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THE EXISTENCE OF WEAK SOLUTIONS FOR A NONLOCAL CAHN-HILLIARD EQUATION WITH DEGENERATE MOBILITY[1]

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The aim of this paper is to study the existence of weak solutions for a nonlocal Cahn-Hilliard equations with degenerate mobility. Based on the uniform Schauder estimates and using the method of continuity, we obtain the existence of classical solutions for non-degenerate problems. After establishing some necessary uniform estimates on the approximate solutions, we prove the existence of weak solutions.

Key words : Nonlocal Cahn-Hilliard equation; degenerate mobility; Existence of weak solutions.

1. INTRODUCTION

In this paper, we investigate the nonlocal Cahn-Hilliard equations with degenerate mobility

$$u_t = \text{div}(|u|^m \nabla \mu), \quad \text{in} \quad \Omega \times (0, T), \tag{1.1}$$

where $m > 0$, $\Omega \subset \mathbb{R}^N$ is a bounded domain and $T > 0$ is a time horizon. u is the order parameter (corresponding to a density of atoms). The chemical potential μ is the first variation of the nonlocal free energy functional (see [6, 17, 18, 26])

$$\chi(u) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)(u(x)-u(y))^2 dxdy + \int_\Omega F(u(x))dx, \tag{1.2}$$

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where $J : \mathbb{R}^N \to \mathbb{R}$ is a smooth function such that $J(x) = J(-x)$, F is bistable. We do not assume that J is nonnegative but make its integral positive. Taking the first variation of χ we can define the chemical potential associated with the nonlocal model (see [6, 8, 17-19])

$$\mu = \int_\Omega J(x-y)dyu(x) - \int_\Omega J(x-y)u(y)dy + F'(u), \tag{1.3}$$

where $x \in \Omega$.

Eq. (1.1) is supplemented by the boundary value condition

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{1.4}$$

The boundary value conditions (1.4) is reasonable for the nonlocal Cahn-Hilliard equation (see [5]), and the initial value condition

$$u(x,0) = u_0(x). \tag{1.5}$$

Now, we consider the interdiffusion of two components A and B in thermodynamics. According to the linear phenomenological relations of thermodynamics for forces and fluxes (see [8, 10, 30]).

$$u_t - \text{div}(m(u)\nabla\mu) = 0, \quad \text{in } \Omega \times (0,T). \tag{1.6}$$

Taking $m(u) = |u|^m, m > 0$, we then obtain an expression form as it shown in equaiton (1.1).

In fact, (1.1) models the evolution of phase separation process, to which a two-phase material (for example, a binary alloy or a mixture), occupying the domain Ω, is subject. In this connection, the variable u, usually referred to as order parameter, stands for the local concentration of one of the two components.

In (1.2), making the approximation

$$u(x) - u(y) \simeq \nabla u(x)(x-y),$$

and assuming J to be isotropic, the Eq. (1.1) leads to

$$u_t - \text{div}(|u|^m(\nabla\Delta u + F'(u))) = 0, \quad \text{in} \quad \Omega \times (0,T), \tag{1.7}$$

which is a classical Cahn-Hilliard equation with u-dependent mobility (see [8, 23-25, 30, 31]). For example, in [27] Otto *et al*. study an equation similar to (1.7) with u-dependet mobility

$$u_t - \text{div}(u(\nabla\Delta u + F'(u))) = 0,$$

which is modelling the thin layers of viscous liquid. Besides, one of the examples is that Cahn-Hilliard equation thanks the form

$$u_t + \frac{1}{3} D(u^3(\gamma_1 D^3 u + F'(u))) = 0,$$

which describes the behavior of the spreading of an oil film when surface tension is included, see in [4] and [28] Chap. 4. In this case, $m(u) = \frac{1}{3}s^3$ with one point of degeneracy at $u = 0$. In fact, we replace $|u|^m$ with a constant in (1.7), then we get the classical Cahn-Hilliard equation (see [2, 3, 11]).

In (1.1), let $|u|^m = 1$, we get the classical nonlocal Cahn-Hilliard equation

$$u_t - \Delta(\int_\Omega J(x-y)dyu - \int_\Omega J(x-y)u(y)dy + F'(u)) = 0, \quad \text{in} \quad \Omega \times (0,T). \tag{1.8}$$

During the past years, many authors have paid much attention to the equation (1.8). Bates and Han considered the equation (1.8) in [5] and [6]. On the one hand, they studied the existence, uniqueness and continuous dependence on initial data of the solution for a nonlocal Cahn-Hilliard eqaution with Dirichlet boundary condition on a bounded domain. Then, they proved that there existed a global attractor in some metric space. On the other hand, they proved the existence, uniqueness and continuous dependence on initial data of the solution for the Neumann boundary condition, and they studied the long-term behavior of the solution in L^∞ norm and in H^1 norm. It is worth noting that they gived a stronger condition to study a degenerate problem in [5], and get some result. In this article, however, a wider degenerate problem which contains the degenerate problem in [5] is considered.

In [20], Gal and Grasselli proved that the dynamical system which is associated with the equation (1.8) has an exponential attractor, provided that the potential is regular. In addition to this, there are some results related to the nonlocal problem ([7, 13-16, 21, 26]).

Because of the degeneracy, the equation (1.1) does not admit classical solutions in general. Hence, we introduce the weak solutions in the following sense.

Definition 1.1 — A function $u \in L^\infty(Q_T), |u|^{\frac{m}{2}}|\nabla u| \in L^2(Q_T)$, $u_t \in L^2(Q_T)$, is said to be a weak solution of the problem (1.1), (1.4) and (1.5), if for all $\psi \in C^1(\overline{Q}_T)$ such that $\psi(x,T) = 0$, and $\psi = 0$ on $\partial\Omega \times (0,T)$ satisfies the following:

$$\begin{aligned} &\iint_{Q_T} (u\psi_t - m(u,\nabla u)(\int_\Omega J(x-y)dy + F''(u))\nabla u \nabla\psi)dxdt \\ &+ \int_\Omega u_0\psi(x,0)dx - \iint_{Q_T} m(u,\nabla u)\left(\nabla a(x)u - (\nabla J) * u\right)\nabla\psi dxdt = 0, \end{aligned} \tag{1.9}$$

and

$$u|_{\partial\Omega\times(0,T)} = 0 \text{ in the sence of traces.} \tag{1.10}$$

In this article, we need the following two propositions.

Proposition 1.1 — (See [9], p.7). There exists a constant γ depending only upon N, p, q, such that $v \in L^q(\Omega) \cap W_0^{1,p}(\Omega)$, we have

$$\iint_{Q_T} |v(x,t)|^h dxdt \leq \quad \gamma \left(\iint_{Q_T} |\nabla v(x,t)|^p dxdt \right) \left(\text{ess} \sup_{0<t<T} \int_\Omega |v(x,t)|^q dx \right)^{p/N},$$

where $h = \frac{p(q+N)}{N}$.

Proposition 1.2 — (See [9], p.12). Let $\{Y_n\}, n = 0, 1, 2, \cdots$, be a sequence of positive numbers, satisfying the recursive inequalities

$$Y_{n+1} \leq Bb^n Y_n^{1+\beta},$$

where $B, b > 1$ and $\beta > 0$ are given numbers. If

$$Y_0 \leq B^{-1/\beta} b^{-1/\beta^2}.$$

Then, $\{Y_n\}$ converges to zero as $n \to \infty$.

Throughout this paper, we make the following assumptions

(A_1) $J \in C^{2+\alpha}(\mathbb{R}^N), F \in C^{3+\alpha}(\mathbb{R})$ for some $\alpha > 0$.

(A_2) There exist $c_1, c_2 > 0$ and $r \geq m$ such that $h(x,u) \equiv a(x) + F''(u) \geq c_1|u|^r + c_2$, where $a(x) = \int_\Omega J(x-y)dy$.

(A_3) $\partial\Omega$ is of class $C^{2+\alpha}$.

(A_4) There exist $c_3 > \frac{1}{2}||J||^2_{L^2(\Omega)}|\Omega| + \frac{1}{2}$ and $c_4 \in \mathbb{R}$ such that

$$F(s) \geq c_3 s^2 - c_4, \quad \forall s \in \mathbb{R}.$$

In this article, we discuss the Dirichlet boundary problem (1.1),(1.4) and (1.5).

To this problem, comparison principle lose efficacy due to the nonlocal term, so it is difficult to get L^∞ norm of the solutions. To overcome these difficulties, we will first consider the non-degenerate problems. Based on the uniform Schauder estimates and using the method of continuity, we obtain the existence of classical solutions for non-degenerate problems. After applying weighted inequality, Sobolev embedding theorem and iteration lemma (Proposition 1.2) to establish some necessary uniform estimates on the approximate solutions, we prove the existence of weak solutions for degenerate problem.

This paper is arranged as follows. We first consider the existence of classical solutions for the regularized problem corresponding to the Dirichlet boundary problem (1.1), (1.4) and (1.5) in Section 2. And then, we give the prove of the global existence of the solutions for the degenerate problem (1.1), (1.4) and (1.5) in Section 3.

2. REGULARIZED PROBLEM

To discuss the existence, we adopt the method of parabolic regularization, namely, the desired solution will be obtained as the limit of some subsequence of solutions of the following regularized problem

$$\begin{cases} \dfrac{du_\varepsilon}{dt} = \text{div}(m_\varepsilon(u_\varepsilon)\nabla(H(u_\varepsilon) + F'(u_\varepsilon))), \quad \text{in } \Omega \times (0,T), \\ u_\varepsilon = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u_\varepsilon = u_0, \quad \text{in } \Omega \times \{0\}, \end{cases} \tag{2.1}$$

where $m_\varepsilon(u_\varepsilon) = (u_\varepsilon^2 + \varepsilon^{\frac{2}{m}})^{\frac{m}{2}}$.

***Theorem* 2.1** — *Suppose conditions (A_1)-(A_3) hold, $u_0 \in C^{2+\alpha}(\bar{\Omega})$ with compact support. Then there exists a solution $u_\varepsilon(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ of the equation (2.1).*

In order to prove Theorem 2.1, we first establish a priori estimate for the solutions of the problem (2.1).

Proposition 2.1 — Assume (A_1)-(A_3). If $u_\varepsilon(x,t) \in C(\overline{Q}_T) \cap C^{2,1}(Q_T)$ is a solution of (2.1), then

$$\|u_\varepsilon\|^2 \le C, \tag{2.2}$$

for some positive constant $C(m, T, \Omega, u_0)$.

PROOF : Multiplying (2.1) by u_ε and integrating over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 dx + \int_\Omega m_\varepsilon(u_\varepsilon) h(x, u_\varepsilon)|\nabla u_\varepsilon|^2 dx \\ &= -\int_\Omega m_\varepsilon(u_\varepsilon) u_\varepsilon \nabla a(x) \nabla u_\varepsilon dx + \int_\Omega m_\varepsilon(u_\varepsilon)(\nabla J) * u_\varepsilon \cdot \nabla u_\varepsilon dx. \end{aligned} \tag{2.3}$$

By condition (A_1) and (A_2), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 dx + c_1 \int_\Omega |u_\varepsilon|^{m+r} |\nabla u_\varepsilon|^2 dx + c_2 \int_\Omega |u_\varepsilon|^m |\nabla u_\varepsilon|^2 dx$$

$$
\begin{aligned}
&+c_1\varepsilon\int_\Omega |u_\varepsilon|^r|\nabla u_\varepsilon|^2dx+c_2\varepsilon\int_\Omega|\nabla u_\varepsilon|^2dx\\
\le& C\int_\Omega |u_\varepsilon|^{m+1}|\nabla u_\varepsilon|dx+C\int_\Omega\int_\Omega \nabla J(x-y)u(y)|u_\varepsilon(x)|^m|\nabla u_\varepsilon(x)|dxdy\\
&+C\varepsilon\int_\Omega|u_\varepsilon||\nabla u_\varepsilon|dx+C\varepsilon\int_\Omega\int_\Omega\nabla J(x-y)u_\varepsilon(y)|\nabla u_\varepsilon(x)|dxdy, \qquad (2.4)
\end{aligned}
$$

where $C>0$ is independent of ε. In this proof, C may change from line to line even if in the same inequality.

From the proof of Proposition 4.5 in [5] (4.38-4.43), we deduce,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|u_\varepsilon|^2dx+c_1\int_\Omega|u_\varepsilon|^{m+r}|\nabla u_\varepsilon|^2dx+c_2\int_\Omega|u_\varepsilon|^m|\nabla u|^2dx\\
&+\frac{4c_1\varepsilon}{(r+2)^2}\int_\Omega|\nabla|u_\varepsilon|^{\frac{r+2}{2}}|^2dx+\frac{c_2\varepsilon}{2}\int_\Omega|\nabla u_\varepsilon|^2dx\\
\le& C\int_\Omega|u_\varepsilon|^{m+1}|\nabla u_\varepsilon|dx+C\int_\Omega\int_\Omega\nabla J(x-y)u(y)|u_\varepsilon(x)|^m|\nabla u_\varepsilon(x)|dxdy\\
&+C\varepsilon\int_\Omega|u_\varepsilon|^2dx. \qquad (2.5)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|u_\varepsilon|^2dx+\frac{4c_1}{(m+r+2)^2}\int_\Omega|\nabla|u_\varepsilon|^{\frac{m+r+2}{2}}|^2dx\\
&+\frac{4c_2}{(m+2)^2}\int_\Omega|\nabla|u_\varepsilon|^{\frac{m+2}{2}}|^2dx+\frac{c_2\varepsilon}{2}\int_\Omega|\nabla u_\varepsilon|^2dx\\
&+\frac{4c_1\varepsilon}{(r+2)^2}\int_\Omega|\nabla|u_\varepsilon|^{\frac{r+2}{2}}|^2dx\\
\le& C\int_\Omega\int_\Omega\nabla J(x-y)u_\varepsilon(y)|u_\varepsilon(x)|^m|\nabla u(x)|dxdy\\
&+C\int_\Omega|u_\varepsilon|^{m+1}|\nabla u_\varepsilon|dx+C\varepsilon\int_\Omega|u_\varepsilon|^2dx. \qquad (2.6)
\end{aligned}
$$

For $\int_\Omega |u|^{m+1}|\nabla u|dx$, if $r-m\le 2$, using Hölder's and Young's inequalities and condition (A_2), we obtain

$$
\begin{aligned}
&C\int_\Omega|u_\varepsilon|^{m+1}|\nabla u_\varepsilon|dx\\
\le& C\int_\Omega|\nabla u_\varepsilon|\cdot|u_\varepsilon|^{\frac{m+r}{2}}\cdot|u_\varepsilon|^{\frac{m-r+2}{2}}dx\\
\le&\frac{c_1}{(m+r+2)^2}\int_\Omega|\nabla|u_\varepsilon|^{\frac{m+r+2}{2}}|^2dx+C\int_\Omega|u_\varepsilon|^{m-r+2}dx; \qquad (2.7)
\end{aligned}
$$

if $r-m>2$, by Hölder's and Young's inequalities and condition (A_2), we get

$$\begin{aligned}
&C\int_\Omega |u_\varepsilon|^{m+1}|\nabla u_\varepsilon|dx\\
\le&\frac{c_1}{(m+r+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+r+2}{2}}|^2dx+pC\int_\Omega |u_\varepsilon|^{m+1-\frac{b(r-1)}{2-b}}\cdot|\nabla u_\varepsilon|^{\frac{2(1-b)}{2-b}}dx\\
\le&\frac{c_1}{(m+r+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+r+2}{2}}|^2dx+\frac{c_2}{(m+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+2}{2}}|^2dx\\
&+C\int_\Omega |u_\varepsilon|^2dx, \qquad (2.8)
\end{aligned}$$

where $b=\frac{m}{r}$.

Applying Hölder's and Young's inequalities and condition (A_2), we have

$$\begin{aligned}
&\int_\Omega\int_\Omega \nabla J(x-y)u_\varepsilon(y)|u_\varepsilon(x)|^m|\nabla u_\varepsilon(x)|dxdy\\
\le&\frac{c_1}{(m+r+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+r+2}{2}}|^2dx\\
&+C\int_\Omega(\int_\Omega |\nabla J(x-y)u_\varepsilon(y)|dy)^{\frac{2}{2-b}}|u_\varepsilon(x)|^{m-\frac{br}{2-b}}|\nabla u_\varepsilon(x)|^{\frac{2(1-b)}{2-b}}dx\\
\le&\frac{c_1}{(m+r+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+r+2}{2}}|^2dx+\frac{c_2}{(m+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+2}{2}}|^2dx\\
&+C\int_\Omega(\int_\Omega \nabla|J(x-y)u_\varepsilon(y)|dy)^2\\
\le&\frac{c_1}{(m+r+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+r+2}{2}}|^2dx+\frac{c_2}{(m+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+2}{2}}|^2dx\\
&+C\int_\Omega |u_\varepsilon|^2dx, \qquad (2.9)
\end{aligned}$$

where $b=\frac{m}{r}$.

Note that $\varepsilon<1$, thanks to inequalities (2.6)-(2.9) and condition (A_2), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2dx+\frac{c_1}{(m+2)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+2}{2}}|^2dx\le C\int_\Omega |u_\varepsilon|^2dx. \qquad (2.10)$$

By Gronwall's inequality, we obtain

$$\sup_{0\le t\le T}\int_\Omega |u_\varepsilon|^2dx\le C(m,T,\Omega,u_0). \qquad (2.11)$$

The prove is completed. □

Next, we rewrite the equation (2.1) as

$$\begin{aligned}\frac{du_\varepsilon}{dt} =& mm'_\varepsilon(u_\varepsilon)((a(x)+F''(u_\varepsilon))(\nabla u_\varepsilon)^2+\nabla a(x)\cdot\nabla u_\varepsilon u_\varepsilon-(\nabla J)*u_\varepsilon\cdot\nabla u_\varepsilon)\\ &+m_\varepsilon(u_\varepsilon)(a(x)+F''(u_\varepsilon))\Delta u_\varepsilon+2m_\varepsilon(u_\varepsilon)\nabla a(x)\cdot\nabla u_\varepsilon\\ &+m_\varepsilon(u_\varepsilon)F^{(3)}(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon+m_\varepsilon(u_\varepsilon)u_\varepsilon\Delta a(x)\\ &-m_\varepsilon(u_\varepsilon)(\Delta J)*u_\varepsilon, \end{aligned}\tag{2.12}$$

where

$$\begin{aligned}(\nabla J)*u_\varepsilon &\equiv \int_\Omega \nabla J(x-y)u_\varepsilon(y)dy,\\ (\Delta J)*u_\varepsilon &\equiv \int_\Omega \Delta J(x-y)u_\varepsilon(y)dy.\end{aligned}$$

We need the following proposition.

Proposition 2.2 — Assume (A_1)-(A_3). If $u_\varepsilon(x,t)\in C(\overline{Q}_T)\cap C^{2,1}(Q_T)$ is a solution of (2.12), then

$$\max_{\overline{Q}_T}|u_\varepsilon|\le C(m,\Omega,T,u_0),\tag{2.13}$$

for some positive constant $C(m,T,\Omega,u_0)$.

PROOF : Set $u_\varepsilon(x,t)=v_\varepsilon e^{\sigma t}$, where σ is to be determined. Then $\nabla u_\varepsilon=e^{\sigma t}\nabla v_\varepsilon,\Delta u_\varepsilon= e^{\sigma t}\Delta v_\varepsilon$, and (2.12) becomes

$$\begin{aligned}e^{\sigma t}v_{\varepsilon t}+v_\varepsilon e^{\sigma t}\sigma =& mm'_\varepsilon(u_\varepsilon)h(x,u_\varepsilon)e^{2\sigma t}(\nabla v_\varepsilon)^2+mm'_\varepsilon(u_\varepsilon)e^{2\sigma t}\nabla a(x)\cdot\nabla v_\varepsilon v_\varepsilon\\ &-mm'_\varepsilon(u_\varepsilon)e^{2\sigma t}(\nabla J)*v_\varepsilon\cdot\nabla v_\varepsilon+m_\varepsilon(u_\varepsilon)h(x,u_\varepsilon)e^{\sigma t}\Delta v_\varepsilon\\ &+2m_\varepsilon(u_\varepsilon)\nabla a(x)\cdot\nabla v_\varepsilon e^{\sigma t}+m_\varepsilon(u_\varepsilon)F^{(3)}(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla v_\varepsilon e^{2\sigma t}\\ &+m_\varepsilon(u_\varepsilon)\Delta a(x)v_\varepsilon e^{\sigma t}-m_\varepsilon(u_\varepsilon)(\Delta J)*v_\varepsilon e^{\sigma t}.\end{aligned}\tag{2.14}$$

Multiplying (2.14) by v_ε and using $v_\varepsilon\Delta v_\varepsilon=\frac{1}{2}\Delta v_\varepsilon^2-|\nabla v_\varepsilon|^2$, we get

$$\begin{aligned}\frac{1}{2}(v_\varepsilon^2)_t+v_\varepsilon^2\sigma =& \frac{m}{2}m'_\varepsilon(u_\varepsilon)h(x,u_\varepsilon)e^{\sigma t}\nabla v_\varepsilon\cdot\nabla v_\varepsilon^2+\frac{m}{2}m'_\varepsilon(u_\varepsilon)e^{\sigma t}\nabla a(x)\cdot\nabla v_\varepsilon^2 v_\varepsilon\\ &-\frac{m}{2}m'_\varepsilon(u_\varepsilon)e^{\sigma t}(\nabla J)*v_\varepsilon\cdot\nabla v_\varepsilon^2+\frac{1}{2}m_\varepsilon(u_\varepsilon)h(x,u_\varepsilon)\Delta v_\varepsilon^2\\ &-m_\varepsilon(u_\varepsilon)h(x,u_\varepsilon)|\nabla v_\varepsilon|^2+m_\varepsilon(u_\varepsilon)\nabla a(x)\cdot\nabla v_\varepsilon^2\\ &+\frac{1}{2}m_\varepsilon(u_\varepsilon)F^{(3)}(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla v_\varepsilon^2 e^{\sigma t}+m_\varepsilon(u_\varepsilon)\Delta a(x)v_\varepsilon^2\\ &-m_\varepsilon(u_\varepsilon)(\Delta J)*v_\varepsilon v_\varepsilon.\end{aligned}\tag{2.15}$$

If there exists $(p_0, t_0) \in \overline{Q}_T$ with $t_0 > 0$ such that $v_\varepsilon^2(p_0, t_0) = \max v_\varepsilon^2$, then $\Delta v_\varepsilon^2(p_0, t_0) \leq 0, \nabla v_\varepsilon^2(p_0, t_0) = 0, (v_\varepsilon^2)_t(p_0, t_0) \geq 0$, and (2.15) yields

$$\begin{aligned} &(\sigma - m_\varepsilon(u_\varepsilon(p_0, t_0))\Delta a)v_\varepsilon^2(p_0, t_0) \\ &\leq - \int_\Omega \Delta J(p_0 - y)v_\varepsilon(y, t_0)dy m_\varepsilon(u_\varepsilon(p_0, t_0))v_\varepsilon(p_0, t_0). \end{aligned} \tag{2.16}$$

Choose σ large enough such that that $\sigma - m_\varepsilon(u_\varepsilon(p_0, t_0)) \max(\Delta a) > \delta > 0$. We obtain

$$\max |v_\varepsilon| \leq \frac{M_0}{\delta} \int_\Omega |v_\varepsilon(y, t_0)|dy \leq M_1 e^{-\sigma t_0} \int_\Omega |u_\varepsilon(y, t_0)|dy \leq M_2 e^{-\sigma t_0}\|u_\varepsilon\|, \tag{2.17}$$

for some positive constants M_0, M_1 and M_2. Hence, by estimate (2.2) and inequality (217), we obtain

$$|v_\varepsilon(P_0, t_0)| \leq C_1\|u_0\|. \tag{2.18}$$

Therefore,

$$\max |v_\varepsilon| \leq \max\{C_1\|u_0\|, \max |u_0|\}. \tag{2.19}$$

Since $\max |u_\varepsilon| \leq e^{\sigma T} \max |v_\varepsilon|$, (2.13) follows from (2.19). □

Proposition 2.3 — For any solution $u_\varepsilon(x, t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{Q}_T)$ of the equation (2.12) satisfying $\max\limits_{\overline{Q}_T} |u_\varepsilon| \leq C$ one has the estimates

$$\max_{\overline{Q}_T} |\nabla u_\varepsilon| \leq K_1, \quad |u_\varepsilon|_{Q_T}^{1+\alpha} \leq K_2, \tag{2.20}$$

where constants K_1, K_2 depend only on C, J, c_1, u_0 and the boundary of Ω.

The proof of Proposition 2.3 is quite similar to the Theorem 4.1, chapter VI of [22], so we omit the details.

Note that it is not difficult to prove Theorem 2.1 with Schaefer's Fixed Point Theorem [12]. The prove is quite similar to the Theorem 2.3 in [26], so we don't give the processes.

3. The Existence of Weak Solutions

Now, we are going to prove the existence of weak solutions for the degenerate problem (1.1), (1.4) and (1.5). First of all, we consider the following regularized problem

$$\begin{cases} \dfrac{du_\varepsilon}{dt} = \text{div}(m_\varepsilon(u_\varepsilon)\nabla(H(u_\varepsilon) + F'(u_\varepsilon))), \quad (x, t) \in \Omega \times (0, T), \\ u_\varepsilon = 0, \quad (x, t) \in \partial\Omega \times (0, T), \\ u_\varepsilon = u_{\varepsilon 0}, \quad (x, t) \in \Omega \times \{0\}, \end{cases} \tag{3.1}$$

where $m_\varepsilon(u_\varepsilon) = (u_\varepsilon^2 + \varepsilon^{\frac{2}{m}})^{\frac{m}{2}}$.

We have the following lemma.

Lemma 3.1 — Let (A_1)-(A_3) hold. Assume that $u_0 \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ and $p \in [1, \infty)$, then

$$||u_\varepsilon||_{p+1} \leq C, \tag{3.2}$$

where the constant $C > 0$ is only dependent on m, T, Ω, p, u_0.

PROOF : For the regularized problem (3.1), here $u_0 \in C_0^\infty(\Omega)$ such that $\|u_{\varepsilon 0}\|_\infty \leq \|u_0\|_\infty + 1$, $\|\nabla u_{\varepsilon 0}\| \leq \|\nabla u_0\| + 1$ and

$$u_{\varepsilon 0} \to u_0 \quad \text{strongly} \quad \text{in} \quad W_0^{1,2}.$$

Next, we prove the estimate $\|u_\varepsilon\|_{p+1}^{p+1}$, where $1 \leq p < \infty$.

Multiplying (3.1) by $u_\varepsilon |u_\varepsilon|^{p-1}$ and integrating over Ω, we get

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u_\varepsilon|^{p+1}dx + \int_\Omega m_\varepsilon(u)h(x,u_\varepsilon)\nabla u_\varepsilon \cdot \nabla(u_\varepsilon|u_\varepsilon|^{p-1})dx \\
= &-\int_\Omega m_\varepsilon(u_\varepsilon)u_\varepsilon \nabla a(x) \cdot \nabla(u_\varepsilon|u_\varepsilon|^{p-1})dx \\
&+ \int_\Omega m_\varepsilon(u_\varepsilon)(\nabla J) * u \cdot \nabla(u_\varepsilon|u_\varepsilon|^{p-1})dx.
\end{aligned} \tag{3.3}$$

Note that $\nabla(u_\varepsilon|u_\varepsilon|^{p-1}) = p|u_\varepsilon|^{p-1}\nabla u_\varepsilon$, by condition ($A_1$) and ($A_2$), we have

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u_\varepsilon|^{p+1}dx + pc_1\int_\Omega |u_\varepsilon|^{m+r+p-1}|\nabla u_\varepsilon|^2dx + pc_2\int_\Omega |u_\varepsilon|^{m+p-1}|\nabla u_\varepsilon|^2dx \\
&+ pc_1\varepsilon\int_\Omega |u_\varepsilon|^{r+p-1}|\nabla u_\varepsilon|^2dx + pc_2\varepsilon\int_\Omega |u_\varepsilon|^{p-1}|\nabla u_\varepsilon|^2dx \\
\leq & pC\int_\Omega |u_\varepsilon|^{m+p}|\nabla u_\varepsilon|dx + 2pC\int_\Omega\int_\Omega |\nabla J(x-y)u_\varepsilon(y)||u_\varepsilon(x)|^{m+p-1}|\nabla u_\varepsilon(x)|dxdy \\
&+ pC\varepsilon\int_\Omega |u_\varepsilon|^p|\nabla u_\varepsilon|dx \\
&+ 2pC\varepsilon\int_\Omega\int_\Omega |\nabla J(x-y)u_\varepsilon(y)||u_\varepsilon(x)|^{p-1}|\nabla u_\varepsilon(x)|dxdy,
\end{aligned} \tag{3.4}$$

where $C > 0$ is independent of p, T and ε. In this part, C may change from line to line even if in the same inequality.

From the Proposition 4.5 in [5], we deduce,

$$
\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u_\varepsilon|^{p+1}dx + pc_1\int_\Omega |u_\varepsilon|^{m+r+p-1}|\nabla u_\varepsilon|^2dx + pc_2\int_\Omega |u_\varepsilon|^{m+p-1}|\nabla u_\varepsilon|^2dx\\
&+\frac{4pc_1\varepsilon}{(p+r+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{p+r+1}{2}}|^2dx + \frac{2pc_2\varepsilon}{(p+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{p+1}{2}}|^2dx\\
\le& pC\int_\Omega |u_\varepsilon|^{m+p}|\nabla u_\varepsilon|dx + 2pC\int_\Omega\int_\Omega |\nabla J(x-y)u_\varepsilon(y)||u_\varepsilon(x)|^{m+p-1}|\nabla u_\varepsilon(x)|dxdy\\
&+pC\varepsilon\int_\Omega |u_\varepsilon|^{p+1}dx.
\end{aligned}
\tag{3.5}
$$

It is easy to get

$$
\begin{aligned}
&pc_1\int_\Omega |u_\varepsilon|^{m+r+p-1}|\nabla u_\varepsilon|^2dx + pc_2\int_\Omega |u_\varepsilon|^{m+p-1}|\nabla u_\varepsilon|^2dx\\
=&\frac{4pc_1}{(m+r+p+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{m+p+r+1}{2}}|^2dx\\
&+\frac{4pc_2}{(m+p+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{m+p+1}{2}}|^2dx.
\end{aligned}
\tag{3.6}
$$

By inequality (3.5) and equation (3.6), we get

$$
\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u_\varepsilon|^{p+1}dx + \frac{4pc_1}{(m+r+p+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{m+p+r+1}{2}}|^2dx\\
&+\frac{4pc_2}{(m+p+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{m+p+1}{2}}|^2dx + \frac{2pc_2\varepsilon}{(p+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{p+1}{2}}|^2dx\\
&+\frac{4pc_1\varepsilon}{(p+r+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{p+r+1}{2}}|^2dx\\
\le& 2pC\int_\Omega\int_\Omega |\nabla J(x-y)u_\varepsilon(y)||u_\varepsilon(x)|^{m+p-1}|\nabla u_\varepsilon(x)|dxdy\\
&+pC\int_\Omega |u_\varepsilon|^{m+p}|\nabla u_\varepsilon|dx + pC\varepsilon\int_\Omega |u_\varepsilon|^{p+1}dx.
\end{aligned}
\tag{3.7}
$$

For $pC\int_\Omega |u_\varepsilon|^{m+p}|\nabla u_\varepsilon|dx$, if $r-m\le p+1$, using Hölder's and Young's inequalities and condition (A_2), we have

$$
\begin{aligned}
&pC\int_\Omega |u_\varepsilon|^{m+p}|\nabla u_\varepsilon|dx\\
\le& pC\int_\Omega |\nabla u_\varepsilon|\cdot |u_\varepsilon|^{\frac{p+m+r+1}{2}}\cdot |u_\varepsilon|^{\frac{p+m-r+1}{2}}dx\\
\le&\frac{pc_1}{(p+m+r+1)^2}\int_\Omega |\nabla |u_\varepsilon|^{\frac{p+m+r+1}{2}}|^2dx + pC\int_\Omega |u_\varepsilon|^{p+m-r+1}dx;
\end{aligned}
\tag{3.8}
$$

if $r-m>p+1$, by Hölder's and Young's inequalities and condition (A_2), we get

$$\begin{aligned}
&pC\int_\Omega |u_\varepsilon|^{m+p}|\nabla u_\varepsilon|dx\\
\leq&\frac{pc_1}{(p+m+r+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{p+m+r+1}{2}}|^2dx+pC\int_\Omega |u_\varepsilon|^{m+p-\frac{br-b}{2-b}}\cdot|\nabla u_\varepsilon|^{\frac{2(1-b)}{2-b}}dx\\
\leq&\frac{pc_1}{(p+m+r+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{p+m+r+1}{2}}|^2dx+\frac{pc_2}{(m+p+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+p+1}{2}}|^2dx\\
&+pC\int_\Omega |u_\varepsilon|^{p+1}dx, \qquad (3.9)
\end{aligned}$$

where $b=\frac{m}{r}$.

Applying Hölder's and Young's inequalities and condition (A_2), we have

$$\begin{aligned}
&p\int_\Omega\int_\Omega |\nabla J(x-y)u_\varepsilon(y)||u_\varepsilon(x)|^{m+p-1}|\nabla u_\varepsilon(x)|dxdy\\
=&p\int_\Omega |u_\varepsilon|^{\frac{b(m+p+r-1)}{2}}|\nabla u_\varepsilon|^b\cdot|(\nabla J)*u_\varepsilon||u_\varepsilon|^{\frac{(m+p-1)(2-b)-br}{2}}|\nabla u_\varepsilon|^{1-b}dx\\
\leq&\frac{pc_1}{(p+m+r+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{p+m+r+1}{2}}|^2dx\\
&+pC\int_\Omega |(\nabla J)*u_\varepsilon|^{\frac{2}{2-b}}|u_\varepsilon|^{p+m-1-\frac{br}{2-b}}|\nabla u_\varepsilon|^{\frac{2(1-d)}{2-d}}dx\\
\leq&\frac{pc_1}{(p+m+r+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{p+m+r+1}{2}}|^2dx\\
&+\frac{pc_2}{(p+m+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+p+1}{2}}|^2dx+pC\int_\Omega |(\nabla J)*u_\varepsilon|^2|u_\varepsilon|^{p-1}dx\\
\leq&\frac{pc_1}{(p+m+r+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{p+m+r+1}{2}}|^2dx\\
&+\frac{pc_2}{(p+m+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+p+1}{2}}|^2dx\\
&+pC\left(\int_\Omega |u_\varepsilon(x)|^{p+1}dx\right)^{\frac{p-1}{p+1}}\left(\int_\Omega(\int_\Omega |\nabla J(x-y)||u_\varepsilon(y)|dy)^{p+1}dx\right)^{\frac{2}{p+1}}\\
\leq&\frac{pc_1}{(p+m+r+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{p+m+r+1}{2}}|^2dx\\
&+\frac{pc_2}{(p+m+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+p+1}{2}}|^2dx+pC\int_\Omega |u_\varepsilon|^{p+1}dx, \qquad (3.10)
\end{aligned}$$

where $b=\frac{m}{r}$.

Note that $\varepsilon<1$, thanks to inequalities (3.7)-(3.10) and condition (A_2), we get

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{dt}\int_\Omega |u_\varepsilon|^{p+1}dx+\frac{pc_2}{(m+p+1)^2}\int_\Omega |\nabla|u_\varepsilon|^{\frac{m+p+1}{2}}|^2dx\\
\leq&pC\int_\Omega |u_\varepsilon|^{p+1}dx. \qquad (3.11)
\end{aligned}$$

Using Gronwall's inequality, we obtain

$$\sup_{0\le t\le T}\int_{\Omega}|u_\varepsilon|^{p+1}dx \le C(m,p,T,\Omega,u_0).\square \tag{3.12}$$

Next, we give the main theorem in this section.

***Theorem* 3.1** — *Let (A_1)-(A_3) hold. Assume $u_0\in L^\infty(\Omega)\cap W_0^{1,2}(\Omega)$. Then problem (1.1), (1.4) and (1.5) has a solution $u\in L^\infty(Q_T)$ for any $T>0$.*

PROOF : Note that,

$$u_{\varepsilon 0}\to u_0 \quad \text{strongly in} \quad W_0^{1,2}.$$

Multiplying the equation (3.1) by $(u_\varepsilon-k)_+^2$ and integrating over $Q_t=\Omega\times(0,t)$, we have

$$\begin{aligned}
&\frac{1}{3}\sup_{t\in(0,T)}\int_{\Omega}(u_\varepsilon-k)_+^3dx+2\iint_{Q_t}|u_\varepsilon|^m h(x,u_\varepsilon)|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
&+2\varepsilon\iint_{Q_t}h(x,u_\varepsilon)|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
\le& C\iint_{Q_t}(|u_\varepsilon|^m+\varepsilon)|\nabla a(x)u_\varepsilon-(\nabla J)*u_\varepsilon)|(u_\varepsilon-k)_+|\nabla u_\varepsilon|dxd\tau
\end{aligned} \tag{3.13}$$

By the estimate (3.2), the Young's and Hölder's inequalities, we get

$$\begin{aligned}
&C\iint_{Q_t}(|u_\varepsilon|^m+\varepsilon)|\nabla a(x)u_\varepsilon-(\nabla J)*u_\varepsilon)|(u_\varepsilon-k)_+|\nabla u_\varepsilon|\\
\le& C\iint_{Q_t}(|u_\varepsilon|^m+\varepsilon)(|u_\varepsilon|+C)(u_\varepsilon-k)_+|\nabla u_\varepsilon|dxd\tau\\
\le& C\iint_{Q_t}(|u_\varepsilon|^{m+1}+C|u|^m)(u_\varepsilon-k)_+|\nabla u_\varepsilon|dxd\tau\\
\le& C\varepsilon\iint_{Q_t}(|u_\varepsilon|+C)(u_\varepsilon-k)_+|\nabla u_\varepsilon|dxd\tau\\
\le& c_2\iint_{Q_t}(|u_\varepsilon|^m(u_\varepsilon-k)_+|\nabla u_\varepsilon|^2dxd\tau\\
&+C\iint_{Q_t}(|u_\varepsilon|^{m+2}+|u_\varepsilon|^m)(u_\varepsilon-k)_+dxd\tau\\
&+\varepsilon c_1\iint_{Q_t}|\nabla u_\varepsilon|^2dxd\tau+\varepsilon C\iint_{Q_t}(u_\varepsilon-k)_+^2dxd\tau\\
&+\varepsilon C\iint_{Q_t}|u_\varepsilon|^2(u_\varepsilon-k)_+^2dxd\tau,
\end{aligned} \tag{3.14}$$

where $C>0$ is independent of ε.

Note that,

$$
\begin{aligned}
&2\iint_{Q_t}|u_\varepsilon|^m h(x,u_\varepsilon)|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
\geq&2\iint_{Q_t}|u_\varepsilon|^m(c_1|u_\varepsilon|^r+c_2)|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
=&2c_1\iint_{Q_t}|u_\varepsilon|^{m+r}|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau+2c_2\iint_{Q_t}|u_\varepsilon|^m|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau,
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
&2\varepsilon\iint_{Q_t}|h(x,u_\varepsilon)|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
\geq&2\varepsilon\iint_{Q_t}(c_1|u_\varepsilon|^r+c_2)|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
=&2c_1\iint_{Q_t}|u_\varepsilon|^r|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau+2c_2\iint_{Q_t}|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau.
\end{aligned}
\tag{3.16}
$$

It is obvious that $\varepsilon<1$, thanks to (3.13)-(3.16) and estimate (3.2), we obtain

$$
\begin{aligned}
&\frac{1}{3}\sup_{t\in(0,T)}\int_\Omega(u_\varepsilon-k)_+^3dx+2c_1\iint_{Q_t}|u_\varepsilon|^{m+r}|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
&+c_2\iint_{Q_t}|u_\varepsilon|^m|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
\leq&C\iint_{Q_t}(|u_\varepsilon|^{m+2}+|u_\varepsilon|^m)(u_\varepsilon-k)_+dxd\tau\\
&+C\iint_{Q_t}(u_\varepsilon-k)_+^2dxd\tau+C\iint_{Q_t}|u_\varepsilon|^2(u_\varepsilon-k)_+^2dxd\tau.
\end{aligned}
\tag{3.17}
$$

Assume that $s>\frac{6+N(m+3)}{6+N(m+2)}>1$, by Hölder's and Young's inequalities, condition (A_2) and estimate (3.2), we obtain

$$
\begin{aligned}
&C\iint_{Q_t}(|u_\varepsilon|^{m+2}+|u_\varepsilon|^m)(u_\varepsilon-k)_+dxd\tau\\
\leq&C\left(\iint_{Q_t}|u_\varepsilon|^{(m+2)s}dxd\tau\right)^{\frac{1}{s}}\left(\iint_{Q_t}(u_\varepsilon-k)_+^{\frac{s}{s-1}}dxd\tau\right)^{\frac{s-1}{s}}\\
&+C\left(\iint_{Q_t}|u_\varepsilon|^{ms}dxd\tau\right)^{\frac{1}{s}}\left(\iint_{Q_t}(u_\varepsilon-k)_+^{\frac{s}{s-1}}dxd\tau\right)^{\frac{s-1}{s}}\\
\leq&C\left(\iint_{Q_t}(u_\varepsilon-k)_+^{\frac{s}{s-1}}dxd\tau\right)^{\frac{s-1}{s}},
\end{aligned}
\tag{3.18}
$$

$$\begin{aligned}
&C\iint_{Q_t}|u_\varepsilon|^2(u_\varepsilon-k)_+^2dxd\tau\\
\leq &C\left(\iint_{Q_t}|u_\varepsilon|^{3s}dxd\tau\right)^{\frac{1}{s}}\left(\iint_{Q_t}(u_\varepsilon-k)_+^{\frac{s}{s-1}}dxd\tau\right)^{\frac{s-1}{s}}\\
\leq &C\left(\iint_{Q_t}(u_\varepsilon-k)_+^{\frac{s}{s-1}}dxd\tau\right)^{\frac{s-1}{s}}, \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
&C\int_0^T\int_\Omega(u_\varepsilon-k)_+^2dxdt\\
\leq &C\left(\int_0^T\int_\Omega|u_\varepsilon|^sdxdt\right)^{1/s}\left(\int_0^T\int_\Omega(u_\varepsilon-k)_+^{s/(s-1)}dxdt\right)^{(s-1)/s}\\
\leq &C\left(\int_0^T\int_\Omega(u_\varepsilon-k)_+^{s/(s-1)}dxdt\right)^{(s-1)/s}. \qquad (3.20)
\end{aligned}$$

Note that

$$\begin{aligned}
&2c_1\iint_{Q_t}|u_\varepsilon|^{m+r}|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
&+c_2\iint_{Q_t}|u_\varepsilon|^m|\nabla u_\varepsilon|^2(u_\varepsilon-k)_+dxd\tau\\
\geq&\frac{c_1}{(m+r+3)^2}\iint_{Q_t}|\nabla(u_\varepsilon-k)_+^{\frac{m+r+3}{2}}|^2dxd\tau\\
&+\frac{c_2}{(m+3)^2}\iint_{Q_t}|\nabla(u_\varepsilon-k)_+^{\frac{m+3}{2}}|^2dxd\tau, \qquad (3.21)
\end{aligned}$$

Using inequalities (3.17)-(3.21), we have

$$\begin{aligned}
&\sup_{t\in(0,T)}\int_\Omega(u_\varepsilon-k)_+^3dx+\frac{c_1}{(m+r+3)^2}\iint_{Q_t}|\nabla(u_\varepsilon-k)_+^{\frac{m+r+3}{2}}|^2dxd\tau\\
&+\frac{c_2}{(m+3)^2}\iint_{Q_t}|\nabla(u_\varepsilon-k)_+^{\frac{m+3}{2}}|^2dxd\tau\\
\leq&C\left(\iint_{Q_t}(u_\varepsilon-k)_+^{\frac{s}{s-1}}dxd\tau\right)^{\frac{s-1}{s}}. \qquad (3.22)
\end{aligned}$$

Denote

$$\begin{aligned}
J_k=&\sup_{t\in(0,T)}\int_\Omega(u_\varepsilon-k)_+^3(\cdot,t)dx+\frac{c_2}{(m+3)^2}\iint_{Q_T}|\nabla(u_\varepsilon-k)_+^{\frac{m+3}{2}}|^2dxdt\\
&+\frac{c_1}{(m+r+3)^2}\iint_{Q_T}|\nabla(u_\varepsilon-k)_+^{\frac{m+r+3}{2}}|^2dxdt.
\end{aligned}$$

Then

$$J_k \le C \iint_{Q_T} \left(\iint_{Q_T} (u_\varepsilon - k)_+^{\frac{s}{s-1}} dx d\tau \right)^{\frac{s-1}{s}}.$$

By Hölder's inequality, we get

$$\begin{aligned} & C \left(\int_0^T \int_\Omega (u_\varepsilon - k)_+^{s/(s-1)} dx dt \right)^{(s-1)/s} \\ \le & C \left(\int_0^T \int_\Omega ((u_\varepsilon - k)_+^{\frac{m+3}{2}})^{(12+2N(m+3))/(N(m+3))} dx dt \right)^{N/(6+N(m+3))} \\ & \cdot \mu(k)^{(s-1)/s - N/(6+N(m+3))}, \end{aligned}$$

where $\mu(k)$ denotes the measure of the set $\{(x,t) \in Q_T; u_\varepsilon \ge k\}$.

In the Proposition 1.1, let $v_\varepsilon(x,t) = (u_\varepsilon - k)_+^{\frac{m+3}{2}}, p = 2, h = \frac{12+2N(m+3)}{N(m+3)}, q = \frac{6}{m+3}$, by the imbedding inequality, we deduce

$$\begin{aligned} & C \left(\int_0^T \int_\Omega (u_\varepsilon - k)_+^{s/(s-1)} dx dt \right)^{(s-1)/s} \\ \le & C J_k^{(N+2)/(6+N(m+3))} \mu(k)^{(s-1)/s - N/(6+N(m+3))}. \end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22), we get

$$J_k \le C J_k^{(N+2)/(6+N(m+3))} \mu(k)^{(s-1)/s - N/(6+N(m+3))}.$$

By Young's inequality, we obtain

$$J_k \le C \mu(k)^{1+(2s-N(m+3)-6)/s(4+N(m+2))}.$$

So, for all $k^{**} \ge k^*$, we have

$$\begin{aligned} & (k^{**} - k^*)(\mu(k^{**}))^{N/(6+N(m+3))} \\ \le & \left(\int_0^T \int_\Omega ((u_\varepsilon - k^*)_+^{\frac{m+3}{2}})^{(12+2N(m+3))/(N(m+3))} dx dt \right)^{N/(6+N(m+3))} \\ \le & C J_{k^*}^{(N+2)/(6+N(m+3))} \\ \le & C \left(\mu(k^*)^{1+(2s-N(m+3)-6)/s(4+N(m+2))} \right)^{(N+2)/(6+N(m+3))}. \end{aligned} \tag{3.24}$$

If we take $k^{**} = \|u_0\|_{L^\infty(\Omega)} + j (j > 1)$ and $k^* = \|u_0\|_{L^\infty(\Omega)} + 1$, then

$$\begin{aligned} & \mu(k^{**})^{N/(6+N(m+3))} \\ \le & \frac{C}{j-1} \left(((T+1)|\Omega|)^{1+(2s-N(m+3)-6)/s(4+N(m+2))} \right)^{(N+2)/(6+N(m+3))}. \end{aligned}$$

Hence, there exists a constant $j_0 > 0$ depending only on $m, T, |\Omega|, N, p, s$ such that

$$\mu(k^{**}) \leq 1, \quad \text{as } j \geq j_0.$$

We take $k_n = M(2 - 2^{-n}), n = 0, 1, 2, \cdots$, where $M \geq \|u_0\|_{L^\infty(\Omega)} + j_0$ is a constant.

Then it is easy to see that

$$\mu(k_n) \leq 1, \qquad \text{as} \quad n = 0, 1, \cdots. \tag{3.25}$$

Now we consider the following two cases.

if

$$(i) \qquad 1 + (2s - N(m+3) - 6)/s(4 + N(m+2)) \geq 1,$$

then by (3.24) and (3.25)

$$(k_{n+1} - k_n)(\mu(k_{n+1}))^{N/(6+N(m+3))} \leq C_1(\mu(k_n))^{(N+2)/(6+N(m+3))},$$

i.e.,

$$\mu(k_{n+1}) \leq \left(\frac{2C_1}{M}\right)^{(6+N(m+3))/N} 2^{n(6+N(m+3))/N} \mu(k_n)^{1+2/N}.$$

If

$$(ii) \qquad 1 + (2s - N(m+3) - 6)/s(4 + N(m+2)) < 1,$$

then

$$\begin{aligned} & (k_{n+1} - k_n)(\mu(k_{n+1}))^{N/(6+N(m+3))} \\ \leq \quad & C_2(\mu(k_n)^{1+(2s-N(m+3)-6)/s(4+N(m+2))})^{(N+2)/(6+N(m+3))}. \end{aligned}$$

i.e.,

$$\mu(k_{n+1}) \leq \left(\frac{2C_2}{M}\right)^{(6+N(m+3))/N} 2^{n(6+N(m+3))/N} \mu(k_n)^{1+\delta_1},$$

where

$$\delta_1 = \frac{2}{N} + \frac{(N+2)(2s - N(m+3) - 6)}{sN(4 + N(m+2))} > 0.$$

Now, take

$$M = \max\{C_1 2^{N/2+1}, C_2 2^{N/2\delta_1(N+2)+1}, \|u_0\|_{L^\infty(\Omega)} + j_0\}.$$

Hence by Proposition 1.2, we have that

$$u_\varepsilon \leq 2M.$$

Similarly, we may derive a lower bound. So, for a not relabeled subsequence, we obtain

$$u_\varepsilon \to u \quad \text{weakly} \quad \text{in} \quad L^\infty(Q_T).$$

Here the prove completes. □

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