2^8 \cdot 3^2$. By Magma [2], N_v has no subgroups of this order, a contradiction.

Let $N \cong PSL(2, 3^4)$. If N is transitive on $V(\Gamma)$, then $|N_v| = 360$. By Proposition 2.6, $\Gamma \cong Cos(N, N_v, D)$, where g is a 2-element in N such that $g^2 \in N_v$ and $|N : N_v \cap N_v^g| = |D|/|N_v| = 5$. Thus $N_v \cap N_v^g \leqslant N$ and $|N_v \cap N_v^g| = 2^8 \cdot 3^2$. By Magma [2], N_v has no subgroups of this order, a contradiction. Therefore, N has two orbits on $V(\Gamma)$ and $Aut(\Gamma)$ is transitive on $V(\Gamma)$. By a similar argument, we can conclude that $Aut(\Gamma)$ isn't transitive on $V(\Gamma)$ and hence $N \neq PSL(2, 3^4)$.

Now assume that $N \cong A_9$. If N has two orbits on $V(\Gamma)$, then $|N_v| = 2^6 \cdot 3^2 \cdot 5$. By Atlas [4, PP. 71], A_9 has no subgroups of order $2^6 \cdot 3^2 \cdot 5$. Hence N is transitive on $V(\Gamma)$ and $|N_v| = 2^5 \cdot 3^2 \cdot 5$. By Proposition 2.4, $N_v \cong (A_4 \times A_5) \rtimes \mathbb{Z}_2$. Since $N_v \trianglelefteq A_v$ for any $v \in V(\Gamma)$, by primitivity of A_v on $N(v)$, we have N_v is transitive on $N(v)$ and therefore N is transitive on $A(\Gamma)$. Thus by Example 3.2 and Lemma 3.3, $\Gamma \cong \mathcal{G}_{126}$.

Finally, Suppose that $N \cong \mathrm{PSL}(2, 19)$. If N is transitive on $V(\Gamma)$, then $|N_v| = 10$, for $v \in V(\Gamma)$ and by Proposition 2.4, $N_v \cong D_{10}$. By a similar argument as in the above paragraph, N is arc-transitive. So by Example 3.4 and Lemma 3.5, $\Gamma \cong \mathcal{G}_{PSL(2,19)}$.

If N has two orbits on $V(\Gamma)$, then $|N_v| = 20$ and Γ is bipartite. Recall that A is almost simple and $N < A \leq \mathrm{Aut}(N)$. Since $\mathrm{Aut}(N) \cong \mathrm{PGL}(2, 19)$, we have $A \cong \mathrm{PGL}(2, 11)$. By Example 3.4 and Lemma 3.6, $\Gamma \cong \mathcal{G}_{PGL(2,19)}$. This completes the proof of the main theorem. □

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