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CONSTRUCTIONS OF r-IDENTIFYING CODES AND $(r, \leq l)$-IDENTIFYING CODES[1]

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In binary Hamming space, we construct r-identifying codes from a $\sum_{i=0}^{r-1}\binom{n}{i} + \binom{n-1}{r-1} + 1$-fold r-covering code. By using this construction, we modify the construction of r-identifying codes of Charon *et al*. which helps to find codes of greater length. From this modified construction, we improve the upper bound of $M_5(23)$ which is the smallest possible cardinality of a 5-identifying code of length 23. We also construct $(2, \leq l)$-identifying codes and we define the length function $L(r, l)$ as the smallest positive integer n for which there exists an $(r, \leq l)$-identifying code in $\mathbb{F}_2^n$. By using the construction of $(2, \leq l)$-identifying codes, we improve the upper bounds of $L(2, l)$ for all $l \geq 6$. We also improve the upper bounds of $M_2^{(\leq l)}(n)$ for all $l \geq 6$ and when n is one more than the improved upper bound of $L(2, l)$. We give the construction for $(r, \leq l)$-identifying codes. From this result, we prove that $M_r^{(\leq l)}(2r + 1) \leq 2^{2r}$ for certain values of r and l.

Key words : Binary Hamming spaces; covering codes; code construction; identifying codes; direct sum.

1. INTRODUCTION

Let $\mathbb{F}_2$ denote the binary field of two elements $\{0, 1\}$. The binary Hamming space $\mathbb{F}_2^n$ is the cartesian product of $\mathbb{F}_2$ taken n times. The Hamming distance $d(x, y)$ between the words $x, y \in \mathbb{F}_2^n$ is the number of coordinates in which they differ. We say that x r-covers y if $d(x, y) \leq r$. Since the distance function is a metric, x r-covers y is equivalent to y r-covers x. We denote

$$B_r(x) = \{y \in \mathbb{F}_2^n \mid d(x, y) \leq r\}$$

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as the closed ball in $\mathbb{F}_2^n$ with center at x and radius r and it is called as Hamming ball with center at x and radius r, and

$$S_r(x) = \{y \in \mathbb{F}_2^n \mid d(x,y) = r\}.$$

Also, we denote zeroword by $\mathbf{0}$. The Hamming weight $w(x)$ of $x \in \mathbb{F}_2^n$ is defined as $d(x, \mathbf{0})$.

A non-empty subset C of $\mathbb{F}_2^n$ is called a *code of length* n. Each element of a code is called a codeword and any element of $\mathbb{F}_2^n$ is called as a word. The covering radius of C is the least positive integer r with $\mathbb{F}_2^n = \bigcup_{x \in C} B_r(x)$.

In 1998, Karpovsky *et al*. [9] introduced identifying codes. The concept of identifying code was motivated for finding malfunctioning processors in a multiprocessor system. In this application, testers are positioned in the processors which are the elements of the identifying code. These testers monitor their neighboring processors and send the number 1 to the host if it finds any faulty processor in its neighbors, otherwise it sends the number 0 to the host. From this information, one can identify the faulty processors. Fault identification is also applied in environmental monitoring, sensor networks and location detection in hostile environments. More details of applications and theoretical connections of identifying codes are given in [11].

Let $C \subseteq \mathbb{F}_2^n$ be a code of length n. The I_r-set of a word $x \in \mathbb{F}_2^n$ with respect to the code C is defined to be

$$I_r(x) = B_r(x) \cap C.$$

Moreover, for $X = \{x_1, \ldots, x_k\} \subseteq \mathbb{F}_2^n$,

$$I_r(X) = \bigcup_{i=1}^{k} I_r(x_i) \text{ and we set } I_r(\varnothing) = \varnothing.$$

A code C of length n is called an r-*covering code* if $I_r(x) \neq \varnothing$ for all $x \in \mathbb{F}_2^n$. Let μ be a positive integer. Code C is said to be a μ-fold r-covering code if $|I_r(x)| \geq \mu$ for all $x \in \mathbb{F}_2^n$. A code is said to be an r-separating code if $I_r(x) \neq I_r(y)$ for all $x, y \in \mathbb{F}_2^n$ with $x \neq y$ and an r-*identifying code* if it is both an r-covering code and an r-separating code.

Let r and l be positive integers. A code $C \subseteq \mathbb{F}_2^n$ is said to be an $(r, \leq l)$-*identifying code* if for all $X, Y \subseteq \mathbb{F}_2^n$ such that $|X| \leq l, |Y| \leq l$ and $X \neq Y$, we have $I_r(X) \neq I_r(Y)$. For example, $\{0000, 0010, 0001, 1100, 1010, 1001, 0101, 0011, 1110, 1101, 0111\}$ is an $(1, \leq 2)$-identifying code of length 4 given in [8]. If $l = 1$, then the code is an r-identifying code.

The smallest possible cardinality of an $(r, \leq l)$-identifying code of length n is denoted by $M_r^{(\leq l)}(n)$ whenever such a code exists for these parameters. Note that it is possible that such a code may not exists for these parameters (for a specific example, see Theorem 4 of [8]). For an r-identifying code, we denote the smallest cardinality by $M_r(n)$.

Suppose $X \subseteq \mathbb{F}_2^n$ is an unknown set of faulty processors with $|X| \leq l$. Testers positioned in the processors belonging to an $(r, \leq l)$-identifying code check their r-radius neighborhoods and give us $I_r(X)$. Hence, we can determine X. Using $(r, \leq l)$-identifying code, we can find at most l faulty processors while an r-identifying code determines at most one faulty processor.

Identifying codes now form a topic of their own. In the website [13], there are many papers dealing with identifying codes and closely related topics. For recent development of identifying codes in binary Hamming spaces, see [1, 2, 6, 10].

The structure of our paper is as follows: In Section 2, we give a sufficient condition for a set to be an r-identifying code and using this result, we modify the construction in [2]. By the use of this modification, we can obtain codes of greater lengths and also we improve the upper bound of $M_5(23)$. In Section 3, we give a sufficient condition for a set to be a $(2, \leq l)$-identifying code. Given r and l, we define the length function $L(r, l)$ to be the smallest positive integer for which there exists an $(r, \leq l)$-identifying code in $\mathbb{F}_2^n$. Using this result, we improve the upper bound of $L(r, l)$ for $r = 2$ and $l \geq 6$. The lower bound of $L(r, l)$ is obtained from Theorem 4 of [8]. Also, we improve the upper bounds of $M_2^{(\leq l)}(n)$ for all $l \geq 6$ and n is one more than the improved upper bound of $L(2, l)$. We also give the construction for an $(r, \leq l)$-identifying code. From this result, we prove that $M_r^{(\leq l)}(2r + 1) \leq 2^{2r}$ for certain values of r and l.

2. Preliminaries

***Theorem* 2.1** — *(Theorem 2.4.8 of [3]). If $r, s \in \{0, 1, \ldots, n\}, c_1, c_2, c_3, c_4 \in \mathbb{F}_2^n$, and $d(c_1, c_2) \leq d(c_3, c_4)$, then*

$$|B_r(c_1) \cap B_s(c_2)| \geq |B_r(c_3) \cap B_s(c_4)|.$$

Futhermore, if $d(c_3, c_4) = 1 + d(c_1, c_2)$ and $r + s + d(c_1, c_2)$ is odd, equality holds.

***Theorem* 2.2** — *(Theorem 2 of [2]). Let $r \geq 1, p \geq 1$ and let C be an r-identifying code of length n and $X_p = \{x \in \mathbb{F}_2^n \mid d(x, c) \leq r - p$ or $d(x, c) > r$, for all $c \in C\}$. Let $Y_p \subseteq \mathbb{F}_2^n$ be a set such that for every $x \in X_p$, there exists $y \in Y_p$ with $r - p + 1 \leq d(x, y) \leq r$. Then $C' = (C \oplus \mathbb{F}_2^p) \cup (Y_p \oplus (\mathbb{F}_2^p \setminus \{\mathbf{0}\}))$ is an r-identifying code of length $n + p$.*

***Theorem* 2.3** — *(Theorem 3 of [2]). Let $r_1 \geq p \geq r_2 \geq 0$ and let C be an r_1-identifying code of length n and $X_{p,r_2} = \{x \in \mathbb{F}_2^n \mid d(x, c) \leq r_1 - p + r_2$ or $d(x, c) > r_1 + r_2$, for all $c \in C\}$. Let $Y_{p,r_2} \subseteq \mathbb{F}_2^n$ be a set such that for every $x \in X_{p,r_2}$, there exists $y \in Y_{p,r_2}$ with $r_1 - p + r_2 + 1 \leq$*

$d(x,y) \leq r_1 + r_2$. Then $C' = (C \oplus \mathbb{F}_2^p) \cup (Y_{p,r_2} \oplus (\mathbb{F}_2^p \setminus \{\mathbf{0}\}))$ is an ($r_1 + r_2$)-identifying code of length $n + p$.

Corollary 2.4 — (Corollary 4 of [2]).

1. Let $r \geq 1, p \geq 1$, we have

$$M_r(n+p) \leq \begin{cases} 2^p M_r(n) & \text{if } p \geq r+1 \\ 2^p M_r(n) + (2^p - 1)|Y_p| & \end{cases}$$

   where Y_p is the same as in Theorem 2 of [2].

2. Let $r_1 \geq p \geq r_2 > 0$, we have

$$M_{r_1+r_2}(n+p) \leq \begin{cases} 2^p M_{r_1}(n) & \text{if } 0 < r_2 < p \\ 2^p M_{r_1}(n) + (2^p - 1)|Y_{p,p}| & \text{if } r_2 = p \end{cases}$$

   where $Y_{p,p}$ is the same as in Theorem 3 of [2].

***Theorem* 2.5** — *(Theorem 2 of [12]). Let $l \geq 3$ and let C be $(2l-1)$-fold 1-covering code of length n. Then C is $(1, \leq l)$-identifying code. Hence, $M_1^{(\leq l)}(n) \leq K(n, 1, 2l-1)$.*

***Theorem* 2.6** — *(Theorem 4 of [8]). Let $r(n,K)$ denote the smallest covering radius among all binary codes of length n and cardinality K. Let $l < 2^n$. If $t \geq r(n,l)$, then there does not exist a $(t, \leq l)$-identifying code of length n.*

***Theorem* 2.7** — *(Theorem 21 of [4]). Suppose $r \geq 1$. Let $C_i \subseteq \mathbb{F}_2^{n_i}$ for $1 \leq i \leq r$ be $(1, \leq 2)$-identifying codes. Then $C = C_1 \oplus \cdots \oplus C_r$ is an $(r, \leq 2)$-identifying code.*

***Theorem* 2.8** — *(Theorem 24 of [4]). Suppose $l \geq 3$. Let $C_i \subseteq \mathbb{F}_2^{n_i}$ for $1 \leq i \leq r$ be $(1, \leq l)$-identifying codes. Then $C = C_1 \oplus \cdots \oplus C_r$ is an $(r, \leq l)$-identifying code.*

***Theorem* 2.9** — *(Theorem 14.2.2 of [3]). Suppose r is any positive integer. Then there is an μ-fold r-covering code of length $2r+1$ with cardinality 2^{2r} where $\mu = \frac{1}{2}\sum_{i=0}^{r}\binom{n}{i}$.*

3. Construction of r-identifying Codes

Lemmas 3.1 and 3.2 given below will be used to construct r-identifying codes. In [4, 5, 7, 14], the authors have used these two lemmas for $r = 1$ to construct 1-identifying codes. These two lemmas generalize the two lemmas obtained in [4, 5, 7, 14] for the case $r = 1$ and they are used here to construct r-identifying codes.

Lemma 3.1 — Let $x, y \in \mathbb{F}_2^n$ and $r \geq 1$. Then

$$|B_r(x) \cap B_r(y)| = \begin{cases} \sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1} & \text{if } d(x,y) = 1 \text{ or } 2 \\ \sum_{i=0}^{r} \binom{n}{i} - \left(\binom{n-3}{r-2} + 4\binom{n-3}{r-1} + \binom{n-3}{r}\right) & \text{if } d(x,y) = 3 \text{ or } 4 \\ \binom{2r}{r} & \text{if } d(x,y) = 2r-1 \\ & \text{or } 2r \\ 0 & \text{if } d(x,y) \geq 2r+1 \end{cases}$$

PROOF : Let $x, y \in \mathbb{F}_2^n$, then clearly $B_r(x) \cap B_r(y) = \varnothing$ for $d(x,y) \geq 2r+1$. Now, let us determine the cardinality of $B_r(x) \cap B_r(y)$. Without loss of generality, we can take x as the all-zero word. Let $d(x,y) = w(y) = k, 1 \leq k \leq 2r$. If we change p of the coordinates of y where it has 1 and also change q of the coordinates of y where it has 0 and leave the remaining coordinates of y unchanged, with $0 \leq p \leq k, 0 \leq q \leq n-k$ and $0 \leq p+q \leq r$, then the resulting word z belongs to $B_r(y)$. In addition, if p and q satisfy $k - p + q \leq r$, then the word z belongs to $B_r(x) \cap B_r(y)$. Notice that these words are the only words of $B_r(y)$ which also belong to $B_r(x)$ and no other words of $\mathbb{F}_2^n$ belong to the intersection because $B_r(x) \cap B_r(y) \subseteq B_r(y)$.

If $k = 1$, we have to find the pairs (p,q) which satisfy $p + q \leq r$ and $1 - p + q \leq r$. If $p = 0$, then $1 + q \leq r$, that is, $q \leq r - 1$. If $p = 1$, then $1 - p + q \leq r$, that is, $q \leq r - 1 + p$. Therefore, all values of q satisfy the above two relations when $p = 1$. Hence all combinations of p and q which satisfy $p + q \leq r$ also satisfy $k - p + q \leq r$ except when $p = 0$ and $q = r$. Thus we have,

$$|B_r(x) \cap B_r(y)| = \sum_{i=0}^{r} \binom{n}{i} - \binom{1}{0}\binom{n-1}{r} = \sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1}.$$

If $k = 3$, the values of (p,q) that satisfy $p+q \leq r$ but not $3-p+q \leq r$ are $(0,r), (0,r-1), (0,r-2)$ and $(1, r-1)$ because if $p = 0$ then $3 + q \leq r$, that is, $q \leq r-3$, and if $p = 1$ then $3 - 1 + q \leq r$, that is, $q \leq r - 2$. Therefore,

$$\begin{aligned} |B_r(x) \cap B_r(y)| &= \sum_{i=0}^{r} \binom{n}{i} - \left(\binom{3}{0}\binom{n-3}{r-2} + \binom{3}{0}\binom{n-3}{r-1} + \binom{3}{0}\binom{n-3}{r} \right. \\ &\qquad \left. + \binom{3}{1}\binom{n-3}{r-1}\right) \\ &= \sum_{i=0}^{r} \binom{n}{i} - \left(\binom{n-3}{r-2} + 4\binom{n-3}{r-1} + \binom{n-3}{r}\right). \end{aligned}$$

If $k = 2r-1$, we have to find the pairs (p, q) which satisfy $p+q \leq r$ and $k-p+q \leq r$. If $q > 0$ then $p \leq r-q$ but $2r-1-p+q \geq 2r-1-r+q+q = r-1+2q > r$. Therefore $q = 0$. This implies that $p \leq r$ and $2r-1-p \leq r$, that is, $p \geq r-1$. Therefore $(r-1, 0)$ and $(r, 0)$ are the only values of (p, q) which satisfy $p+q \leq r$ and $k-p+q \leq r$. Therefore,

$$|B_r(x) \cap B_r(y)| = \binom{2r-1}{r-1}\binom{n-(2r-1)}{0} + \binom{2r-1}{r}\binom{n-(2r-1)}{0} = \binom{2r}{r}$$

The proof of the result for $k = 2, 4$ and $2r$ are immediately follows from Theorem 2.4.8 of [3] with $r = s$. □

Lemma 3.2 — If the intersection of any $\sum_{i=0}^{r-1}\binom{n}{i} + \binom{n-1}{r-1} + 1$ distinct Hamming balls of radius r is nonempty, then their intersection consists of a single point.

PROOF : Let $k = \sum_{i=0}^{r-1}\binom{n}{i} + \binom{n-1}{r-1} + 1$. Let $c_1, c_2, \ldots, c_k$ be any k words in $\mathbb{F}_2^n$ with $\bigcap_{i=1}^{k} B_r(c_i) \neq \varnothing$. Suppose $x, y \in \bigcap_{i=1}^{k} B_r(c_i)$ with $x \neq y$. Then $x, y \in B_r(c_i)$ for all i and hence $c_i \in B_r(x) \cap B_r(y)$ for all i with $1 \leq i \leq k$. This implies that $|B_r(x) \cap B_r(y)| \geq k$. By Theorem 2.4.8 of [3], we know that the size of the intersection of two Hamming balls of the same radius does not increases when the distance between their centers increases. From this fact and by Lemma 3.1, we have

$$|B_r(x) \cap B_r(y)| \leq \sum_{i=0}^{r-1}\binom{n}{i} + \binom{n-1}{r-1} < k,$$

a contradiction. Therefore, our supposition is wrong and hence $\bigcap_{i=1}^{k} B_r(c_i)$ consists of a single point. □

The following example exhibits the sharpness of the above lemma.

Example 3.3 :

1. For $r = 2$ and $n = 5$, let us take $X = \{00000, 10000, 01000, 00100, 00010, 00001, 11000, 10100, 10010, 10001\}$, then $\bigcap_{x \in X} B_2(x) = \{00000, 10000\}$.
   Here $|X| = 10 < 11 = \sum_{i=0}^{1}\binom{5}{i} + \binom{4}{1} + 1$.
2. For $r = 3$ and $n = 4$, let us take $Y = \{0000, 1000, 0100, 0010, 0001, 1100, 1010, 1001, 0110, 0101, 0011, 1110, 1101, 1011\}$, then $\bigcap_{y \in Y} B_3(y) = \{0000, 1000\}$.
   Here $|Y| = 14 < 15 = \sum_{i=0}^{2}\binom{4}{i} + \binom{3}{2} + 1$.

***Theorem* 3.4**— *Let n and r be two positive integers such that $r < n$. Let $C \subseteq \mathbb{F}_2^n$ be such that $|I_r(x)| \geq \sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1} + 1$ for all $x \in \mathbb{F}_2^n$, then C is an r-identifying code.*

PROOF : Let $k = \sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1} + 1$ and $x \in \mathbb{F}_2^n$. Since $|I_r(x)| \geq k$, $I_r(x) \neq \varnothing$ and hence $I_r(x)$ contains at least k codewords, say, $c_1, c_2, \ldots, c_k$. Therefore, by Lemma 3.2, $\bigcap_{i=1}^{k} B_r(c_i) = \{x\}$. Suppose that there exists y not equal to x in $\mathbb{F}_2^n$ such that $I_r(y) = I_r(x)$. Then $\{x, y\} \subset \bigcap_{i=1}^{k} B_r(c_i)$, a contradiction to the Lemma 3.2. Therefore, $I_r(y) \neq I_r(x)$ for every $x \neq y$. Hence, C is an r-identifying code. □

As an immediate consequence of the above theorem, we have

Corollary 3.5 — (1) For $n \geq 2, \mathbb{F}_2^n$ is an r-identifying code for all r with $1 \leq r \leq n-1$.

(2) Let n and r be two integers such that $n \geq 2$ and $1 \leq r \leq n-1$. If $X \subseteq \mathbb{F}_2^n$ with $|X| \leq \binom{n-1}{r} - 1$, then $\mathbb{F}_2^n \setminus X$ is an r-identifying code.

(3) For all $n \geq 3$ and $r \in \{1, 2, \ldots, n-2\}$, the set $\mathbb{F}_2^n \setminus Y$ is an r-identifying code where $Y \subseteq \mathbb{F}_2^n$ with $|Y| \leq n-2$.

PROOF : (1) Let $C = \mathbb{F}_2^n$ and let $x \in \mathbb{F}_2^n$. Then $|I_r(x)| = |B_r(x)| = \sum_{i=0}^{r} \binom{n}{i} \geq \sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1} + 1$. By Theorem 3.4, $\mathbb{F}_2^n$ is an r-identifying code.

(2) Let $X \subseteq \mathbb{F}_2^n$ with $|X| \leq \binom{n-1}{r} - 1$ and let $C = \mathbb{F}_2^n \setminus X$. Let $x \in \mathbb{F}_2^n$. Then $I_r(x) = B_r(x) \cap C$. If $X \subseteq B_r(x)$ then $|B_r(x) \cap C| \geq \sum_{i=0}^{r} \binom{n}{i} - \binom{n-1}{r} + 1$. Otherwise $|B_r(x) \cap C| > \sum_{i=0}^{r} \binom{n}{i} - \binom{n-1}{r} + 1$. Therefore $|I_r(x)| \geq \sum_{i=0}^{r} \binom{n}{i} - \binom{n-1}{r} + 1$. Again by Theorem 3.4, C is an r-identifying code.

(2) Let n be an integer such that $n \geq 3$ and $Y \subseteq \mathbb{F}_2^n$ with $|Y| \leq n-2$ and let $C = \mathbb{F}_2^n \setminus Y$. Let r be an integer with $1 \leq r \leq n-2$. Then $|Y| \leq n-2 \leq \binom{n-1}{r} - 1$. By (2) of Corollary 3.5, C is an r-identifying code. □

Using $\mathbb{F}_2^n \setminus \{\mathbf{0}\}$ is an r-separating code for all $r \in \{0, 1, 2, \ldots, n-1\}$ and $n \geq 1$, the construction of an r-identifying code in [2] relaxes many previous known upper bounds of $M_r(n)$ for various r and n. But (3) of Corollary 3.5 above gives better r-separating codes in terms of size and using these r-separating codes in Theorem 3 of [2], we get the following construction which helps to find codes of greater lengths.

***Theorem* 3.6** — *Let $r_1 \geq p \geq r_2 \geq 0,\ p \geq 3$ and let C be an r_1-identifying code of length n and $X_{p,r_2} = \{x \in \mathbb{F}_2^n \mid d(x,c) \leq r_1 - p + r_2 \text{ or } d(x,c) > r_1 + r_2,$ for all $c \in C\}$. Let $Y'_{p,r_2} \subseteq \mathbb{F}_2^n$ be a set such that for every $x \in X_{p,r_2}$, there exists $y \in Y'_{p,r_2}$ with $r_1 + r_2 - (p-2) \leq d(x,y) \leq r_1 + r_2 - 1$. Then $C' = (C \oplus \mathbb{F}_2^p) \cup (Y'_{p,r_2} \oplus (\mathbb{F}_2^p \setminus X))$ is an $(r_1 + r_2)$-identifying code of length $n + p$ where $X \subset \mathbb{F}_2^p$ with $|X| = p - 2$.*

PROOF : The proof that C' is $(r_1 + r_2)$-covering code is the same as in Theorem 3 of [2] and $(r_1 + r_2)$-separating code is also the same except case (iv) of Theorem 3 of [2].

In case (iv) $x_2 \neq y_2$ and $x_1 = y_1 \in X_{p,r_2}$. By the construction, there is a vector $z \in Y'_{p,r_2}$ such that $r_1 + r_2 - (p-2) \leq d(z, x_1) \leq r_1 + r_2 - 1$. Let $r = r_1 + r_2 - d(z, x_1)$, then $1 \leq r \leq p - 2$. By (3) of Corollary 3.5, there is a vector $v \in \mathbb{F}_2^p \setminus X$ which is within distance r from x_2 and not from y_2, or the other way round. Then $d(z|v, x_1|x_2) \leq d(z, x_1) + r = r_1 + r_2$ and $d(z|v, x_1|y_2) > d(z, x_1) + r = r_1 + r_2$ or the other way round, with $z|v \in Y'_{p,r_2} \oplus (\mathbb{F}_2^p \setminus X) \subseteq C'$ and we have proved that $I_{r_1+r_2}(x) \neq I_{r_1+r_2}(y)$. Hence the result. □

In particular, if $r_2 = 0$, we have

***Theorem* 3.7** — *Let $r_1 \geq 1, p \geq 3$ and let C be an r_1-identifying code of length n and $X_p = \{x \in \mathbb{F}_2^n \mid d(x,c) \leq r_1 - p \text{ or } d(x,c) > r_1$ for all $c \in C\}$. Let $Y'_p \subseteq \mathbb{F}_2^n$ be a set such that for every $x \in X_p$, there exists $y \in Y'_p$ with $r_1 - p + 2 \leq d(x,y) \leq r_1 - 1$. Then $C' = (C \oplus \mathbb{F}_2^p) \cup (Y'_p \oplus (\mathbb{F}_2^p \setminus X))$ is an r_1-identifying code of length $n + p$ where $X \subseteq \mathbb{F}_2^p$ with $|X| = p - 2$.*

Using Corollary 4 of [2], our Theorem 3.6 and our Theorem 3.7, we have the following

Corollary 3.8 — 1. Let $r_1 \geq 1, p \geq 3$, we have

$$M_{r_1}(n+p) \leq \begin{cases} 2^p M_{r_1}(n) + (2^p - 1)|Y_p| \\ 2^p M_{r_1}(n) + (2^p - p + 2)|Y'_p| \end{cases}$$

where Y_p and Y'_p are the same as in Theorem 2 of [2] and our Theorem 3.7, respectively.

2. Let $r_1 \geq p = r_2 \geq 3$, we have

$$M_{r_1+r_2}(n+p) \leq \begin{cases} 2^p M_{r_1}(n) + (2^p - 1)|Y_{p,p}| \\ 2^p M_{r_1}(n) + (2^p - p + 2)|Y'_{p,p}| \end{cases}$$

where $Y_{p,p}$ and $Y'_{p,p}$ are the same as in Theorem 3 of [2] and our Theorem 3.6, respectively.

Observe that the second inequality in (1) and (2) of Corollary 3.8 are direct consequence of our Theorem 3.7 and our Theorem 3.6 respectively and the first inequalities came from Corollary 4 of [2].

By the following example, we improve the upper bound of $M_5(23)$ calculated from Theorem 2 of [2] and also we find the situation that our Theorem 3.7 gives a better upper bound than Theorem 2 of [2]. Always $|Y_p| \leq |Y'_p|$. If $|Y_p| = |Y'_p|$, then our Theorem 3.7 gives obviously a better upper bound than Theorem 2 of [2]. Otherwise, it is very difficult to compare both the results because it is fully dependent on the cardinality of the two sets Y_p and Y'_p and the value of p

Example 3.9 : Take $p = 4$. By using the 5-identifying code of length 19 in *http://perso.telecom-paristech.fr/hudry/newIdentifyingNcube.html* with 326 codewords, we have

$$X_4 = \{181588\}, Y_4 = \{509268\}, Y'_4 = \{517460\}$$

Here elements of $\mathbb{F}_2^{19}$ are represented by the corresponding decimal numbers. By Theorem 2 of [2], $M_5(23) \leq 2^4 \times 326 + (2^4 - 1)(1) = 5231$ and by Theorem 3.7, $M_5(23) \leq 2^4 \times 326 + (2^4 - (4 - 2))(1) = 5230$. (Note that $|Y_4| = 1 = |Y'_4|$)

4. ON $(r, \leq l)$-IDENTIFYING CODES

Our Theorems 4.1 and 4.4 below are inspired by [12]. Multiple covering codes are a well studied topic (see Chapter 14 in [3]). In this section, we prove that the upper bound of $M_r^{(\leq l)}(n)$ can be obtained from the upper bound of $K(n, r, \mu)$ which is the smallest possible cardinality of a μ-fold r-covering code of length n. The importance of these theorems is given in this section.

***Theorem* 4.1** — *Let $l \geq 6$. Let $C \subseteq \mathbb{F}_2^n$ be such that $|I_2(x)| \geq 2n(l-1)+1$ for all $x \in \mathbb{F}_2^n$. Then C is a $(2, \leq l)$-identifying code. Hence, $M_2^{(\leq l)}(n) \leq K(n, 2, \mu)$ where $\mu = 2n(l-1)+1$*

PROOF : We have to show that $I_2(X) \neq I_2(Y)$ for any two distinct subsets X and Y of $\mathbb{F}_2^n$ where $|X| \leq l$ and $|Y| \leq l$. Observe that $I_2(X) \neq \emptyset$ for all $X \subseteq \mathbb{F}_2^n, X \neq \emptyset$, since there exists an $x \in X$ and by our assumtion that $|I_2(x)| \geq 2n(l-1)+1$.

If without loss of generality $|Y| < |X|$ or $|X| = |Y| \leq l-1$, then $|Y| \leq l-1$ and there exists an element $x \in X$ such that $x \notin Y$. Now x is 2-covered by at least $2n(l-1)+1$ codewords of C. By Lemmas 3.1 and 3.2, any element of Y can 2-cover at most $2n$ of the words in $I_2(x)$. This implies that $2n|Y| \leq 2n(l-1) < 2n(l-1)+1 \leq |I_2(x)|$. Therefore there exists $c \in I_2(x)$ with $c \notin I_2(Y)$. Since $I_2(x) \subseteq I_2(X)$, we get $I_2(X) \neq I_2(Y)$.

Suppose now $|X| = |Y| = l$. If $I_2(X) = I_2(Y)$, then as X and Y are distinct, there exist elements x and y such that $x \in X$ but $x \notin Y$ and $y \in Y$ but $y \notin X$. By Lemma 3.1, $|B_2(x) \cap B_2(a)| \leq 2n$ for all $a \in Y$ and hence $I_2(a)$ contains at most $2n$ of the words in $I_2(x)$.

Suppose there exists an element $a' \in Y$ such that $d(x, a') \geq 5$, then $|B_2(x) \cap B_2(a')| = 0$ and hence $I_2(a')$ does not contain any word of $I_2(x)$. Therefore $2n(|Y| - 1) \leq 2n(l - 1) < 2n(l-1)+1 \leq |I_2(x)|$. Therefore there exists a $c \in I_2(x)$ with $c \notin I_2(Y) = I_2(X)$, a contradiction. Hence, $1 \leq d(x, a) \leq 4$ for all $a \in Y$. Analogously, $1 \leq d(y, b) \leq 4$ for all $b \in X$. Without loss of generality we may assume that x is the all-zero word. Denote S_i by $S_i(\mathbf{0})$.

Case 1 : Let $d(x, y) = 1$. Then $w(y) = 1$ and there exists an $i \in \{1, 2, \ldots, n\}$ such that the i-th coordinate of y is 1. Since $|I_2(y)| \geq 2n(l - 1) + 1$ and $|I_2(y) \cap (S_0 \cup S_1 \cup S_2)| = |I_2(y) \cap S_0| + |I_2(y) \cap S_1| + |I_2(y) \cap S_2| \leq 1 + n + n - 1 \leq 2n$, we have $|I_2(y) \cap S_3| \geq 2nl - 4n + 1$.

We now claim that every element of X 2-covers at most $2n - 6$ elements of $I_2(y) \cap S_3$. If α is any element of X with $d(\alpha, y) \geq 3$, by Lemma 3.1, α can 2-cover at most 6 elements in $I_2(y) \cap S_3$. If $d(\alpha, y) = 1$ or 2, then $w(\alpha) \in \{0, 1, 2, 3\}$. If $w(\alpha) = 0$, then α can not 2-cover elements of S_3. If $w(\alpha) = 1$, then there exists $j \neq i$ such that the j-th coordinate of α is 1 because $\alpha \neq y$. We must change the i-th coordinate of α because it is zero and the resulting word belongs to $I_2(y) \cap S_3$. If we change p of the coordinates of α which are having 1 and also change q of the coordinates of α except i which are zero and leave the remaining coordinates unchanged, then p and q must satisfy the two relations $p + q \leq 1$ and $1 - p + q = 2$ because the resulting word belongs to $I_2(y) \cap S_3$. If $p = 1$, then $q = 2$ but $p + q = 3 > 1$. Therefore $p = 0$. This impies that $q = 1$. Therefore α 2-covers at most $\binom{1}{1}\binom{1}{0}\binom{n-2}{1} = n - 2$ words of $I_2(y) \cap S_3$. Similarly we can prove that if $w(\alpha) = 2$ then α can 2-cover at most $n - 2$ words of $I_2(y) \cap S_3$ and if $w(\alpha) = 3, \alpha$ can 2-cover at most $2n - 6$ words of $I_2(y) \cap S_3$. Therefore every element of X can 2-cover at most $2n - 6$ words of $I_2(y) \cap S_3$.

Next we claim that every element of X except x 2-covers at least six words of $I_2(y) \cap S_3$. Otherwise, there exists $\alpha' \in X, \alpha' \neq x$ such that α' 2-covers at most five words of $I_2(y) \cap S_3$. Since x does not 2-cover any word of $I_2(y) \cap S_3$ because x is the zero word and α' 2-covers at most 5 words of $I_2(y) \cap S_3$ and we have already shown that every element of X 2-covers at most $2n - 6$ elements of $I_2(y) \cap S_3$, we get

$$\begin{aligned}
(2n - 6)(|X| - 2) + 5 &= (2n - 6)(l - 2) + 5 \\
&= 2nl - 4n - 6l + 17 \\
&\leq 2nl - 4n - 36 + 17 \\
&= 2nl - 4n - 19 \\
&< 2nl - 4n + 1 \leq |I_2(y) \cap S_3|.
\end{aligned}$$

Therefore there exists a codeword $c \in I_2(y) \cap S_3$ with $c \notin I_2(X)$. Since $I_2(y) \cap S_3 \subseteq I_2(Y)$, we get a contradiction to $I_2(X) = I_2(Y)$.

Now we are going to find an element in X which has distance greater than or equal to 3 from y. Since $|I_2(x)| \geq 2n(l-1)+1$ and $I_2(y)$ contains at most $2n$ words of $I_2(x)$, the remaining at least $2nl-4n+1$ words of $I_2(x)$ are from the set $A := \{z = (z_1 z_2 \dots z_n) \in \mathbb{F}_2^n : w(z) = 2, z_i = 0\}$(where i has already been fixed). These words must be 2-covered by some elements of Y except y. If all elements of Y except y are having weight either 1 or greater than or equal to 2 with i-th coordinate 1, then we have to prove that any one of these elements can 2-cover at most $n-2$ elements of A.

If $y' \in Y$ with $w(y') = 1$, then there exists $j \neq i$ such that the j-th coordinate of y' is 1 because $y' \neq y$. We can not change the i-th coordinate of y' because the resulting word must be in A, so its i-th coordinate must be zero. If we change p of the coordinates of y' which are having 1 and also change q of the coordinates of y' except i which are 0 and leave the remaining coordinates unchanged, then p and q must satisfy these two relations $p+q \leq 1$ and $1-p+q = 2$ because the resulting word belongs to A. If $p = 1$ then $q = 2$ but $p+q = 3 > 1$. Therefore $p = 0$. This impies that $q = 1$. Therefore y' 2-covers at most $\binom{1}{0}\binom{1}{0}\binom{n-2}{1} = n-2$ words of A. Similarly, we can prove that y' 2-covers at most $n-2$ words of A if $w(y') = 2$ with i-th coordinate of y' being 1. If $w(y') \geq 3$ with i-th coordinate of y' being 1, then y' can 2-cover at most 6 words of A because $d(y', x) \geq 3$. Because $l \geq 6$, $(n-2)(|Y|-1) = nl-n-2l+2 < 2nl-4n+1$, a contradiction to $I_2(X) = I_2(Y)$. Therefore there exists an element $\beta \in Y$ with $2 \leq w(\beta) \leq 4$ with the i-th coordinate of β being zero.

If $\beta \in X$, we have obtained an element in X which has distance greater than or equal to 3 from y because $d(\beta, y) \geq 3$. If $\beta \notin X$, then we have to find an element in X which has distance greater than or equal to 3 from y. We do this now,

If $w(\beta) = 2$, then we have to prove that in $I_2(\beta)$, there are at most $4n-5$ words such that either these words or some of their 2-covers have distance less than or equal to two with y. Since $w(\beta) = 2$, there exist j and k in $\{1, 2, \dots, n\}$ with j-th and k-th coordinates of β being 1. If we change two of the coordinates of β except the i-th, j-th and k-th coordinates, then the resulting word belongs to $B_2(\beta)$ and it has distance 5 from y and any of its 2-covers have distance at least 3 from y. There are $\binom{3}{0}\binom{n-3}{2} = \binom{n-3}{2}$ words in $B_2(\beta)$ such that either these words or any of their 2-covers have distance ≥ 3 from y. Therefore there are at most $\sum_{i=0}^{2} \binom{n}{i} - \binom{n-3}{2} = 4n-5$ words in $I_2(\beta)$ such that either these words or some of their 2-covers have distance less than or equal to two from y because $I_2(\beta) \subseteq B_2(\beta)$.

Since $I_2(\beta) \geq 2n(l-1)+1 \geq 10n+1$, the remaining at least $10n+1-(4n-5) = 6n+6$ words or any of its 2-covers have distance greater than or equal to three from y. These words belong to $I_2(Y)$. Because $I_2(X) = I_2(Y)$, these words or some of their 2-covers must be in X. Therefore there exists an element $\gamma \in X$ such that $d(\gamma, y) \geq 3$. Similarly we can prove that there exists an element $\gamma \in X$ such that $d(\gamma, y) \geq 3$ if $w(\beta) = 3 \ or \ 4$.

Therefore $w(\gamma) \in \{2, 3, 4, 5\}$ because $1 \leq d(y, b) \leq 4$ for all $b \in X$. If $w(\gamma) = 2$, then we have to prove that γ can 2-cover at most one element of $I_2(y) \cap S_3$. Since $d(\gamma, y) \geq 3$ and $w(\gamma) = 2$, the i-th coordinate of γ is zero, so we must change it because the resulting word must be in $I_2(y) \cap S_3$ and if we change p of the coordinates of γ which are having one and also change q of the coordinates of γ except i which are zero and leave the remaining coordinates unchanged, then p and q must satisfy the two relations $p + q \leq 1$ and $2 - p + q = 2$ and hence $p = q$. Therefore $p = 0$ and $q = 0$. Similarly we can prove that γ can 2-cover at most three words of $I_2(y) \cap S_3$ if $w(\gamma) = 3 \ or \ 4$. Because every element of X except x 2-covers at least six elements of $I_2(y) \cap S_3$, we have $w(\gamma) = 5$. But now $|I_2(\gamma)| \geq 2n(l-1) + 1 \geq 10n + 1, |B_2(\gamma) \cap S_7| = \binom{n-5}{2}$ and

$$|I_2(\gamma) \cap (\cup_{i=3}^{6} S_i)| \leq |B_2(\gamma) \cap (\cup_{i=3}^{6} S_i)| = \sum_{i=0}^{2} \binom{n}{i} - \binom{n-5}{2} = 6n - 14 < 10n + 1$$

and therefore, $I_2(\gamma) \cap S_7 \neq \emptyset$. Because $w(a) \leq 4$ for all $a \in Y$, the set $I_2(Y)$ can not contain the codewords of $I_2(\gamma) \cap S_7$, a contradiction. Consequently, $I_2(X) \neq I_2(Y)$ if $d(x, y) = 1$.

Similarly we can prove the result if $d(x, y) = 2$.

Case 2 : Let $d(x, y) = 3$. Hence $w(y) = 3$ and there exist i, j and k in $\{1, 2, \ldots, n\}$ such that the i-th, j-th and k-th coordinates of y are 1. Now $|I_2(y) \cap S_5| \geq 2nl - 6n + 6$ due to the facts that $|I_2(y)| \geq 2n(l-1) + 1$ and $|I_2(y) \cap (\cup_{i=1}^{4} S_i)| \leq |B_2(y) \cap (\cup_{i=1}^{4} S_i)| = \sum_{i=0}^{2} \binom{n}{i} - \binom{n-3}{2} = 4n - 5$. Now we claim that any element of X can 2-cover at most $n-4$ elements of $I_2(y) \cap S_5$. If $x' \in X$ with $d(x', y) \geq 3$, then by Lemma 3.1, x' can 2-cover at most six elements of $I_2(y) \cap S_5$. If $x' \in X$ with $d(x', y) = 1$ or 2, Clearly $w(x') \in \{1, 2, 3, 4, 5\}$. If $w(x') = 5$, then, by $|I_2(x')| \geq 2n(l-1) + 1 \geq 10n + 1, |B_2(x') \cap S_7| = \binom{n-5}{2}$ and

$$|I_2(x') \cap (\cup_{i=3}^{6} S_i)| \leq |B_2(x') \cap (\cup_{i=3}^{6} S_i)| = \sum_{i=0}^{2} \binom{n}{i} - \binom{n-5}{2} = 6n - 14 < 10n + 1$$

and therefore, $I_2(x') \cap S_7 \neq \emptyset$. Because $w(a) \leq 4$ for all $a \in Y$, the set $I_2(Y)$ can not contain the codewords of $I_2(x') \cap S_7$, a contradiction to $I_2(X) = I_2(Y)$.

If $w(x') = 1$ or 2, then x' can not 2-cover the elements of S_5. If $w(x') = 3$, then at least one of the i-th, j-th and k-th coordinates of x' is zero because $x' \neq y$ (say i-th coordinate). We must change the i-th coordinate because the resulting word belongs to $I_2(y) \cap S_5$ and if we change p of the coordinates of x' which are having one and also change q of the coordinates of x' except i which are zero and leave the remaining coordinates unchanged, then p and q must satisfy these two relations $p+q \leq 1$ and $3-p+q=4$ because the resulting word belongs to $I_2(y)\cap S_5$. If $p\geq 1$ then $q=4-3+p\geq 2$ but $p+q\geq 1+2>1$. Therefore $p=0$. This implies that $q=1$. Therefore x' can 2-cover at most $\binom{1}{1}\binom{3}{0}\binom{n-4}{1}=n-4$ elements of $I_2(y)\cap S_5$. Similarly we can prove that x' can 2-cover at most $n-4$ elements of $I_2(y)\cap S_5$ if $w(x')=4$. Consequently, any word of X can 2-cover at most $n-4$ words of $I_2(y)\cap S_5$ and x does not 2-cover any element of S_5. Therefore

$$2nl-6n+6 \leq (n-4)(|X|-1)=(n-4)(l-1)$$

$$\Rightarrow l(2n-n+4)\leq 6n-6-n+4$$

$$\Rightarrow l(n+4)\leq 5n-2$$

$$\Rightarrow l\leq 5\frac{n}{n+4}-\frac{2}{n+4}<5\frac{n}{n+4}\leq 5$$

a contradiction to $l\geq 6$. Consequently, $I_2(X)\neq I_2(Y)$ if $d(x,y)=3$.

Similarly we can prove the result if $d(x,y)=4$. □

Now we define the length function $L(r,l)$: Let r and l be any two positive integers. Define $L(r,l)$ to be the smallest positive integer n for which there exists an $(r,\leq l)$-identifying code in $\mathbb{F}_2^n$ (Such a code exists. See Corollary 4.5 below).

From Theorem 4.1, the upper bound in Table 1 is reduced for all $r=2$ and $l\geq 6$. The improved upper bound of $L(2,l)$ is the smallest positive integer satisfying $1+n+\binom{n}{2}\geq 2n(l-1)+1$ by the statement of Theorem 4.1. The improved upper bounds of $L(2,l)$ are given in Table 2 for small values of l.

The following results are immediate consequences of Theorem 4.1.

Corollary 4.2 — Let k be the smallest positive integer satisfying $1+k+\binom{k}{2}\geq 2k(l-1)+1$. We have,

1. $\mathbb{F}_2^n$ is a $(2,\leq l)$-identifying code for all $n\geq k, l\geq 6$.
2. $\mathbb{F}_2^n\setminus X$ is a $(2,\leq l)$-identifying code for all $n\geq k, l\geq 6$ where $X\subseteq\mathbb{F}_2^n$ with $|X|\leq 3n+\binom{n}{2}-2nl$.

PROOF : (1) Let $n\geq k$. Let $C=\mathbb{F}_2^n$ and let $x\in\mathbb{F}_2^n$. Since k is the smallest positive integer satisfying $1+k+\binom{k}{2}\geq 2k(l-1)+1$ and $n\geq k$, $|I_2(x)|=|B_2(x)|=1+n+\binom{n}{2}\geq 2n(l-1)+1$. By Theorem 4.1, C is a $(2,\leq l)$-identifying code.

Table 1: Previous lower and upper bounds for $L(r, l)$

r \ l	6	7	8	9	10	11	12	13
2	11-20	13-24	14-28	15-32	16-36	17-40	19-44	20-48
3	14-30	15-36	16-42	18-48	19-54	20-60	23-66	24-72
4	16-40	17-48	19-56	20-64	21-72	23-80	25-88	26-96
5	19-50	20-60	21-70	22-80	24-90	25-100	27-110	28-120
6	21-60	22-72	23-84	25-96	26-108	27-120	29-132	30-144
7	23-70	24-84	25-98	27-112	28-126	31-140	32-154	33-168

[2] The lower bounds of Table 1 are from Theorem 4 of [8] and Table 7.1 of [3].

[3] The upper bounds in Table 1 are from Theorem 2 of [12] and Theorem 21, Theorem 24 of [4].

Table 2: Improved upper bounds for $L(2, l)$ from our Theorem 4.1

r \ l	6	7	8	9	10	11	12	13
2	19	23	27	31	35	39	43	47

(2) Let $n \geq k$ and let $X \subseteq \mathbb{F}_2^n$ with $|X| \leq 3n + \binom{n}{2} - 2nl$. Let $C = \mathbb{F}_2^n \setminus X$ and let $x \in \mathbb{F}_2^n$. Then $I_2(x) = B_2(x) \cap C$. If $X \subseteq B_2(x)$, then $|B_2(x) \cap C| \geq 1 + n + \binom{n}{2} - 3n - \binom{n}{2} + 2nl = 2n(l-1) + 1$. Otherwise, $|B_2(x) \cap C| > 1 + n + \binom{n}{2} - 3n - \binom{n}{2} + 2nl = 2n(l-1) + 1$. Therefore $|I_2(x)| \geq 2n(l-1) + 1$. Hence by the Theorem 4.1, C is an $(2, \leq l)$-identifying code. □

Remark 4.3 : For every integer $l \geq 6$, using (2) of Corollary 4.2, we can give $(2, \leq l)$-identifying codes having smaller cardinality than previously known (see Theorem 2 of [12] and Theorem 21, Theorem 24 of [4]) for the length one more than the improved upper bounds of $L(2, l)$ in Table 2. This is established in Table 3 for small values of l.

***Theorem* 4.4** — *Let r and l be any two positive integers. Let $C \subseteq \mathbb{F}_2^n$ be such that $|I_r(x)| \geq (\sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1})l + 1$ for all $x \in \mathbb{F}_2^n$. Then C is an $(r, \leq l)$-identifying code. Consequently, $M_r^{(\leq l)}(n) \leq K(n, r, \mu)$ where $\mu = (\sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1})l + 1$.*

PROOF : Let X and Y be any two distinct subsets of $\mathbb{F}_2^n$ with $|X| \leq l$ and $|Y| \leq l$. Without loss of generality we assume that there exists an element $x \in X$ such that $x \notin Y$. Now x is r-covered by at least $\left(\sum_{i=0}^{r-1} \binom{n}{i} + \binom{n-1}{r-1}\right) l + 1$ codewords of C. By Lemmas 3.1 and 3.2, any element

Table 3: Improved upper bounds of $M_2^{(\leq l)}(n)$ using (2) of Corollary 4.2

l	n	Improved upper bound	Previous upper bound
6	20	1048566	1048576
7	24	16777204	16777216
8	28	268435442	268435456
9	32	4294967280	4294967296
10	36	68719476718	68719476736

of Y can r-cover at most $\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1}$ words of $I_r(x)$. This implies that $(\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})|Y| \leq (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l < (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l+1 \leq |I_r(x)|$. Therefore there exists $c \in I_r(x)$ with $c \notin I_r(Y)$. Since $I_r(x) \subseteq I_r(X)$, we get $I_r(X) \neq I_r(Y)$ and hence C is an $(r, \leq l)$-identifying code. The second part is a trivial consequence of the first part. □

Corollary 4.5 — Let k be the smallest positive integer satisfying $\sum_{i=0}^{r}\binom{k}{i} \geq \sum_{i=0}^{r}\binom{k}{i} - (\sum_{i=0}^{r-1}\binom{k}{i} + \binom{k-1}{r-1})l - 1$. Then

1. $\mathbb{F}_2^n$ is an $(r, \leq l)$-identifying code for all $n \geq k$
2. $\mathbb{F}_2^n \setminus X$ is an $(r, \leq l)$-identifying code for all $n \geq k$ where $X \subseteq \mathbb{F}_2^n$ with $|X| \leq \sum_{i=0}^{r}\binom{n}{i} - (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l - 1$.

PROOF : (1) Let $n \geq k$. Let $C = \mathbb{F}_2^n$ and let $x \in \mathbb{F}_2^n$. Since k is the smallest positive integer satisfying $\sum_{i=0}^{r}\binom{k}{i} \geq \sum_{i=0}^{r}\binom{k}{i} - (\sum_{i=0}^{r-1}\binom{k}{i}+\binom{k-1}{r-1})l - 1$ and $n \geq k$, $|I_r(x)| = |B_r(x)| = \sum_{i=0}^{r}\binom{n}{i} \geq \sum_{i=0}^{r}\binom{n}{i} - (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l - 1$. By Theorem 4.4, C is an $(r, \leq l)$-identifying code.

(2) Let $n \geq k$ and let $X \subseteq \mathbb{F}_2^n$ with $|X| \leq \sum_{i=0}^{r}\binom{n}{i} - (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l - 1$. Let $C = \mathbb{F}_2^n \setminus X$ and let $x \in \mathbb{F}_2^n$. Then $I_r(x) = B_r(x) \cap C$. If $X \subseteq B_r(x)$ then $|B_r(x) \cap C| \geq \sum_{i=0}^{r}\binom{n}{i} - (\sum_{i=0}^{r}\binom{n}{i} - (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l - 1) = (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l+1$. Otherwise $|B_r(x) \cap C| > \sum_{i=0}^{r}\binom{n}{i} - (\sum_{i=0}^{r}\binom{n}{i} - (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l - 1) = (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l+1$. Therefore $|I_r(x)| \geq (\sum_{i=0}^{r-1}\binom{n}{i}+\binom{n-1}{r-1})l+1$. So by Theorem 4.4, C is an $(r, \leq l)$-identifying code. □

Table 4: gives the values of k as defined in Corollary 4.5.

l	k	l	k
1	3	8	172
2	7	9	221
3	19	10	277
4	37	11	339
5	61	12	407
6	92	13	482
7	129	14	562

Corollary 4.6 — Let l be any positive integer. Let k be the smallest positive integer such that $(\sum_{i=0}^{k-1}\binom{2k+1}{i}+\binom{2k}{k-1})l+1 \leq \frac{1}{2}\sum_{i=0}^{k}\binom{2k+1}{i}$. Then $M_r^{(\leq l)}(2r+1) \leq 2^{2r}$ for all $r \geq k$.

PROOF : Let $r \geq k$ and let $\mu = (\sum_{i=0}^{r-1}\binom{2r+1}{i}+\binom{2r}{r-1})l+1$ and $\gamma = \frac{1}{2}\sum_{i=0}^{r}\binom{2r+1}{i}$. By Theorem 14.2.2 of [3], there exists an γ-fold r-covering code C of length $2r+1$ with cardinality 2^{2r}. Since $r \geq k$ and k is the smallest positive integer such that $(\sum_{i=0}^{k-1}\binom{2k+1}{i}+\binom{2k}{k-1})l+1 \leq \frac{1}{2}\sum_{i=0}^{k}\binom{2k+1}{i}$, $\mu \leq \gamma$. Since every p-fold r-covering code is also q-fold r-covering code for all $q \leq p$, C is also a μ-fold r-covering code of length $2r+1$ with cardinality 2^{2r}. Therefore $K(2r+1,r,\mu) \leq 2^{2r}$. By Theorem 4.4, $M_r^{(\leq l)}(2r+1) \leq 2^{2r}$. □

In Table 4, we present the smallest possible integer k that satisfies $(\sum_{i=0}^{k-1}\binom{2k+1}{i}+\binom{2k}{k-1})l+1 \leq \frac{1}{2}\sum_{i=0}^{k}\binom{2k+1}{i}$ for small values of l. Then by Corollary 4.6, we have $M_r^{(\leq l)}(2r+1) \leq 2^{2r}$ for all $r \geq k$. For example, $M_r^{(\leq 3)}(2r+1) \leq 2^{2r}$ for all $r \geq 19$.

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