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ON SUMS OF RANGE SYMMETRIC MATRICES WITH REFERENCE TO INDEFINITE INNER PRODUCT

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We give necessary and sufficient condition for the sums of range symmetric matrices to be range symmetric with reference to indefinite inner product. As an application it is shown that the sum and parallel sum of parallel summable range symmetric matrices are range symmetric.

Key words : Indefinite inner product; EP matrix; Range symmetric matrix.

1. Introduction

An indefinite inner product in $\mathbb{C}_J^n$ is a conjugate symmetric sesquilinear form $[x, y]$ together with the regularity condition that $[x, y] = 0, \ \forall y \in \mathbb{C}_J^n$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] = \langle x, Jy \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$. We also make an additional assumption on J, that is, $J^2 = I$, to present the results with much algebraic ease. A complex matrix A is said to be range symmetric if and only if the range space of A and that of its conjugate transpose A^* are equal.

Investigations of linear maps on indefinite inner product employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors [2, 4]. This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [4]. More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is defined to be the matrix $J_nA^*J_m$, where J_m and J_n are weights.

Many properties of this product are similar to that of the usual matrix product [4]. Moreover, it not only rectifies the difficulty indicated earlier, but also enables us to recover some interesting results with reference to indefinite inner product in a manner analagous to that of the Euclidean case. Kamaraj, Ramanathan and Sivakumar also established in [4] that in the setting of indefinite inner product spaces, the indefinite matrix product is more appropriate that the usual matrix product. Recall that the Moore-Penrose inverse exists if and only if $rank(AA^*) = rank(A^*A) = rank(A)$. If we take $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then $AA^{[*]}$ and $A^{[*]}A$ are both the zero matrix and so $rank(AA^{[*]}) < rank(A)$, thereby proving that the Moore-Penrose inverse does not exist with respect to the usual matrix product. However, it can be easily verified that with respect to the indefinite matrix product, $rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A) = rank(A)$. Thus, the Moore-Penrose J-inverse with real or complex entries exists over an indefinite inner product, whereas a similar result is false with respect to the usual matrix multiplication. It is therefore really pertinant to extend the study of generalized inverses to the setting of indefinite inner product.

In this paper we give necessary and sufficient condition for the sums of range symmetric matrices to be range symmetric with reference to indefinite inner product. As an application it is shown that the sum and parallel sum of parallel summable range symmetric matrices are range symmetric.

2. Preliminaries

We first recall the notion of an indefinite multiplication of matrices.

Definition 1 — [3]. Let $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$, $B \in \mathbb{C}^{n\times k}_{J_n,J_k}$. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of A and B (relative to J_n) is defined by $A \circ B = AJ_nB$.

Definition 2 — [3]. For $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$, $A^{[*]} = J_nA^*J_m$ is the adjoint of A relative to J_n and J_m.

Definition 3 — [3]. A matrix $A \in \mathbb{C}^{n\times n}_{J_n}$ is said to be J-invertible if there exists $X \in \mathbb{C}^{n\times n}_{J_n}$, such that $A \circ X = X \circ A = J_n$ such an X is denoted by $A^{[-1]} = JA^{-1}J$.

Remark 1 : [3]. For the identity matrix J, it reduces to a generalized inverse of A and $A_J\{1\} = A\{1\}$. It can be easily verified that X is a generalized inverse of A under the indefinite matrix product if and only if J_nXJ_m is a generalized inverse of A under the usual product of matrices. Hence $A_J\{1\} = \{X : J_nXJ_m$ is a generalized inverse of A.

Definition 4 — [3]. For $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$, and $X \in \mathbb{C}^{n\times m}_{J_n,J_m}$ is called the Moore - Penrose J-inverse of A if it satisfies the following equations:

(i) $A \circ X \circ A = A.$ ($\{1\}\, inverse$)

(ii) $X \circ A \circ X = X.$ ($\{2\}\, inverse$)

(iii) $(A \circ X)^{[*]} = A \circ X.$ ($\{3\}\, inverse$)

(iv) $(X \circ A)^{[*]} = X \circ A.$ ($\{4\}\, inverse$)

such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_n A^{\dagger} J_m$.

Definition 5 — [3]. The range space $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$ is defined by $Ra(A) = \{y = A \circ x \in \mathbb{C}^m : x \in \mathbb{C}^n\}$. The null space of $A \in \mathbb{C}^{m\times n}_{J_m,J_n}$ is defined by $Nu(A) = \{x \in \mathbb{C}^n : A \circ x = 0\}$.

Property 1 : [3]. Let $A \in \mathbb{C}^{m\times n}_{J_n}$. Then

(i) $(A^{[*]})^{[*]} = A$.

(ii) $(A^{[\dagger]})^{[\dagger]} = A$.

(iii) $(AB)^{[*]} = B^{[*]}A^{[*]}$.

(iv) $Ra(A^{[*]}) = Ra(A^{[\dagger]})$.

(v) $Ra(A \circ A^{[*]}) = Ra(A),\ Ra(A^{[*]} \circ A) = Ra(A^{[*]})$.

(vi) $Nu(A \circ A^{[*]}) = Nu(A^{[*]}),\ Nu(A^{[*]} \circ A) = Nu(A)$.

Definition 6 — [3]. A is range symmetric in $\mathbb{C}^{n\times n}_J$ if and only if $Ra(A) = Ra(A^{[*]})$ (or) equivalently $Nu(A) = Nu(A^{[*]})$.

Remark 2 : [3]. In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix.

***Theorem* 1** — [3]. *For $A \in C^{n\times n}_{J_n}$, the following are equivalent*:

(i) *A is range symmetric in $\mathbb{C}^{n\times n}_J$*.

(ii) *AJ is EP*.

(iii) *JA is EP*.

(iv) $Nu(A) = Nu(A^{[*]})$.

(v) $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.

(vi) $(A^{\dagger}A)^{[*]} = JA^{\dagger}AJ = AA^{\dagger}$.

(vii) *A is J-EP*.

3. On Sums of Range Symmetric Matrices

In [3], the concept of a range symmetric matrix with reference to indefinite inner product introduced and developed. In this section, conditions are obtained for sums of range symmetric matrices to be range symmetric with reference to indefinite inner product.

Lemma 1 — Let $A_1, A_2, \cdots, A_m \in C_{J_n}^{n\times n}$. If $A = \sum_{i=1}^{m} A_i$, then $A^{[*]} = \sum_{i=1}^{m} A_i^{[*]}$.

Proof : By Definition $A_i^{[*]} = JA_i^*J$, for $i = 1, 2, \cdots, n$, where J is the weight matrix of order n.

Given $A = \sum_{i=1}^{m} A_i$

$$\begin{aligned} A^{[*]} &= J \sum_{i=1}^{m} A_i^* \ J \\ &= J(A_1^* + A_2^* + \cdots + A_m^*)J \\ &= JA_1^*J + JA_2^*J + \cdots + JA_m^*J \\ &= A_1^{[*]} + A_2^{[*]} + \cdots + A_m^{[*]} \\ &= \sum_{i=1}^{m} A_i^{[*]}. \end{aligned}$$

Example 1 : Let $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If $A = \sum_{i=1}^{2} A_i$, then $A^{[*]} = \sum_{i=1}^{2} A_i^{[*]}$.

Lemma 2 — Let $A_1, A_2, \cdots, A_m \in \mathbb{C}_{J_n}^{n\times n}$ and let $A = \sum_{i=1}^{m} A_i$. Consider the following conditions:

(a) $Nu(A) \subseteq Nu(A_i)$ for $i = 1, 2, \cdots, m$

(b) $Nu(A) = \bigcap_{i=1}^{m} Nu(A_i)$

(c) $rank(A) = rank \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$

(d) $\sum_{i=1}^{m} \sum_{j=1}^{m} A_i^{[*]} \circ A_j = 0$

(e) $rank(A) = \sum_{i=1}^{m} rank(A_i)$.

Then the following statements hold:

(i) Condition $(a),\ (b)$ and (c) are equivalent.

(ii) Condition (d) implies (a), but the converse may or may not hold.

(iii) Condition (e) implies (a), but condition (a) does not imply (e).

PROOF : (i) $(a) \Leftrightarrow (b) \Leftrightarrow (c)$:

Since, $Nu(A) \subseteq Nu(A_i)$, for each i, we know that, $Nu(A) = Nu(\sum A_i) \supseteq Nu(A_1) \cap Nu(A_2) \cdots \cap Nu(A_m) = \bigcap Nu(A_i)$ implies $Nu(A) \supseteq \bigcap Nu(A_i)$. Also $Nu(A) \subseteq \bigcap Nu(A_i)$. Therefore $Nu(A) = \bigcap Nu(A_i)$ and (b) holds.

Now, $Nu(A) = \bigcap\limits_{i=1}^{m} Nu(A_i) = Nu \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$. Hence, $rank(A) = rank \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}$. Thus (c) holds.

Conversely, Since $rank \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} = rank(A)$

$Nu \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} = \bigcap\limits_{i=1}^{m} Nu(A_i) \subseteq Nu(A) \Rightarrow Nu(A) = \bigcap\limits_{i=1}^{m} Nu(A_i)$, hence (b) holds.

Further, $Nu(A) \subseteq Nu(A_i)$ for each i, and therefore (a) holds.

(ii) $(d) \Rightarrow (a)$:

Since $\sum\limits_{i=1}^{m} \sum\limits_{j=1}^{m} A_i^{[*]} \circ A_j = 0$,

$$A^{[*]} \circ A = (A_1 + A_2 + \cdots + A_m)^{[*]} \circ (A_1 + A_2 + \cdots + A_m) = (\sum_{j=1}^{m} A_j)^{[*]} \circ (\sum_{i=1}^{m} A_i) = \sum_{i=1}^{m} \sum_{j=1}^{m} J A_j^* A_i$$

$$\begin{aligned} Nu(A) &= Nu(A^{[*]} \circ A) \\ &= Nu(\sum_{i=1}^{m} \sum_{j=1}^{m} J A_j^* A_i) \\ &= Nu \left[J \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}^* \begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix} \right] \qquad \text{(since } Nu(A_1 A_2) = Nu(A_2)) \end{aligned}$$

$$= Nu\begin{pmatrix} A_1 \\ \vdots \\ A_m \end{pmatrix}$$

$$= Nu(A_1) \cap Nu(A_2) \cap \cdots \cap Nu(A_m)$$

$$= \bigcap_{i=1}^{m} Nu(A_i).$$

Thus $Nu(A) \subseteq Nu(A_i)$ for each i and hence (a) holds.

The converse of $(a) \Rightarrow (d)$ may or may not hold follows from the given examples below:

Example 2 : Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $A_1^{[*]} \circ A_2 + A_2^{[*]} \circ A_1 = 0$.

Example 3 : Consider the example

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_1 + A_2 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. Here, $Nu(A_1 + A_2) \subseteq Nu(A_1)$ and $Nu(A_1 + A_2) \subseteq Nu(A_2)$. But $A_1^{[*]} \circ A_2 + A_2^{[*]} \circ A_1 \neq 0$.

(iii) $(e) \Rightarrow (a)$:

Since $rank(A) = \sum rank(A_i)$, $Ra(A_i) \cap Ra(A_j) = 0, i \neq j$, implies $Nu(A) \subseteq Nu(A_i)$ for each i and so (a) holds.

$(a) \not\Rightarrow (e)$:

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. Clearly, $Nu(A_1 + A_2) \subseteq Nu(A_1)$ and $Nu(A_1 + A_2) \subseteq Nu(A_2)$. But $rank(A_1 + A_2) \neq rank(A_1) + rank(A_2)$.

***Theorem* 2** — *Let A_i be range symmetric in $\mathbb{C}_J^{n\times n}$. If any one of the conditions of Lemma 2 holds, then $A = \sum_{i=1}^{m} A_i$ is range symmetric in $\mathbb{C}_J^{n\times n}$.*

PROOF : Since each A_i is range symmetric in $\mathbb{C}_J^{n\times n}$, by Definition 6, $Nu(A_i) = Nu(A_i^{[*]})$ for each $i = 1, 2, \cdots, m$. By the given condition $Nu(A) \subseteq Nu(A_i)$.

We get $Nu(A) \subseteq \bigcap_{i=1}^{m} Nu(A_i) = \bigcap_{i=1}^{m} Nu(A_i^{[*]})$.

$$\begin{aligned} \text{Now, } x \in \bigcap_{i=1}^{m} Nu(A_i^{[*]}) &\Rightarrow x \in Nu(A_i^{[*]}), \text{ for } i = 1, 2, \cdots, m \\ &\Rightarrow A_i^{[*]} \circ x = 0 \\ &\Rightarrow (A_1^{[*]} + A_2^{[*]} + \cdots + A_m^{[*]}) \circ x = 0 \\ &\Rightarrow (A^{[*]}) \circ x = 0 \qquad \text{(By Lemma 1)} \\ &\Rightarrow x \in Nu(A^{[*]}) \end{aligned}$$

implies $\bigcap_{i=1}^{m} Nu(A_i^{[*]}) \subseteq Nu(A^{[*]})$.

Therefore $Nu(A) \subseteq \bigcap_{i=1}^{m} Nu(A_i^{[*]}) \subseteq Nu(A^{[*]})$ and $rank(A) = rank(A^{[*]})$.

Hence $Nu(A) = Nu(A^{[*]})$. Thus A is range symmetric in $\mathbb{C}_J^{n\times n}$.

Remark 3 : The converse of the Theorem 2 is not true. For $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_1 + A_2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. $A_1,\ A_2,\ A_1 + A_2$ are range symmetric in $\mathbb{C}_J^{n\times n}$ but $Nu(A_1 + A_2) \nsubseteq Nu(A_1)$ and $Nu(A_1 + A_2) \nsubseteq Nu(A_2)$.

Remark 4 : If A_1 and A_2 are range symmetric matrices then by Theorem 1, $A_1^* = H_1JA_1J$ and $A_2^* = H_2JA_2J$ where H_1 and H_2 are non-singular $n \times n$ matrices.

If $H_1 = H_2$, then $A_1^* + A_2^* = H_1JA_1J + H_2JA_2J = H_1JA_1J + H_1JA_2J = H_1J(A_1 + A_2)J$ $A_1 + A_2$ is range symmetric in $\mathbb{C}_J^{n\times n}$.

If $H_1 - H_2$ is non-singular then the above conditions are also necessary for the sum of range symmetric to be range symmetric in $\mathbb{C}_J^{n\times n}$.

***Theorem* 3** — *Let A_1 and A_2 be range symmetric in $\mathbb{C}_J^{n\times n}$. $A_1^* = H_1JA_1J$ and $A_2^* = H_2JA_2J$ such that $H_1 - H_2$ is a non-singular matrix. Then $A_1 + A_2$ is range symmetric in $\mathbb{C}_J^{n\times n}$ iff $Nu(A_1 + A_2) \subseteq Nu(A_i)$ for some (and hence both) $i \in \{1, 2\}$.*

PROOF : Since $A_1^* = H_1JA_1J$ and $A_2^* = H_2JA_2J$, $Nu(A_1 + A_2) \subseteq Nu(A_2)$ we can see that $Nu(A_1 + A_2) \subseteq Nu(A_1)$. Hence by Theorem 2 $(A_1 + A_2)$ is range symmetric in $\mathbb{C}_J^{n\times n}$. Conversely, let us assume that $(A_1 + A_2)$ is range symmetric $\mathbb{C}_J^{n\times n}$. By Remark 4, there exists a non-singular matrix G such that

$$
\begin{aligned}
&(A_1 + A_2)^* = GJ(A_1 + A_2)J \\
&\Rightarrow A_1^* + A_2^* = GJ(A_1 + A_2)J \\
&\Rightarrow H_1JA_1J + H_2JA_2J = GJ(A_1 + A_2)J \\
&\Rightarrow (H_1JA_1 + H_2JA_2)J = GJ(A_1 + A_2)J \\
&\Rightarrow H_1JA_1 + H_2JA_2 = GJA_1 + GJA_2 \\
&\Rightarrow (H_1J - GJ)A_1 = (GJ - H_2J)A_2 \\
&\Rightarrow (H_1 - G)JA_1 = (G - H_2)JA_2 \\
&\Rightarrow LJA_1 = MJA_2,
\end{aligned}
$$

where $L = H_1 - G,\ M = G - H_2$.

Now
$$
\begin{aligned}
(L + M)(JA_1) &= LJA_1 + MJA_1 \\
&= MJA_2 + MJA_1 \\
&= MJ(A_1 + A_2)
\end{aligned}
$$
and $(L + M)(JA_2) = LJ(A_1 + A_2)$.

By hypothesis, $(L + M) = H_1 - G + G - H_2 = H_1 - H_2$ is non-singular.

Therefore, $Nu(A_1+A_2) \subseteq Nu(MJ(A_1+A_2)) = Nu((L+M)JA_1) = Nu(JA_1) = Nu(A_1)$.

Therefore, $Nu(A_1 + A_2) \subseteq Nu(A_1)$. Similarly, $Nu(A_1 + A_2) \subseteq Nu(A_2)$. Hence the theorem.

4. Parallel Summable Range Symmetric Matrices

In the section we show that sum and parallel sum of parallel summable range symmetric matrices are range symmetric. First we shall give the definition and some properties of parallel summable matrices as in (P.188, [5]).

Definition 7 — [5]. A_1 and A_2 are said to be parallel summable if $N(A_1 + A_2) \subseteq N(A_1)$ and $N((A_1 + A_2)^*) \subseteq N(A_1^*)$.

Definition 8 — [5]. If A_1 and A_2 are parallel summable then parallel sum of A_1 and A_2 denoted by $A_1 \pm A_2$ is defined as $A_1 \pm A_2 = A_1(A_1 + A_2)^- A_2$. The product $A_1(A_1 + A_2)^- A_2$ is invariant for all choices of generalized inverse $(A_1 + A_2)^-$of $(A_1 + A_2)$ under the conditions that A_1 and A_2 are parallel summable (p.188, [5]).

Lemma 3 — Let A_1 and A_2 be matrices in $\mathbb{C}_J^{n \times n}$ then $N(A_1^*) \subseteq N(A_2^*)$ if and only if $Nu(A_1^{[*]}) \subseteq Nu(A_2^{[*]})$.

Proof : Let us assume that $N(A_1^*) \subseteq N(A_2^*)$

We need to prove : $Nu(A_1^{[*]}) \subseteq Nu(A_2^{[*]})$

Let us choose $x \in Nu(A_1^{[*]})$

$\Rightarrow A_1^{[*]} \circ x = 0$

$\Rightarrow JA_1^* x = 0$

$\Rightarrow A_1^* x = 0$

$\Rightarrow x \in N(A_1^*) \subseteq N(A_2^*)$

$\Rightarrow A_2^* x = 0$

$\Rightarrow JA_2^* x = 0$

$\Rightarrow A_2^{[*]} \circ x = 0$

$\Rightarrow x \in Nu(A_2^{[*]})$

Thus $N(A_1^{[*]}) \subseteq Nu(A_2^{[*]})$

Conversely, Let us assume that $Nu(A_1^{[*]}) \subseteq Nu(A_2^{[*]})$. We need to prove : $N(A_1^*) \subseteq N(A_2^*)$

Let us choose $x \in N(A_1^*)$

$\Rightarrow A_1^* x = 0$

$\Rightarrow JA_1^* x = 0$

$\Rightarrow JA_1^* JJx = 0$, where $J^2 = I$

$\Rightarrow A_1^{[*]} \circ x = 0$

$\Rightarrow x \in Nu(A_1^{[*]}) \subseteq Nu(A_2^{[*]})$

$\Rightarrow A_2^{[*]} \circ x = 0$

$\Rightarrow JA_2^* x = 0$

$\Rightarrow A_2^* x = 0$

$\Rightarrow x \in N(A_2^*)$

Thus $N(A_1^*) \subseteq N(A_2^*)$.

By using Lemma 3, Definition 7 can be modified as follows.

Definition 9 — A pair of matrices A_1 and A_2 are said to be parallel summable in $\mathbb{C}_J^{n\times n}$ if $N(A_1 + A_2) \subseteq N(A_2)$ and $Nu((A_1 + A_2)^{[*]}) \subseteq Nu(A_2^{[*]})$ or equivalently $N(A_1 + A_2) \subseteq N(A_1)$ and $Nu((A_1 + A_2)^{[*]}) \subseteq Nu(A_1^{[*]})$.

Property 2 : Let A_1 and A_2 be a pair of parallel summable matrices in $\mathbb{C}_J^{n\times n}$. Then the following hold:

P.1. $(A_1 \pm A_2)^{[*]} = (A_2 \pm A_1)^{[*]}$

P.2. $A_1^{[*]}$ and $A_2^{[*]}$ are parallel summable and $(A_1 \pm A_2)^{[*]} = A_1^{[*]} \pm A_2^{[*]}$.

PROOF :

$$
\begin{aligned}
P.1: \quad (A_1 \pm A_2)^{[*]} &= (A_1(A_1+A_2)^{-}A_2)^{[*]} && \text{(By Definition 8)}\\
&= (A_1(A_1+A_2)^{-}A_2 + A_1(A_1+A_2)^{-}A_1 - A_1(A_1+A_2)^{-}A_1)^{[*]}\\
&= (A_1(A_1+A_2)^{-}(A_1+A_2) - A_1(A_1+A_2)^{-}A_1)^{[*]}\\
&= (A_1 - A_1(A_1+A_2)^{-}A_1)^{[*]}\\
(A_1 \pm A_2)^{[*]} &= A_1^{[*]} - (A_1(A_1+A_2)^{-}A_1)^{[*]} && (1)\\
(A_2 \pm A_1)^{[*]} &= (A_2(A_2+A_1)^{-}A_1)^{[*]} && \text{(By Definition 8)}\\
&= (A_2(A_1+A_2)^{-}A_1 + A_1(A_1+A_2)^{-}A_1 - A_1(A_1+A_2)^{-}A_1)^{[*]}\\
&= ((A_1+A_2)(A_1+A_2)^{-}A_1 - A_1(A_1+A_2)^{-}A_1)^{[*]}\\
&= (A_1 - A_1(A_1+A_2)^{-}A_1)^{[*]}\\
(A_2 \pm A_1)^{[*]} &= A_1^{[*]} - (A_1(A_1+A_2)^{-}A_1)^{[*]} && (2)
\end{aligned}
$$

Combining equations (1) and (2), we get $(A_1 \pm A_2)^{[*]} = (A_2 \pm A_1)^{[*]}$.

P.2: A_1 and A_2 are parallel summable it is obvious that $A_1^{[*]}$ and $A_2^{[*]}$ are parallel summable

$$
\begin{aligned}
(A_1 \pm A_2)^{[*]} &= (A_2 \pm A_1)^{[*]} && \text{(By P.1)}\\
&= (A_2(A_1+A_2)^{-}A_1)^{[*]} && \text{(By Definition 8)}\\
&= A_1^{[*]}((A_1+A_2)^{-})^{[*]}A_2^{[*]}\\
&= A_1^{[*]}((A_1+A_2)^{[*]})^{-}A_2^{[*]}\\
&= A_1^{[*]}(A_1^{[*]}+A_2^{[*]})^{-}A_2^{[*]}\\
(A_1 \pm A_2)^{[*]} &= A_1^{[*]} \pm A_2^{[*]} && \text{(By Definition 8)}
\end{aligned}
$$

Lemma 4 — Let A_1 and A_2 be range symmetric matrices in $\mathbb{C}_J^{n\times n}$. Then A_1 and A_2 are parallel summable range symmetric in $\mathbb{C}_J^{n\times n}$ if and only if $Nu(A_1+A_2) \subseteq Nu(A_i)$ for some (and hence both) $i \in \{1, 2\}$.

PROOF : A_1 and A_2 are parallel summable by Definition 7, it follows that $Nu(A_1+A_2) \subseteq Nu(A_i)$, $i \in \{1, 2\}$.

Conversely, since $Nu(A_1+A_2) \subseteq Nu(A_1)$ and $Nu(A_1+A_2) \subseteq Nu(A_2)$, under the condition that A_1 and A_2 are range symmetric in $\mathbb{C}_J^{n\times n}$, by Theorem 2 A_1+A_2 is range symmetric in $\mathbb{C}_J^{n\times n}$. Hence $Nu(A_1+A_2) = Nu((A_1+A_2)^{[*]})$ and $Nu(A_1+A_2) \subseteq Nu(A_1)$ implies $Nu((A_1+A_2)^{[*]}) \subseteq Nu(A_1^{[*]})$, follows that A_1 and A_2 are range symmetric in $\mathbb{C}_J^{n\times n}$ by Definition 9.

Remark 5 : Lemma 4 fails if we relax the condition that A_1 and A_2 are parallel summable range symmetric in $\mathbb{C}_J^{n\times n}$. Let $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then A_1 is range symmetric in $\mathbb{C}_J^{n\times n}$, A_2 is not range symmetric in $\mathbb{C}_J^{n\times n}$. $A_1^{[*]} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A_2^{[*]} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $(A_1 + A_2) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $(A_1 + A_2)^{[*]} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. $Nu(A_1 + A_2) \subseteq Nu(A_1)$ and $Nu(A_2)$ but $Nu((A_1 + A_2)^{[*]}) \not\subseteq Nu(A_1^{[*]})$ and $Nu(A_2^{[*]})$. Hence A_1 and A_2 are not parallel summable.

***Theorem* 4** — *Let A_1 and A_2 are parallel summable range symmetric in $\mathbb{C}_J^{n\times n}$. Then $(A_1 \pm A_2)$ and $(A_1 + A_2)$ are range symmetric in $\mathbb{C}_J^{n\times n}$.*

PROOF : Since A_1 and A_2 be parallel summable range symmetric in $\mathbb{C}_J^{n\times n}$, by Lemma 4, $Nu(A_1 + A_2) \subseteq Nu(A_1)$ and $Nu(A_1 + A_2) \subseteq Nu(A_2)$. Now the fact that $(A_1 + A_2)$ is range symmetric in $\mathbb{C}_J^{n\times n}$ follows from Theorem 2. $A_1 \pm A_2$ is range symmetric in $\mathbb{C}_J^{n\times n}$ runs as follows

$$\begin{aligned}
Nu((A_1 \pm A_2)^{[*]}) &= Nu((A_2 \pm A_1)^{[*]}) && (\text{By Property 2 (P.1)})\\
&= Nu((A_2(A_2 + A_1)^{-}A_1)^{[*]})\\
&= Nu((A_2(A_2^{-} + A_1^{-})A_1)^{[*]})\\
&= Nu((A_2A_2^{-}A_1 + A_2A_1^{-}A_1)^{[*]})\\
&= Nu((A_1 + A_2)^{[*]})\\
&= Nu(A_1 + A_2) && (\text{By Theorem 3})\\
&= Nu(A_2A_2^{-}A_1 + A_2A_1^{-}A_1)\\
&= Nu(A_2(A_2^{-} + A_1^{-})A_1)\\
&= Nu(A_2(A_2 + A_1)^{-}A_1)\\
&= Nu(A_2 \pm A_1)\\
&= Nu(A_1 \pm A_2).
\end{aligned}$$

Thus $A_1 \pm A_2$ is range symmetric in $\mathbb{C}_J^{n\times n}$ whenever A_1 and A_2 are parallel summable range symmetric in $\mathbb{C}_J^{n\times n}$.

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