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HOLOMORPHIC ASPECTS OF MODULI OF REPRESENTATIONS OF QUIVERS

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This article describes some complex-analytic aspects of the moduli space of the finite-dimensional complex representations of a finite quiver, which are stable with respect to a fixed rational weight. We construct a natural structure of a complex manifold on this moduli space, and a Kähler metric on the complex manifold. We then define a Hermitian holomorphic line bundle on the moduli space, and show that its curvature is a rational multiple of the Kähler form.

Key words : Quiver representations; Moduli space; Kähler metric; Hermitian line bundle.

1. INTRODUCTION

Moduli spaces of representations of quivers are of interest because of their relations with the moduli spaces of representations of algebras [3], and with the moduli spaces of sheaves on projective schemes [1]. A general survey about the moduli spaces of representations of quivers is [4].

In this paper, we discuss some complex-analytic aspects of the moduli space of the finite-dimensional complex representations of a finite quiver, which are stable with respect to a fixed rational weight. We describe a natural Kähler metric on this moduli space, and exhibit a Hermitian holomorphic line bundle on it, whose Chern form is essentially an integral multiple of the Kähler form of this metric. This integral multiple depends only on the chosen weight.

The methods of this paper are elementary in nature. They are based on Kähler geometry, and do not use any results from geometric invariant theory. In particular, when the moduli spaces of stable representations are compact, by the Kodaira embedding theorem, the results of this paper give an analytic proof of the projectivity of these moduli spaces.

We view the stability of representations of quivers as a special case of Rudakov's theory of stability structures on an abelian category. Accordingly, we begin by recalling this theory in Section 2. Any finite positive family of additive functions on an abelian category, and a corresponding family of real numbers, called a *weight*, define a stability structure on the category. There is a natural hyperplane arrangement on the space of weights, and the stability condition remains constant within every facet of this hyperplane arrangement. We describe this idea in Section 3. It includes, as a special case, the stability of representations of a finite quiver with respect to a given weight.

We recall some basic notions about quivers and their representations in Section 4. We also describe a theorem of King, which relates stability of a representation of a quiver to the existence of a certain kind of inner product on the representation, which we call an *Einstein-Hermitian metric*, because of its similarity to Einstein-Hermitian metrics on vector bundles. We discuss families of representations in Section 5, and explain a criterion for two representations in a family to be separated from each other.

We construct the moduli space of Schur representations in Section 6. It is, in general, a non-Hausdorff complex manifold. Its open subset of stable representations is Hausdorff, and has a natural Kähler metric, as we explain in Section 7. We end the paper with a description of a natural Hermitian holomorphic line bundle on the moduli space of representations that are stable with respect to a rational weight, and show that its curvature is essentially an integral multiple of the Kähler form on the moduli space.

2. Stability Structures

The stability of representations of a quiver is a special case of the notion of a stability structure that was defined by Rudakov [5, Definition 1.1]. In this section, we recall some properties of such stability structures. We first look at the definition of stability structures. Then, we recall the Schur Lemma about the endomorphisms of stable objects. Lastly, we mention Jordan-Hölder and Harder Narasimhan filtrations, and the notion of S-equivalence of semistable objects.

2.A. *Semistable objects of an abelian category*

Let $\mathcal{A}$ be an abelian category, and $\preceq$ a total preorder on the set of non-zero objects of $\mathcal{A}$. For any two non-zero objects M and N of $\mathcal{A}$, write $M \succeq N$ if $N \preceq M$, and define

$$\begin{aligned} M \prec N &\quad \text{if} \quad M \preceq N \text{ and } M \not\succeq N, \\ M \asymp N &\quad \text{if} \quad M \preceq N \text{ and } M \succeq N, \\ M \succ N &\quad \text{if} \quad M \succeq N \text{ and } M \not\preceq N. \end{aligned}$$

Then, $\succeq$ is a total preorder, $\prec$ and $\succ$ irreflexive transitive relations, and $\asymp$ an equivalence relation, on the set of non-zero objects of $\mathcal{A}$. Moreover, for any two non-zero objects M and N of $\mathcal{A}$, exactly one of the three statements

$$M \prec N, \quad M \asymp N, \quad M \succ N$$

holds. We say that $\preceq$ has the *seesaw property* if for every short exact sequence

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

of non-zero objects of $\mathcal{A}$, exactly one of the three statements

$$M' \prec M \prec M'', \quad M' \asymp M \asymp M'', \quad M' \succ M \succ M''$$

is true. A *stability structure* on $\mathcal{A}$ is a total preorder on the set of non-zero objects of $\mathcal{A}$, which has the seesaw property.

We fix a stability structure $\preceq$ on an abelian category $\mathcal{A}$. The seesaw property implies that if M and M' are two isomorphic non-zero objects of $\mathcal{A}$, then $M \asymp M'$. It follows that if $M \cong M'$ and $N \cong N'$ are isomorphisms of objects in $\mathcal{A}$, then $M \preceq N$ (respectively, $M \prec N$, $M \asymp N$) if and only if $M' \preceq N'$ (respectively, $M' \prec N'$, $M' \asymp N'$). In particular, if $i : N \to M$ and $i' : N' \to M$ are two equivalent non-zero subobjects of M, we have $N \preceq M$ (respectively, $N \prec M$, $N \asymp M$) if and only if $N' \preceq M$ (respectively, $N' \prec M$, $N' \asymp M$).

An object M of $\mathcal{A}$ is called *semistable* (respectively, *stable*) if

$$N \preceq M \quad \text{(respectively, } N \prec M)$$

for every non-zero proper subobject N of M. We say that an object of $\mathcal{A}$ is *polystable* if it is semistable, and is isomorphic to the direct sum of a finite family of stable objects of $\mathcal{A}$. It is obvious that stable $\Rightarrow$ polystable $\Rightarrow$ semistable, and that all three properties are preserved by isomorphisms in $\mathcal{A}$. It is also easy to verify the following statements about semistable objects.

Proposition 2.1 — Let $\preceq$ be a stability structure on an abelian category $\mathcal{A}$. Let S be an $\asymp$-equivalence class in the set of non-zero objects of $\mathcal{A}$, and let $\mathcal{A}(S)$ be the full subcategory of $\mathcal{A}$, whose objects are either zero objects of $\mathcal{A}$, or semistable objects of $\mathcal{A}$ which belong to S. Then:

(1) A non-zero object M of $\mathcal{A}$ is semistable (respectively, stable) if and only if for every non-zero epimorphism $f : M \to N$ which is not an isomorphism, we have

$$M \preceq N \quad \text{(respectively, } M \prec N).$$

(2) Let

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

be a short exact sequence of non-zero objects of $\mathcal{A}$. Suppose that

$$M' \asymp M \asymp M''.$$

Then, M is semistable if and only if both M' and M'' are semistable.

(3) Let M and N be two non-zero objects of $\mathcal{A}$. Then, the object $M \oplus N$ is semistable if and only if both M and N are semistable, and $M \asymp N$. In that case,

$$(M \oplus N) \asymp M \asymp N.$$

(4) Let M and N be two semistable objects of $\mathcal{A}$, and let $f : M \to N$ be a morphism. Suppose that $M \asymp N$. Then, each of the objects $\mathrm{Ker}\,(f)$, $\mathrm{Img}\,(f)$, $\mathrm{Coimg}\,(f)$, and $\mathrm{Coker}\,(f)$, is either zero, or is semistable and $\asymp$-related to M.

(5) The category $\mathcal{A}(S)$ is an abelian subcategory of $\mathcal{A}$.

2.B *The Schur Lemma*

Recall that an object M of an additive category $\mathcal{A}$ is called *simple* if it is non-zero, and has no non-zero proper subobject. We say that M is a *Schur* object, or a *brick*, if the ring $\mathrm{End}(M)$ is a division ring. It is easy to see that every simple object of $\mathcal{A}$ is Schur. The following Proposition follows directly from [5, Theorem 1].

Proposition 2.2 — Let $\mathcal{A}$, $\preceq$, S, and $\mathcal{A}(S)$, be as in Proposition 2.1.

(1) Let M and N be two semistable objects of $\mathcal{A}$, and let $f : M \to N$ be a non-zero morphism. Suppose that $M \succeq N$. Then:

  (a) $M \asymp N$.

  (b) If M is stable, then f is a monomorphism.

  (c) If N is stable, then f is an epimorphism.

  (d) If both M and N are stable, then f is an isomorphism.

(2) An element M of S is a simple object of the abelian category $\mathcal{A}(S)$ if and only if it is stable.

(3) (Schur Lemma) Every stable object of $\mathcal{A}$ is a Schur object of $\mathcal{A}$.

(4) Suppose that $\mathcal{A}$ is a K-linear abelian category, where K is an algebraically closed field, and let M be an object of $\mathcal{A}$. Suppose also that the K-vector space $\mathrm{End}(M)$ is finite-dimensional. Then, M is a Schur object of $\mathcal{A}$ if and only if for every endomorphism f of M in $\mathcal{A}$, there exists a unique element $\lambda \in K$, such that $f = \lambda \mathbf{1}_M$.

2.C. *Jordan-Hölder filtrations*

A sequence $(M_n)_{n\in\mathbf{N}}$ of subobjects of an object M of $\mathcal{A}$ is called *stationary* if there exists $n_0 \in \mathbf{N}$, such that $M_n = M_{n+1}$ for all $n \geq n_0$. We say that an object M of $\mathcal{A}$ is

(1) *Noetherian* (respectively, *Artinian*) if every sequence $(M_n)_{n\in\mathbf{N}}$ of subobjects of M, such that $M_n \subset M_{n+1}$ (respectively, $M_n \supset M_{n+1}$) for all $n \in \mathbf{N}$, is stationary.

(2) *quasi-Noetherian* with respect to $\preceq$ if every sequence $(M_n)_{n\in\mathbf{N}}$ of subobjects of M, such that $M_n \subset M_{n+1}$, and $M_n \preceq M_{n+1}$ for all $n \in \mathbf{N}$, is stationary.

(3) *weakly Artinian* with respect to $\preceq$ if every sequence $(M_n)_{n\in\mathbf{N}}$ of subobjects of M, such that $M_n \supset M_{n+1}$, and $M_n \preceq M_{n+1}$ for all $n \in \mathbf{N}$, is stationary.

(4) *weakly Noetherian* with respect to $\preceq$ if it is quasi-Noetherian with respect to $\preceq$, and if every sequence $(M_n)_{n\in\mathbf{N}}$ of subobjects of M, such that $M_n \subset M_{n+1}$, and $M_n \succeq M_{n+1}$ for all $n \in \mathbf{N}$, is stationary.

The category $\mathcal{A}$ is called *Noetherian* (respectively, *Artinian*) if every object in it is Noetherian (respectively, Artinian). It is called *quasi-Noetherian* (respectively, *weakly Artinian*, *weakly Noetherian*) with respect to $\preceq$ if every object in it is quasi-Noetherian (respectively, weakly Artinian, weakly Noetherian) with respect to $\preceq$.

If M is a semistable object of $\mathcal{A}$, then a *Jordan-Hölder filtration* of M with respect to $\preceq$ is a sequence $(M_i)_{i=0}^n$ of subobjects of M, such that $n \geq 1$, $M_0 = M$, $M_n = 0$, and $M_i \subset M_{i-1}$, M_{i-1}/M_i is stable, and $M_{i-1} \asymp M$, for every $i = 1, \ldots, n$. If M is an arbitrary object of $\mathcal{A}$, then a *Harder-Narasimhan filtration* of M with respect to $\preceq$ is a sequence $(M_i)_{i=0}^n$ of subobjects of M, such that $n \in \mathbf{N}$, $M_0 = M$, $M_n = 0$, $M_i \subset M_{i-1}$ and $G_i = M_{i-1}/M_i$ is semistable for every $i = 1, \ldots, n$, and $G_{i-1} \prec G_i$ for every $i = 2, \ldots, n$.

The statements in the following Proposition are proved in [5, Theorems 2 and 3].

Proposition 2.3 — Let $\preceq$ be a stability structure on an abelian category $\mathcal{A}$.

(1) Suppose that $\mathcal{A}$ is quasi-Noetherian, and weakly Artinian, with respect to $\preceq$. Then, every semistable object M of $\mathcal{A}$ has a Jordan-Hölder filtration with respect to $\preceq$. Moreover, if

$(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ are two Jordan-Hölder filtrations of M with respect to $\preceq$, then $n = m$, and there exists a permutation $\pi \in \mathrm{S}_n$ such that M_{i-1}/M_i is isomorphic to $N_{\pi(i)-1}/N_{\pi(i)}$ for every $i = 1, \ldots, n$.

(2) Suppose that $\mathcal{A}$ is weakly Noetherian, and weakly Artinian, with respect to $\preceq$. Then, every object of $\mathcal{A}$ has a unique Harder-Narasimhan filtration with respect to $\preceq$.

Let M and N be two semistable objects of $\mathcal{A}$. Let $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ be Jordan-Hölder filtrations of M and N, respectively, with respect to $\preceq$, which exist by Proposition 2.3(1). Then, we say that M is *S-equivalent* to N with respect to $\preceq$, if $n = m$, and there exists a permutation $\pi \in \mathrm{S}_n$, such that M_{i-1}/M_i is isomorphic to $N_{\pi(i)-1}/N_{\pi(i)}$ for every $i = 1, \ldots, n$. By the above Proposition, this is independent of the choices of the Jordan-Hölder filtrations, and defines an equivalence relation on the set of all semistable objects of $\mathcal{A}$.

3. Stability with Respect to a Weight

We now consider a special kind of the stability structures defined in Section 2, namely stability structures defined by a finite family of positive additive functions on an abelian category, and a corresponding family of real numbers called weights, which form a finite-dimensional real vector space, the weight space. Fixing the values of the additive functions defines a hyperplane arrangement on the weight space. We describe how semistability and other related notions behave with respect to this hyperplane arrangement.

3.A. *Semistablity with respect to a weight*

Let $\mathcal{A}$ be an abelian category. We say that a family $(\phi_i)_{i \in I}$ of additive functions from the set of objects of $\mathcal{A}$ to an ordered abelian group G is *positive* if $\phi_i(M) \geq 0$ for every object M of $\mathcal{A}$ and for every $i \in I$, and if for every non-zero object M of $\mathcal{A}$, there exists an $i \in I$, such that $\phi_i(M) > 0$. In particular, we say that an additive function ϕ from $\mathcal{A}$ to G is *positive* if the singleton family (ϕ) is positive. If ϕ is a positive additive function from $\mathcal{A}$ to G, then an object M of $\mathcal{A}$ is zero if and only if $\phi(M) = 0$, hence, if M' is a subobject of an object M, then $\phi(M') \leq \phi(M)$, and equality holds if and only if $M' = M$. The category $\mathcal{A}$ is Noetherian and Artinian if there exists a positive additive function from $\mathcal{A}$ to $\mathbf{Z}$.

We now fix a non-empty finite positive family $(\phi_i)_{i \in I}$ of additive functions from $\mathcal{A}$ to $\mathbf{Z}$. The *dimension vector* of any object M of $\mathcal{A}$ is the element $\phi(M)$ of $\mathbf{N}^I$ that is defined by

$$\phi(M) = (\phi_i(M))_{i \in I}.$$

The *rank* of M is the natural number $\mathrm{rk}(M)$ defined by

$$\mathrm{rk}(M) = \sum_{i \in I} \phi_i(M).$$

Since $(\phi_i)_{i \in I}$ is a positive family of additive functions from $\mathcal{A}$ to $\mathbf{Z}$, the function rk is a positive additive function from $\mathcal{A}$ to $\mathbf{Z}$.

An element of the $\mathbf{R}$-vector space $\mathbf{R}^I$ is called a *weight* of $\mathcal{A}$. We say that a weight is *rational* (respectively, *integral*) if it belongs to the subset $\mathbf{Q}^I$ (respectively, $\mathbf{Z}^I$) of $\mathbf{R}^I$. We fix a weight θ of $\mathcal{A}$. We define the θ*-degree* of any object M of $\mathcal{A}$ to be the real number $\deg_\theta(M)$ given by

$$\deg_\theta(M) = \sum_{i \in I} \theta_i \phi_i(M).$$

If $M \neq 0$, we define another real number $\mu_\theta(M)$ by

$$\mu_\theta(M) = \frac{\deg_\theta(M)}{\mathrm{rk}(M)},$$

and call it the θ*-slope* of M.

Proposition 3.1 — Let θ be any weight of $\mathcal{A}$. Define a relation $\preceq_\theta$ on the set of non-zero objects of $\mathcal{A}$, by setting $M \preceq_\theta N$ if $\mu_\theta(M) \leq \mu_\theta(N)$. Then, $\preceq_\theta$ is a stability structure on $\mathcal{A}$.

PROOF : Let $c(M) = \deg_\theta(M)$, and $r(M) = \mathrm{rk}(M)$, for every object M of $\mathcal{A}$. Then, c is an additive function from $\mathcal{A}$ to the ordered abelian group $\mathbf{R}$, and r is a positive additive function from $\mathcal{A}$ to $\mathbf{Z}$. Moreover, in the notation of [5, Definition 3.1], the function μ_θ is the $(c : r)$ slope, and the relation $\preceq_\theta$ the $(c : r)$ preorder, on the set of non-zero objects of $\mathcal{A}$. Therefore, it follows from [5, Lemma 3.2 and Remark] that $\preceq_\theta$ is a stability structure on $\mathcal{A}$. □

Let θ be a weight of $\mathcal{A}$. An object of $\mathcal{A}$ is called θ*-semistable* (respectively, θ*-stable,* θ*-polystable*) if it is semistable (respectively, stable, polystable) with respect to the stability structure $\preceq_\theta$ on $\mathcal{A}$. If ζ is a strictly positive real number, and $\omega = \zeta\theta$, then an object of $\mathcal{A}$ is θ-semistable (respectively, θ-stable, θ-polystable) if and only if it is ω-semistable (respectively, ω-stable, ω-polystable).

There are obvious special versions of the statements in Propositions 2.1-2.2, with semistable (respectively, stable) objects replaced by θ-semistable (respectively, θ-stable) objects of $\mathcal{A}$. Moreover, since rk is a positive additive function from $\mathcal{A}$ to $\mathbf{Z}$, the category $\mathcal{A}$ is Noetherian and Artinian. In particular, by Proposition 2.3, every θ-semistable object of $\mathcal{A}$ has a Jordan-Hölder filtration, and every object of $\mathcal{A}$ has a unique Harder-Narasimhan filtration, with respect to $\preceq_\theta$. We say that two θ-semistable objects of $\mathcal{A}$ are S_θ*-equivalent* if they are S-equivalent with respect to $\preceq_\theta$.

There is a well-known definition of the stability of objects of $\mathcal{A}$ that is defined by King in [3, p. 516]. Let λ be an additive function from $\mathcal{A}$ to $\mathbf{R}$. Then, King defines an object M of $\mathcal{A}$ to be λ-semistable (respectively, θ-stable) if it is non-zero, if $\lambda(M) = 0$, and if $\lambda(N) \geq 0$ (respectively, $\lambda(N) > 0$) for every non-zero proper subobject N of M. The following Proposition shows that the notion of θ-semistability (respectively, θ-stability) defined above is a special case of this definition of λ-semistability (respectively, λ-stability).

Proposition 3.2 — Let θ be a weight of $\mathcal{A}$, μ a real number, and c a strictly positive real number. Define an additive function λ from $\mathcal{A}$ to $\mathbf{R}$, by putting $\lambda(M) = c(\mu\,\mathrm{rk}(M) - \deg_\theta(M))$ for every object M of $\mathcal{A}$. Let $O^{\mathrm{ss}}(\theta, \mu)$ (respectively, $O^{\mathrm{s}}(\theta, \mu)$) be the set of all θ-semistable (respectively, θ-stable) objects M of $\mathcal{A}$, such that $\mu_\theta(M) = \mu$. Let $K^{\mathrm{ss}}(\lambda)$ (respectively, $K^{\mathrm{s}}(\lambda)$) be the set of all λ-semistable (respectively, λ-stable) objects of $\mathcal{A}$, in the sense of King. Then, $O^{\mathrm{ss}}(\theta, \mu) = K^{\mathrm{ss}}(\lambda)$, and $O^{\mathrm{s}}(\theta, \mu) = K^{\mathrm{s}}(\lambda)$.

PROOF : The Proposition follows from the fact that $\mathrm{rk}(N) > 0$ for every non-zero object N of $\mathcal{A}$. □

3.B. *Facets with respect to a hyperplane arrangement*

Let E be an affine space modelled after a finite-dimensional $\mathbf{R}$-vector space T. We will give T its usual topology. A *hyperplane arrangement* in E is a locally finite set of hyperplanes in E. We fix a hyperplane arrangement $\mathcal{H}$ in E. For any subset X of E, we define

$$\mathcal{H}(X) = \{H \in \mathcal{H} \mid H \cap X \neq \emptyset\}.$$

If X is a singleton $\{x\}$, we write $\mathcal{H}(x)$ for $\mathcal{H}(X)$.

For every hyperplane H in E, we define an equivalence relation $\sim_H$ on E by setting $x \sim_H y$ if x and y belong to H, or if x and y are strictly on the same side of H. We define $\sim$ to be the equivalence relation on E, which is the intersection of the relations $\sim_H$ as H runs over $\mathcal{H}$. The $\sim$-equivalence class of an element a of E is called the *facet* of E through a with respect to $\mathcal{H}$.

Let F be a facet of E. Then, for every point a in F, and for any element H of $\mathcal{H}(a)$, we have $F \subset H$, hence

$$\mathcal{H}(F) = \mathcal{H}(a) = \{H \in \mathcal{H} \mid F \subset H\}.$$

In particular, since $\mathcal{H}$ is locally finite, the set $\mathcal{H}(F)$ is finite. The intersection $\mathrm{Supp}(F)$ of all the elements of $\mathcal{H}(F)$ is called the *support* of F. It is an affine subspace of E, whose dimension is called the *dimension* of F, and is denoted by $\dim(F)$. The closure $\overline{F}$ of F in E is a subset of $\mathrm{Supp}(F)$.

Remark 3.3 : Let F be a facet of E, and L its support. Then, F equals the interior of $\overline{F}$ in L. In particular, F is open in L [2, Chapter V, § 1, no. 2, Proposition 3].

3.C. *The hyperplane arrangement on the weight space*

Let $\mathcal{A}$ be an abelian category, $(\phi_i)_{i\in I}$ a non-empty finite positive family of additive functions from $\mathcal{A}$ to $\mathbf{Z}$, and rk a positive additive function from $\mathcal{A}$ to $\mathbf{Z}$, as in Section 2.A. The $\mathbf{R}$-vector space $\mathbf{R}^I$ is called the *weight space* of $\mathcal{A}$. We give it the usual, that is, the product topology.

For all elements $\theta = (\theta_i)_{i\in I}$ of $\mathbf{R}^I$ and $d = (d_i)_{i\in I}$ of $\mathbf{N}^I$, we define a real number $\deg_\theta(d)$, and a natural number $\mathrm{rk}(d)$, by

$$\deg_\theta(d) = \sum_{i\in I} \theta_i d_i, \quad \mathrm{rk}(d) = \sum_{i\in I} d_i.$$

If d is a non-zero element of $\mathbf{N}^I$, then $\mathrm{rk}(d) > 0$, so for each $\theta \in \mathbf{R}^I$, we have a real number $\mu_\theta(d)$, which is defined by

$$\mu_\theta(d) = \frac{\deg_\theta(d)}{\mathrm{rk}(d)}.$$

For any two non-zero elements d and e of $\mathbf{N}^I$, we define an $\mathbf{R}$-linear function $f(d, e) : \mathbf{R}^I \to \mathbf{R}$ by

$$f(d, e)(\theta) = \mu_\theta(d) - \mu_\theta(e).$$

It is obvious that $f(d, e) = 0$ if and only if $e \in \mathbf{Q}d$.

Fix a non-zero element d of $\mathbf{N}^I$. Let S_d denote the set of all elements e of $\mathbf{N}^I \setminus \mathbf{Q}d$, for which there exist an object M of $\mathcal{A}$, and a subobject N of M, such that $\phi(M) = d$ and $\phi(N) = e$. Since the family $(\phi_i)_{i\in I}$ of additive functions on $\mathcal{A}$ is positive, the set S_d is contained in $\prod_{i\in I}(\mathbf{N} \cap [0, d_i])$, and is hence finite. For each $e \in S_d$, let

$$H(d, e) = \mathrm{Ker}(f(d, e)) = \left\{\theta \in \mathbf{R}^I \mid \mu_\theta(e) = \mu_\theta(d)\right\}.$$

Since $e \notin \mathbf{Q}d$, the function $f(d, e)$ is non-zero, hence $H(d, e)$ is a hyperplane in $\mathbf{R}^I$. We thus get a finite hyperplane arrangement

$$\mathcal{H}(d) = \{H(d, e) \mid e \in S_d\}$$

in the affine space $\mathbf{R}^I$.

We define sgn to be the function from $\mathbf{R}$ to the subset $\{-1, 0, 1\}$ of $\mathbf{R}$, which is -1 at every negative real number, vanishes at 0, and is 1 at every positive real number.

Proposition 3.4 — Let F be a facet of $\mathbf{R}^I$ with respect to the hyperplane arrangement $\mathcal{H}(d)$, and let θ and ω be two elements of F. Let M be an object of $\mathcal{A}$, such that $\phi(M) = d$, and let N be a non-zero subobject of M. Then,

$$\operatorname{sgn}(\mu_\theta(M) - \mu_\theta(N)) = \operatorname{sgn}(\mu_\omega(M) - \mu_\omega(N)).$$

PROOF : Let $e = \phi(N)$, and let $f = f(d, e) : \mathbf{R}^I \to \mathbf{R}$. Then, for each weight λ of $\mathcal{A}$, we have

$$\mu_\lambda(M) - \mu_\lambda(N) = \mu_\lambda(d) - \mu_\lambda(e) = f(\lambda).$$

Therefore, we have to prove that $\operatorname{sgn}(f(\theta)) = \operatorname{sgn}(f(\omega))$. If $f(\theta) = f(\omega) = 0$, then the equality to be proved is obvious. Suppose that either $f(\theta)$ or $f(\omega)$ is non-zero. By interchanging θ and ω, we can assume that $f(\theta) \neq 0$. Then, $e \notin \mathbf{Q}d$. As $\phi(M) = d$ and $\phi(N) = e$, this implies that $e \in S_d$. Thus, the hyperplane $H = H(d, e)$ is an element of $\mathcal{H}(d)$. Now, as $H = \operatorname{Ker}(f)$, f is a defining function of H. Moreover, $\theta \notin H$, since $f(\theta) \neq 0$. As θ and ω both belong to the same facet F, we have $\theta \sim_H \omega$, so θ and ω are strictly on the same side of H. Therefore, $f(\theta)f(\omega) > 0$. It follows that $\operatorname{sgn}(f(\theta)) = \operatorname{sgn}(f(\omega))$. □

Proposition 3.5 — Let d be a non-zero element of N^I, F a facet of $\mathbf{R}^I$ with respect to the hyperplane arrangement $\mathcal{H}(d)$, and θ and ω two elements of F. Let M be an object of $\mathcal{A}$, such that $\phi(M) = d$. Then, M is θ-semistable (respectively, θ-stable) if and only if it is ω-semistable (respectively, ω-stable).

PROOF : As $\phi(M) = d$ is non-zero, and the family $(\phi_i)_{i \in I}$ is positive, M is non-zero. For each weight λ of $\mathcal{A}$, and for each non-zero subobject N of M, define

$$g_\lambda(N) = \mu_\lambda(M) - \mu_\lambda(N).$$

Then, M is λ-semistable if and only if $g_\lambda(N) \geq 0$, or equivalently, $\operatorname{sgn}(g_\lambda(N))$ belongs to $\{0, 1\}$, for every non-zero subobject N of M. Similarly, M is λ-stable if and only if $\operatorname{sgn}(g_\lambda(N)) = 1$ for every non-zero proper subobject N of M. Therefore, it suffices to check that $\operatorname{sgn}(g_\theta(N)) = \operatorname{sgn}(g_\omega(N))$ for every non-zero proper subobject N of M. As θ and ω belong to the same facet F, this is a consequence of Proposition 3.4. □

Proposition 3.6 — Let d be a non-zero element of N^I, F a facet of $\mathbf{R}^I$ with respect to the hyperplane arrangement $\mathcal{H}(d)$, and θ and ω two elements of F. Let M be an object of $\mathcal{A}$, such that $\phi(M) = d$. Suppose that M is θ-semistable, and let $(M_i)_{i=0}^n$ be a Jordan-Hölder filtration of M with respect to θ. Then, M is ω-semistable, and $(M_i)_{i=0}^n$ is a Jordan-Hölder filtration of M with respect to ω also.

PROOF : The fact that M is ω-semistable has already been proved in Proposition 3.5. For every $i = 1, \dots, n$, we have $\mu_\theta(M_{i-1}) = \mu_\theta(M)$, hence, by Proposition 3.4,

$$\operatorname{sgn}(\mu_\omega(M) - \mu_\omega(M_{i-1})) = \operatorname{sgn}(\mu_\theta(M) - \mu_\theta(M_{i-1})) = 0,$$

so $\mu_\omega(M_{i-1}) = \mu_\omega(M)$. It remains to prove that the quotient object $N_i = M_{i-1}/M_i$ is ω-stable for every $i = 1, \dots, n$.

We will first verify that $\mu_\omega(N_i) = \mu_\omega(M)$ for all $i = 1, \dots, n$. If $i = n$, this follows from the above paragraph, since $N_n = M_{n-1}$. Suppose $1 \le i \le n-1$. Then, both i and $i+1$ belong to $\{1, \dots, n\}$, hence, by the above paragraph,

$$\mu_\omega(M_i) = \mu_\omega(M_{i-1}) = \mu_\omega(M).$$

We also have a short exact sequence

$$0 \to M_i \to M_{i-1} \to N_i \to 0,$$

of non-zero objects of $\mathcal{A}$. Therefore, by the seesaw property of $\preceq_\omega$, we get

$$\mu_\omega(N_i) = \mu_\omega(M_{i-1}) = \mu_\omega(M).$$

This proves that $\mu_\omega(N_i) = \mu_\omega(M)$ for all $i = 1, \dots, n$.

Let $i \in \{1, \dots, n\}$. In view of the previous paragraph, to show that N_i is ω-stable, it suffices to show that $\mu_\omega(X) < \mu_\omega(M)$ for every proper non-zero subobject X of N_i. To begin with, since N_i is θ-stable, we have

$$\mu_\theta(X) < \mu_\theta(N_i) = \mu_\theta(M).$$

Suppose first that $i = n$. Then, $N_i = M_{n-1}$, and X is a non-zero subobject of M, hence, by Proposition 3.4 and the above inequality,

$$\operatorname{sgn}(\mu_\omega(M) - \mu_\omega(X)) = \operatorname{sgn}(\mu_\theta(M) - \mu_\theta(X)) = 1.$$

It follows that $\mu_\omega(X) < \mu_\omega(M)$. Suppose next that $1 \le i \le n-1$. Let $\pi : M_{i-1} \to N_i$ be the canonical projection, and let $Y = \pi^{-1}(X)$, that is, the kernel of the composite

$$M_{i-1} \xrightarrow{\pi} N_i \to N_i/X.$$

Thus, Y is a non-zero subobject of M_{i-1}, and we have a short exact sequence

$$0 \to M_i \to Y \to X \to 0$$

of non-zero objects of $\mathcal{A}$. By the above paragraphs,

$$\mu_\theta(M_i) = \mu_\theta(M) = \mu_\theta(N_i) > \mu_\theta(X).$$

Therefore, by the seesaw property of $\preceq_\theta$,

$$\mu_\theta(M) = \mu_\theta(M_i) > \mu_\theta(Y),$$

hence, by Proposition 3.4,

$$\operatorname{sgn}(\mu_\omega(M) - \mu_\omega(Y)) = \operatorname{sgn}(\mu_\theta(M) - \mu_\theta(Y)) = 1,$$

so

$$\mu_\omega(M_i) = \mu_\omega(M) > \mu_\omega(Y).$$

Again, by the seesaw property of $\preceq_\omega$, we get $\mu_\omega(M_i) > \mu_\omega(X)$, hence $\mu_\omega(M) > \mu_\omega(X)$. This proves that N_i is ω-stable. □

Proposition 3.7 — Let d be a non-zero element of N^I, F a facet of $\mathbf{R}^I$ with respect to the hyperplane arrangement $\mathcal{H}(d)$, and θ and ω two elements of F. Let M be an object of $\mathcal{A}$, such that $\phi(M) = d$. Then, M is θ-polystable if and only if it is ω-polystable.

PROOF : Suppose M is θ-polystable. Then, M is θ-semistable, and there exists a sequence $(M_i)_{i=1}^n$ of objects of $\mathcal{A}$, such that $n \in \mathbf{N}$, M_i is θ-stable for each $i = 1, \ldots, n$, and M is isomorphic to $\bigoplus_{i=1}^n M_i$. As M is non-zero, we in fact have $n \geq 1$. Also, by Proposition 3.5, M is ω-semistable. Let $N = \bigoplus_{i=1}^n M_i$. Then, since N is isomorphic to M, it is both θ-semistable and ω-semistable, and $\phi(N) = d$. For each $i = 0, \ldots, n$, define $N_i = \bigoplus_{j=i+1}^n M_i$. Then, $(N_i)_{i=0}^n$ is a decreasing sequence of subobjects of N, $N_0 = N$, $N_n = 0$, and for each $i = 1, \ldots, n$, N_{i-1}/N_i is isomorphic to M_i, and is hence θ-stable. Moreover, since N is θ-semistable, by Proposition 2.1(3), for every $i = 1, \ldots, n$, we get

$$\mu_\theta(N) = \mu_\theta(M_i) = \mu_\theta(N_{i-1}).$$

Therefore, $(N_i)_{i=0}^n$ is a Jordan-Hölder filtration of N with respect to θ. By Proposition 3.6, it is a Jordan-Hölder filtration of N with respect to ω also. In particular, for each $i = 1, \ldots, n$, N_{i-1}/N_i is ω-stable, hence M_i is ω-stable. As M is ω-semistable, and isomorphic to $\bigoplus_{i=1}^n M_i$, it follows that M is ω-polystable. □

Proposition 3.8 — Let d be a non-zero element of N^I, F a facet of $\mathbf{R}^I$ with respect to the hyperplane arrangement $\mathcal{H}(d)$, and θ and ω two elements of F. Let M and N be two objects of $\mathcal{A}$, such that $\phi(M) = \phi(N) = d$. Suppose that M and N are θ-semistable, and that M is S_θ-equivalent to N. Then, M and N are ω-semistable, and M is S_ω-equivalent to N. □

PROOF : Let $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ be Jordan-Hölder filtrations of M and N, respectively, with respect to θ. Then, by Proposition 3.6, M and N are ω-semistable, and $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ are Jordan-Hölder filtrations of M and N, respectively, with respect to ω. Now, because M is S_θ-equivalent to N, $n = m$, and there exists a permutation $\pi \in \mathrm{S}_n$, such that M_{i-1}/M_i is isomorphic $N_{\pi(i)-1}/N_{\pi(i)}$ for every $i = 1, \ldots, n$. As $(M_i)_{i=0}^n$ and $(N_j)_{j=0}^m$ are Jordan-Hölder filtrations of M and N, respectively, with respect to ω, it follows that M is S_ω-equivalent to N. □

Remark 3.9 : We use the following fact in the next proof. Let V be a finite-dimensional $\mathbf{R}$-vector space, and V' a $\mathbf{Q}$-structure on V. Let $(f_i)_{i\in I}$ be a family of $\mathbf{R}$-linear functions from V to $\mathbf{R}$, which are $\mathbf{Q}$-rational. Let $L = \bigcap_{i\in I} \mathrm{Ker}(f_i)$. Then, the closure of $V' \cap L$ in V equals L.

Proposition 3.10 — Every facet of $\mathbf{R}^I$ with respect to the hyperplane arrangement $\mathcal{H}(d)$ contains an integral weight, that is, an element of $\mathbf{Z}^I$.

PROOF : Every element e of S_d is an element of $\mathbf{N}^I$, hence $f(d,e)$ is $\mathbf{Q}$-rational. Let F be a facet of $\mathbf{R}^I$ with respect to $\mathcal{H}(d)$, and let $L = \mathrm{Supp}(F)$. Let K denote the set of all elements e of S_d such that $F \subset H(d,e)$. Then,

$$L = \bigcap_{e\in K} H(d,e) = \mathrm{Ker}(f(d,e)),$$

hence, by Remark 3.9, the closure of $\mathbf{Q}^I \cap L$ in $\mathbf{R}^I$ equals L. Now, by Remark 3.3, F is open in L. Therefore, there exists an open subset U of $\mathbf{R}^I$, such that $F = U \cap L$. Let θ be an element of the non-empty set F. Then, θ belongs to the closure L of $\mathbf{Q}^I \cap L$ in $\mathbf{R}^I$, and U is an open neighbourhood of θ in $\mathbf{R}^I$, hence there exists an element ξ in $(\mathbf{Q}^I \cap L) \cap U = \mathbf{Q}^I \cap F$. Let n be a strictly positive integer such that $\omega = n\xi$ belongs to $\mathbf{Z}^I$. We claim that $\omega \in F$. To see this, let $e \in S_d$. Then, $f(d,e)$ is $\mathbf{R}$-linear, hence $f(d,e)(\omega) = nf(d,e)(\xi)$. As $n > 0$, this implies that $\mathrm{sgn}(f(d,e)(\omega)) = \mathrm{sgn}(f(d,e)(\xi))$. Since $f(d,e)$ is a defining function of $H(d,e)$, it follows that $\omega \sim_{H(d,e)} \xi$. As this is true for all $e \in S_d$, $\omega \sim \xi$, hence ω belongs to the facet F of $\mathbf{R}^I$ through ξ. Thus, ω is an element of $\mathbf{Z}^I \cap F$. □

Proposition 3.11 — Let d be a non-zero element of $\mathbf{N}^I$, and let $\theta \in \mathbf{R}^I$. Then, there exists an integral weight ω of $\mathcal{A}$, with the following properties:

(1) Any object M of $\mathcal{A}$, such that $\phi(M) = d$, is θ-semistable (respectively, θ-stable, θ-polystable) if and only if it is ω-semistable (respectively, ω-stable, ω-polystable).

(2) If M is a θ-semistable object of $\mathcal{A}$ such that $\phi(M) = d$, then every Jordan-Hölder filtration of M with respect to θ is a Jordan-Hölder filtration of M with respect to ω, and conversely.

(3) Two θ-semistable objects M and N of $\mathcal{A}$, such that $\phi(M) = \phi(N) = d$, are S_θ-equivalent if and only if they are S_ω-equivalent.

PROOF : This Proposition follows immediately from Propositions 3.5-3.8 and 3.10. □

4. REPRESENTATIONS OF QUIVERS

In this section, we will specialise the constructs of the previous sections to the specific abelian category of the representations of a quiver over a field. We begin by defining quivers and their representations. We then formulate the notion of the semistability of a representation of a quiver with respect to a weight, as an instance of the general theory described in Section 3.A. In the last part of the Section, we look at special Hermitian metrics on complex representations of a quiver.

4.A. *The category of representations*

A *quiver* Q is a quadruple $(Q_0, Q_1, \mathrm{s}, \mathrm{t})$, where Q_0 and Q_1 are sets, and $\mathrm{s} : Q_1 \to Q_0$, and $\mathrm{t} : Q_1 \to Q_0$ are functions. The elements of Q_0 are called the *vertices of* Q, and those of Q_1 are called the *arrows of* Q. For any arrow α of Q, the vertex $\mathrm{s}(\alpha)$ is called the *source* of α, and the vertex $t(\alpha)$ is called the *target* of α. If $\mathrm{s}(\alpha) = a$ and $\mathrm{t}(\alpha) = b$, then we say that α is an arrow *from* a *to* b, and write $\alpha : a \to b$. We say that Q is *vertex-finite* if the set Q_0 is finite, *arrow-finite* if the set Q_1 is finite, and *finite* if it is both vertex-finite and arrow-finite. The quiver $(\emptyset, \emptyset, \mathrm{s}, \mathrm{t})$, where s and t are the empty functions, is called the *empty* quiver. We say that a quiver Q is *non-empty* if it is not equal to the empty quiver, or equivalently, if the set Q_0 of its vertices is non-empty.

Let k be a field. A *representation of* Q *over* k is a pair (V, ρ), where $V = (V_a)_{a \in Q_0}$ is a family of finite-dimensional k-vector spaces, and $\rho = (\rho_\alpha)_{\alpha \in Q_1}$ is a family of k-linear maps $\rho_\alpha : V_{\mathrm{s}(\alpha)} \to V_{\mathrm{t}(\alpha)}$. We will often drop the base field k from the terminology. If (V, ρ) and (W, σ) are two representations of Q, then a *morphism from* (V, ρ) *to* (W, σ) is a family $f = (f_a)_{a \in Q_0}$ of k-linear maps $f_a : V_a \to W_a$, such that for every $\alpha \in Q_1$, the diagram

$$\begin{array}{ccc} V_{\mathrm{s}(\alpha)} & \xrightarrow{\rho_\alpha} & V_{\mathrm{t}(\alpha)} \\ {\scriptstyle f_{\mathrm{s}(\alpha)}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{t}(\alpha)}} \\ W_{\mathrm{s}(\alpha)} & \xrightarrow[\sigma_\alpha]{} & W_{\mathrm{t}(\alpha)} \end{array}$$

commutes. If (V,ρ), (W,σ), and (X,τ) are three representations of Q, f a morphism from (V,ρ) to (W,σ), and g a morphism from (W,σ) to (X,τ), then the *composite* of f and g is the family $g\circ f$ defined by $g\circ f = (g_a\circ f_a)_{a\in Q_0}$. It is easy to verify that $g\circ f$ is a morphism from (V,ρ) to (X,τ). We thus get a category $\mathbf{Rep}_k(Q)$, whose objects are representations of Q over k, and whose morphisms are defined as above.

For any two representations (V,ρ) and (W,σ) of Q, the set $\mathrm{Hom}((V,\rho),(W,\sigma))$ is a k-subspace of the k-vector space $\bigoplus_{a\in Q_0}\mathrm{Hom}_k(V_a,W_a)$. If (X,τ) is another representation of Q, then the composition operator

$$\mathrm{Hom}((W,\sigma),(X,\tau))\times\mathrm{Hom}((V,\rho),(W,\sigma))\to\mathrm{Hom}((V,\rho),(X,\tau))$$

is k-bilinear. Any representation (V,ρ) such that the k-vector space V_a is zero for all $a\in Q_0$ is a zero object in this category. For every finite family $(V_i,\rho_i)_{i\in I}$ of representations of Q, the pair (V,ρ), which is defined by

$$V_a=\bigoplus_{i\in I}V_{i,a},\quad \rho_\alpha=\bigoplus_{i\in I}\rho_{i,\alpha},$$

for all $a\in Q_0$ and $\alpha\in Q_1$, is a coproduct of $(V_i,\rho_i)_{i\in I}$ in $\mathbf{Rep}_k(Q)$. Thus, $\mathbf{Rep}_k(Q)$ is a k-linear additive category.

If $f:(V,\rho)\to(W,\sigma)$ is a morphism of representations of Q, then the pair (V',ρ'), where $V'_a=\mathrm{Ker}(f_a)$ for all $a\in Q_0$, and $\rho'_\alpha: V'_{\mathrm{s}(\alpha)}\to V'_{\mathrm{t}(\alpha)}$ is the restriction of ρ_α for each $\alpha\in Q_1$, is a representation of Q, and there is an obvious morphism $i:(V',\rho')\to(V,\rho)$, which is given by the inclusion maps $i_a: V'_a\to V_a$ for all $a\in Q_0$. The representation (V',ρ'), together with the morphism i, is a kernel of f in $\mathbf{Rep}_k(Q)$. Similarly, the pair (W',σ'), where $W'_a=\mathrm{Coker}(f_a)$ for all $a\in Q_0$, and $\sigma'_\alpha: W'_{\mathrm{s}(\alpha)}\to W'_{\mathrm{t}(\alpha)}$ is the k-linear map induced by σ_α for each $\alpha\in Q_1$, is a representation of Q, and there is a morphism $\pi:(W,\sigma)\to(W',\sigma')$, which is given by the canonical projections $\pi_a: W_a\to W'_a$ for all $a\in Q_0$. The representation (W',σ'), together with the morphism π, is a cokernel of f in $\mathbf{Rep}_k(Q)$. Thus, every morphism in this additive category has a kernel and a cokernel. It is obvious that the canonical morphism from $\mathrm{Coker}(i)$ to $\mathrm{Ker}(\pi)$ is an isomorphism. It follows that $\mathbf{Rep}_k(Q)$ is a k-linear abelian category.

A *subrepresentation* of a representation (V,ρ) of Q is a subobject of (V,ρ) in the category $\mathbf{Rep}_k(Q)$. Every family $(V_i,\rho_i)_{i\in I}$ of subrepresentations of (V,ρ) has a meet $\bigcap_{i\in I}(V_i,\rho_i)$, and a join $\sum_{i\in I}(V_i,\rho_i)$. Thus, the category $\mathbf{Rep}_k(Q)$ has all meets and joins of subobjects.

Remark 4.1 : For every representation (V,ρ) of Q, and for any element c of k, we have a representation $(V,c\rho)$ of Q, where $c\rho=(c\rho_\alpha)_{\alpha\in Q_1}$. If (W,σ) is a subrepresentation of (V,ρ), then $(W,c\sigma)$ is a subrepresentation of $(V, c\rho)$. It is also obvious that if $(V_i, \rho_i)_{i\in I}$ is a finite family of representations of Q, then

$$(\bigoplus V_i, c(\bigoplus_{i\in I} \rho_i)) = (\bigoplus V_i, \bigoplus_{i\in I}(c\rho_i)).$$

Lastly, if $f : (V, \rho) \to (W, \sigma)$ is a morphism of representations of Q, then f is also a morphism of representations from $(V, c\rho)$ to $(W, c\sigma)$.

4.B *Semistability and stability of representations*

Fix a non-empty vertex-finite quiver Q, and a field k. For every representation (V, ρ) of Q over k, and $a \in Q_0$, let

$$\dim_a (V, \rho) = \dim_k (V_a).$$

Then, $(\dim_a(V, \rho))_{a\in Q_0}$ is a non-empty finite positive family of additive functions from the abelian category $\mathbf{Rep}_k(Q)$ to $\mathbf{Z}$. Therefore, the statements of Section 3 are applicable here. The following are the versions for representations of some of the notions defined there.

For every representation (V, ρ), the element

$$\dim (V, \rho) = (\dim_k(V_a))_{a\in Q_0}$$

of $\mathbf{N}^{Q_0}$ is called the dimension vector of (V, ρ), and the natural number

$$\mathrm{rk}(V, \rho) = \sum_{a\in Q_0} \dim_k (V_a)$$

is called the rank of (V, ρ).

An element of the $\mathbf{R}$-vector space $\mathbf{R}^{Q_0}$ is called a weight of Q. We say that a weight is rational (respectively, integral) if it belongs to the subset $\mathbf{Q}^{Q_0}$ (respectively, $\mathbf{Z}^{Q_0}$) of $\mathbf{R}^{Q_0}$. We fix a weight θ of Q.

For any representation (V, ρ) of Q, the θ-degree of (V, ρ) is the real number

$$\deg_\theta(V, \rho) = \sum_{a\in Q_0} \theta_a \dim_k (V_a).$$

If $(V, \rho) \neq 0$, the real number

$$\mu_\theta(V, \rho) = \frac{\deg_\theta(V, \rho)}{\mathrm{rk}(V, \rho)},$$

is called the θ-slope of (V, ρ).

A representation (V, ρ) of Q is called θ-semistable (respectively, θ-stable) if it is non-zero, and if

$$\mu_\theta(W, \sigma) \leq \mu_\theta(V, \rho) \quad \text{(respectively, } \mu_\theta(W, \sigma) < \mu_\theta(V, \rho)\text{)}$$

for every non-zero proper subrepresentation (W, σ) of (V, ρ). We say that a representation of Q is θ-polystable if it is θ-semistable, and is isomorphic to the direct sum of a finite family of θ-stable representations of Q. There are obvious versions for representations of all the results of Section 3.

Remark 4.2 : For any non-zero element c of k, the θ-semistability (respectively, θ-stability, θ-polystability) of a representation (V, ρ) of Q over k is equivalent to the θ-semistability (respectively, θ-stability, θ-polystability) of $(V, c\rho)$. Also, if ζ is a strictly positive real number, and $\omega = \zeta\theta$, then a representation of Q is θ-semistable (respectively, θ-stable, θ-polystable) if and only if it is ω-semistable (respectively, ω-stable, ω-polystable).

4.C. *Einstein-Hermitian metrics on complex representations*

Let Q be a non-empty finite quiver, and fix a weight θ of Q. All the representations of Q considered in this subsection will be over $\mathbf{C}$.

A *Hermitian metric* on a representation (V, ρ) of Q is a family $h = (h_a)_{a \in Q_0}$ of Hermitian inner products $h_a : V_a \times V_a \to \mathbf{C}$. Given a Hermitian metric h on (V, ρ), for every vertex a of Q_0, we have an endomorphism $K_\theta(V, \rho)_a$ of the $\mathbf{C}$-vector space V_a, which is defined by

$$K_\theta(V, \rho)_a = \theta_a \mathbf{1}_{V_a} + \sum_{\alpha \in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha,$$

where, for each $\alpha \in Q_1$, $\rho_\alpha^* : V_{\mathrm{t}(\alpha)} \to V_{\mathrm{s}(\alpha)}$ is the adjoint of $\rho_\alpha : V_{\mathrm{s}(\alpha)} \to V_{\mathrm{t}(\alpha)}$ with respect to the Hermitian inner products $h_{\mathrm{s}(\alpha)}$ and $h_{\mathrm{t}(\alpha)}$ on $V_{\mathrm{s}(\alpha)}$ and $V_{\mathrm{t}(\alpha)}$, respectively. We say that the metric h is *Einstein-Hermitian* with respect to θ if there exists a constant $c \in \mathbf{C}$, such that

$$K_\theta(V, \rho)_a = c \mathbf{1}_{V_a}$$

for all $a \in Q_0$. If this is the case, and if (V, ρ) is non-zero, then it is easy to see that $c = \mu_\theta(V, \rho)$, hence

$$\sum_{\alpha \in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha = \left(\mu_\theta(V, \rho) - \theta_a\right) \mathbf{1}_{V_a}$$

for all $a \in Q_0$. If we are considering more than one Hermitian metric on (V, ρ), and want to indicate the dependence of $K_\theta(V, \rho)$ on the metric, we will write $K_\theta(V, \rho, h)$ instead of $K_\theta(V, \rho)$.

The following Proposition is a consequence of [3, Proposition 6.5]. The restriction to rational weights here is due to the fact that the cited result is proved in that reference only for integral weights.

Proposition 4.3 — Let θ be a rational weight of Q, and (V, ρ) a non-zero representation of Q. Then, (V, ρ) has an Einstein-Hermitian metric with respect to θ if and only if it is θ-polystable. Moreover, if h_1 and h_2 are two Einstein-Hermitian metrics on (V, ρ) with respect to θ, then there exists an automorphism f of (V, ρ), such that

$$h_{1,a}(v, w) = h_{2,a}(f_a(v), f_a(w))$$

for all $a \in Q_0$ and $v, w \in V_a$.

Given a Hermitian metric h on a representation (V, ρ) of Q, we say that two subrepresentations (V_1, ρ_1) and (V_2, ρ_2) of (V, ρ) are *orthogonal* with respect to h if for every $a \in Q_0$, the subspaces $V_{1,a}$ and $V_{2,a}$ of V_a are orthogonal with respect to the Hermitian inner product h_a on V_a.

Corollary 4.4 — Let θ be a rational weight of Q, (V, ρ) a non-zero representation of Q, and h an Einstein-Hermitian metric on (V, ρ) with respect to θ. Then, there exists a finite family $(V_i, \rho_i)_{i \in I}$ of θ-stable subrepresentations of Q, such that $(V, \rho) = \bigoplus_{i \in I}(V_i, \rho_i)$, and such that, for all $i, j \in I$ with $i \neq j$, (V_i, ρ_i) and (V_j, ρ_j) are orthogonal with respect to h.

PROOF : By Proposition 4.3, (V, ρ) is θ-polystable, hence there exists a finite family $(W_i, \sigma_i)_{i \in I}$ of θ-stable subrepresentations of Q, such that $(V, \rho) = \bigoplus_{i \in I}(W_i, \sigma_i)$. For each $i \in I$, there exists an Einstein-Hermitian metric h_i on (W_i, σ_i). Let h' denote the Hermitian metric $\oplus_{i \in I} h_i$ on (V, ρ). Then, h' is an Einstein-Hermitian metric on (V, ρ). Therefore, there exists an automorphism f of (V, ρ), such that $h'_a(v, w) = h_a(f_a(v), f_a(w))$ for all $a \in Q_0$ and $v, w \in V_a$. Let $(V_i, \rho_i) = f((W_i, \sigma_i))$ for every $i \in I$. Then, $(V_i, \rho_i)_{i \in I}$ is the desired family of subrepresentations of (V, ρ). $\square$

Let h be a Hermitian metric on (V, ρ). We say that an endomorphism f of (V, ρ) is *skew-Hermitian* with respect to h if for every $a \in Q_0$, the endomorphism f_a of V_a is skew-Hermitian with respect to h_a, that is,

$$h_a(f_a(v), w) + h_a(v, f_a(w)) = 0$$

for all $v, w \in V_a$. We denote the set of all skew-Hermitian endomorphisms of (V, ρ) with respect to h by $\mathrm{End}(V, \rho, h)$. It is an $\mathbf{R}$-subspace of the $\mathbf{C}$-vector space $\mathrm{End}(V, \rho)$. We say that h is *irreducible* if for every endomorphism f of (V, ρ) that is skew-Hermitian with respect to h, there exists $\lambda \in \mathbf{C}$, such that $f = \lambda 1_{(V,\rho)}$. The complex number λ is then purely imaginary, hence h is irreducible if and only if

$$\mathrm{End}(V, \rho, h) = \sqrt{-1}\,\mathbf{R}\,1_{(V,\rho)}.$$

The following result on irreducible representations can be proved easily.

Proposition 4.5 — Let (V, ρ) be a non-zero representation of Q, and h a Hermitian metric on (V, ρ). Then, the following are equivalent:

(1) h is irreducible.

(2) If $(V_i, \rho_i)_{i \in I}$ is a finite family of subrepresentations of (V, ρ),

$$(V, \rho) = \bigoplus_{i \in I} (V_i, \rho_i),$$

and (V_i, ρ_i) and (V_j, ρ_j) are orthogonal with respect to h for all $i, j \in I$ with $i \neq j$, then there exists $i \in I$, such that $(V_i, \rho_i) = (V, \rho)$.

Proposition 4.6 — Let θ be a rational weight of Q, (V, ρ) a non-zero representation of Q, and h an Einstein-Hermitian metric on (V, ρ) with respect to θ. Then, the following are equivalent:

(1) h is irreducible.

(2) (V, ρ) is θ-stable.

(3) (V, ρ) is Schur.

PROOF (1)⇒(2) : Suppose h is irreducible. Since h is Einstein-Hermitian with respect to θ, by Corollary 4.4, there exists a finite family $(V_i, \rho_i)_{i \in I}$ of θ-stable subrepresentations of Q, such that $(V, \rho) = \bigoplus_{i \in I} (V_i, \rho_i)$, and such that, for all $i, j \in I$ with $i \neq j$, (V_i, ρ_i) and (V_j, ρ_j) are orthogonal with respect to h. As h is irreducible, by Proposition 4.5, there exists $i \in I$ such that $(V, \rho) = (V_i, \rho_i)$. Therefore, (V, ρ) is θ-stable.

(2)⇒(3): Follows from Proposition 2.2(3).

(3)⇒(1): Suppose f is a skew-Hermitian endomorphism of (V, ρ). Then, by Proposition 2.2(3)-(4), there exists $\lambda \in \mathbf{C}$ such that $f = \lambda 1_{(V,\rho)}$. Therefore, h is irreducible. □

5. FAMILIES OF REPRESENTATIONS

Fix a non-empty finite quiver Q. We will consider only complex representations of Q in this section.

5.A. *Families parametrised by complex spaces*

Let T be a complex space. By the *Zariski topology* on T, we mean the topology whose closed sets are the analytic subsets of T. It is obviously coarser than the given topology on T, which we will call the *strong topology*. In this context, the terms “open”, “continuous”, etc., without any qualifiers, are with respect to the strong topology. As usual, we denote the structure sheaf of T by $\mathcal{O}_T$. If E is any

$\mathcal{O}_T$-module, then for any $t \in T$, we denote by E_t the $\mathcal{O}_{T,t}$-module which is the stalk of E at t, and by $E(t)$ the fibre of E at t, that is, the $\mathbf{C}$-vector space $\mathbf{C} \otimes \mathcal{O}_{T,t} E_t$. For any element γ of E_t, we denote the value of γ at t, that is, the canonical image of γ in $E(t)$, by $\gamma(t)$. If U is an open neighbourhood of t, and $s \in E(U)$, we denote by s_t the germ of s at t, that is, the canonical image of s in E_t, and by $s(t)$ the value of s at t, that is, the element $s_t(t)$ of $E(t)$. If $f : E \to F$ is a morphism of $\mathcal{O}_T$-modules, then for each $t \in T$, we have a canonical $\mathcal{O}_{T,t}$-linear map $f_t : E_t \to F_t$, and a canonical $\mathbf{C}$-linear map $f(t) : E(t) \to F(t)$. We will identify any holomorphic vector bundle on E with the $\mathcal{O}_T$-module of its holomorphic sections.

A *family of representations* of Q parametrised by a complex space T is a pair (V, ρ), where $V = (V_a)_{a \in Q_0}$ is a family of holomorphic vector bundles on T, and $\rho = (\rho_\alpha)_{\alpha \in Q_1}$ is a family of morphisms $\rho_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ of holomorphic vector bundles on T. There are obvious notions of a morphism $f : (V, \rho) \to (W, \sigma)$ between two families of representations of Q parametrised by T, and the restriction $(E, \rho)|_U$ of a family of representations of Q parametrised by T to an open subspace U of T. We thus get the category $\mathbf{Rep}_T(Q)$ of families of representations of Q parametrised by T, and the sheaf $\mathcal{H}om((V, \rho), (W, \sigma))$ of morphisms between two families of representations of Q parametrised by T. The latter is an $\mathcal{O}_T$-submodule of the $\mathcal{O}_T$-module $\oplus_{a \in Q_0} \mathcal{H}om_{\mathcal{O}_T}(V_a, W_a)$. For every morphism $f : (V, \rho) \to (W, \sigma)$ of families of representations of Q parametrised by T, and open subset U of T, we have an obvious restriction $f|_U : (V, \rho)|_U \to (W, \sigma)|_U$.

Remark 5.1 : Given two families of representations (V, ρ) and (W, σ) of Q parametrised by T, we define two $\mathcal{O}_T$-modules E and F by

$$E = \oplus_{a \in Q_0} \mathcal{H}om_{\mathcal{O}_T}(V_a, W_a), \quad F = \oplus_{\alpha \in Q_1} \mathcal{H}om_{\mathcal{O}_T}(V_{s(\alpha)}, W_{t(\alpha)}),$$

and a morphism $u : E \to F$ of $\mathcal{O}_T$-modules by

$$u_U(f) = (f_{\mathrm{t}(\alpha)} \circ \rho_\alpha|_U - \sigma_\alpha|_U \circ f_{\mathrm{s}(\alpha)})_{\alpha \in Q_1},$$

for every open subset U of T, and for every $f = (f_a)_{a \in Q_0}$ in

$$E(U) = \oplus_{a \in Q_0} \mathrm{Hom}(V_a|_U, W_a|_U).$$

By the definition of u, we have $(\mathrm{Ker}(u))(U) = \mathrm{Hom}((V, \rho)|_U, (W, \sigma)|_U)$ for every open subset U of T, hence $\mathcal{H}om((V, \rho), (W, \sigma)) = \mathrm{Ker}(u)$. The assumption that Q is finite, and the fact that V_a is locally free for every $a \in Q_0$, imply that E and F are locally free, hence coherent, $\mathcal{O}_T$-modules. It follows that the $\mathcal{O}_T$-module $\mathcal{H}om((V, \rho), (W, \sigma))$ is coherent.

Let $f : T' \to T$ be a morphism of complex spaces, and (V, ρ) a family of representations of Q parametrised by T. For each $a \in Q_0$, define a holomorphic vector bundle M_a on T', by $M_a = f^*(V_a)$. Then, for each $\alpha \in Q_1$, we have a morphism $\phi_\alpha = f^*(\rho_\alpha) : M_{s(\alpha)} \to M_{t(\alpha)}$ of holomorphic vector bundles on T'. We thus, get a family (M, ϕ) of representations of Q parametrised by T'. We call it the *pullback* of (V, ρ) by f, and will denote it by $f^*(V, \rho)$. In particular, if U is an open subspace of T, and if $f : U \to T$ is the canonical morphism of ringed spaces, then $f^*(V, \rho)$ is canonically isomorphic to the restriction $(V, \rho)|_U$. If A is any complex subspace of T, and $f : A \to T$ the canonical morphism, we will call $f^*(V, \rho)$ the *restriction* of (V, ρ) to A, and will denote it by $(V, \rho)|_A$.

Suppose (V, ρ) is a family of representations of Q parametrised by T. Then, for each point $t \in T$, we get a representation $(V(t), \rho(t))$ of Q over $\mathbf{C}$, which is defined by $V(t) = (V_a(t))_{a \in Q_0}$ and $\rho(t) = (\rho_\alpha(t))_{\alpha \in Q_1}$. If P is any property of representations of Q over an arbitrary field, we say that (V, ρ) *has the property* P if for every $t \in T$, the representation $(V(t), \rho(t))$ of Q over $\mathbf{C}$ has the property P. We can thus speak of a *family of non-zero representations* of Q parametrised by T, a *family of Schur representations* of Q parametrised by T, etc. If θ is a weight in $\mathbf{R}^{Q_0}$, we can speak of a *family of θ-stable representations* of Q parametrised by T, a *family of θ-semistable representations* of Q parametrised by T, etc.

Proposition 5.2 — Let T be a complex space, (V, ρ) and (W, σ) two families of representations of Q parametrised by T, and $u : E \to F$ the morphism of $\mathcal{O}_T$-modules defined in Remark 5.1. Then, for every $t \in T$, there is a canonical isomorphism

$$\mathrm{Hom}((V(t), \rho(t)), (W(t), \sigma(t))) \cong \mathrm{Ker}(u(t))$$

of $\mathbf{C}$-vector spaces. In particular, the function

$$t \mapsto \dim_{\mathbf{C}} \left(\mathrm{Hom}((V(t), \rho(t)), (W(t), \sigma(t)))\right) : T \to \mathbf{N}$$

is upper semi-continuous with respect to the Zariski topology on T.

PROOF : For every point $t \in T$, we have canonical identifications

$$E(t) = \oplus_{a \in Q_0} \mathrm{Hom}_{\mathbf{C}}(V_a(t), W_a(t)), \quad F(t) = \oplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbf{C}}(V_{\mathrm{s}(\alpha)}(t), W_{\mathrm{t}(\alpha)}(t)),$$

since the $\mathcal{O}_T$-modules V_a and W_a are locally free for every $a \in Q_0$. Under these identifications, the $\mathbf{C}$-linear map $u(t) : E(t) \to F(t)$ takes any element $f = (f_a)_{a \in Q_0}$ of $E(t)$ to the element $(f_{\mathrm{t}(\alpha)} \circ \rho_\alpha(t) - \sigma_\alpha(t) \circ f_{\mathrm{s}(\alpha)})_{\alpha \in Q_1}$ of $F(t)$. Therefore, we have a canonical $\mathbf{C}$-isomorphism $\mathrm{Ker}(u(t)) \cong \mathrm{Hom}((V(t), \rho(t)), (W(t), \sigma(t)))$. But, as both E and F are locally free, the function $t \mapsto \dim_{\mathbf{C}} (\mathrm{Ker}(u(t)))$ from T to $\mathbf{N}$ is upper semi-continuous with respect to the Zariski topology on T. □

Corollary 5.3 — Let T be a complex space, and (V,ρ) a family of representations of Q parametrised by T. Then, the subset

$$\{t \in T \mid \dim_{\mathbf{C}}(\mathrm{End}(V(t),\rho(t))) \neq 1\}$$

of T is analytic.

Corollary 5.4 — Let T be a complex space, and (V,ρ) and (W,σ) two families of representations of Q parametrised by T. Suppose that T is reduced, and that the function

$$t \mapsto \dim_{\mathbf{C}}\left(\mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t))\right) : T \to \mathbf{N}$$

is locally constant. Then, the $\mathcal{O}_T$-module $\mathcal{H}om((V,\rho),(W,\sigma))$ is locally free. Moreover, for every $t \in T$, there is a canonical $\mathbf{C}$-isomorphism

$$\mathcal{H}om((V,\rho),(W,\sigma))(t) \cong \mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t))).$$

PROOF : Let $u : E \to F$ be the morphism of $\mathcal{O}_T$-modules defined in Remark 5.1. Then, $\mathcal{H}om((V,\rho),(W,\sigma)) = \mathrm{Ker}(u)$. Moreover, by Proposition 5.2, for every $t \in T$, the $\mathbf{C}$-vector spaces $\mathrm{Hom}((V(t),\rho(t)),(W(t),\sigma(t)))$ and $\mathrm{Ker}(u(t))$ are canonically isomorphic. Therefore, the function $t \mapsto \dim_{\mathbf{C}}(\mathrm{Ker}(u(t)))$ from T to $\mathbf{N}$ is locally constant. Since the $\mathcal{O}_T$-modules E and F are locally free, and T is reduced, this implies that the $\mathcal{O}_T$-module $\mathrm{Ker}(u)$ is locally free, and that for every $t \in T$, there is a canonical $\mathbf{C}$-isomorphism $\mathrm{Ker}(u)(t) \cong \mathrm{Ker}(u(t))$. □

Corollary 5.5 — Let (V,ρ) be a family of representations of Q parametrised by a complex space T. Then, the subset of T, consisting of all the points $t \in T$ such that the representation $(V(t),\rho(t))$ of Q over $\mathbf{C}$ is Schur, is open with respect to the Zariski topology on T.

PROOF : Let T_{schur} denote the said subset of T. By Proposition 2.2(4), T_{schur} equals the set of all the points $t \in T$, such that $\dim_{\mathbf{C}}(\mathrm{End}(V(t),\rho(t))) = 1$. Therefore, by Corollary 5.3, $T \setminus T_{\mathrm{schur}}$ is an analytic subset of T, and is hence Zariski closed. □

Corollary 5.6 — Let $f_1 : S \to T_1$ and $f_2 : S \to T_2$ be morphisms of complex analytic spaces, (V,ρ) a family of representations of Q parametrised by T_1, and (W,σ) a family of representations of Q parametrised by T_2. Then, the function

$$s \mapsto \dim_{\mathbf{C}}\left(\mathrm{Hom}((V(f_1(s)),\rho(f_1(s))),(W(f_2(s)),\sigma(f_2(s))))\right) : S \to \mathbf{N}$$

is upper semi-continuous with respect to the Zariski topology on S.

PROOF : Let $(M,\phi) = f_1^*(V,\rho)$ and $(N,\psi) = f_2^*(W,\sigma)$. Then, for every $s \in S$, we have canonical isomorphisms

$$(M(s),\phi(s)) \cong (V(f_1(s)),\rho(f_1(s))), \quad (N(s),\psi(s)) \cong (W(f_2(s)),\sigma(f_2(s)))$$

of representations of Q over $\mathbf{C}$, hence we get a canonical $\mathbf{C}$-isomorphism

$$\mathrm{Hom}((M(s),\phi(s)),(N(s),\psi(s))) \cong \mathrm{Hom}((V(f_1(s)),\rho(f_1(s))),(W(f_2(s)),\sigma(f_2(s)))).$$

The Corollary now follows from Proposition 5.2. □

5.B. *The Hausdorff property*

Let R be an equivalence relation on a topological space T. We say that two points t_1 and t_2 in T are *separated* with respect to R if there exist an open neighbourhood U_1 of t_1, and an open neighbourhood U_2 of t_2, in T, such that $U_1 \cap U_2 = \emptyset$, and both U_1 and U_2 are saturated with respect to R. This is equivalent to the condition that there exist an open neighbourhood U_1' of $\pi(t_1)$, and an open neighbourhood U_2' of $\pi(t_2)$, in T', such that $U_1' \cap U_2' = \emptyset$, where $T' = T/R$ is the quotient topological space of T by R, $\pi : T \to T'$ the canonical projection. We say that R is *open* if the saturation with respect to R of any open subset of T is open in T. This is equivalent to the condition that $\pi : T \to T'$ is an open map.

Remark 5.7 : It is easy to verify the following facts:

(1) The closure of R in $T \times T$ equals the set of all points (t_1, t_2) in $T \times T$ such that t_1 and t_2 are not separated with respect to R.

(2) The quotient topological space $T' = T/R$ is Hausdorff if and only if R is closed in $T \times T$.

Let T be a complex space, and (V,ρ) a family of representations of Q parametrised by T. Define a relation R on T by setting $t_1 R t_2$ if the representations $(V(t_1),\rho(t_1))$ and $(V(t_2),\rho(t_2))$ of Q over $\mathbf{C}$ are isomorphic. This is an equivalence relation on T. We will call it the equivalence relation on T *induced* by (V,ρ).

Lemma 5.8 — Let T be a complex space, (V,ρ) a family of non-zero representations of Q parametrised by T, and R the equivalence relation on T induced by (V,ρ). Let Z denote the closure of R with respect to the Zariski topology on the product complex space $T \times T$. Then, for every point $(t_1, t_2) \in Z$, there exist non-zero morphisms

$$f : (V(t_1),\rho(t_1)) \to (V(t_2),\rho(t_2)), \quad g : (V(t_2),\rho(t_2)) \to (V(t_1),\rho(t_1))$$

of representations of Q over $\mathbf{C}$.

PROOF: Let

$$A_1 = \{(t_1, t_2) \in T \times T \mid \mathrm{Hom}((V(t_1), \rho(t_1)), (V(t_2), \rho(t_2))) \neq 0\},$$

and

$$A_2 = \{(t_1, t_2) \in T \times T \mid \mathrm{Hom}((V(t_2), \rho(t_2)), (V(t_1), \rho(t_1))) \neq 0\}.$$

Then, it is obvious that R is a subset of $A_1 \cap A_2$. On the other hand, by Corollary 5.6, A_1 and A_2 are closed in the Zariski topology on $T \times T$, and hence so is $A_1 \cap A_2$. It follows that $Z \subset A_1 \cap A_2$. □

Proposition 5.9 — Let T be a complex space, $\theta : T \to \mathbf{R}^{Q_0}$ a continuous function, and (V, ρ) a family of representations of Q parametrised by T. Suppose that the equivalence relation R on T induced by (V, ρ) is open, that the representation $(V(t), \rho(t))$ over $\mathbf{C}$ is $\theta(t)$-stable for every $t \in T$, and that the function θ is R-invariant. Then, the quotient topological space T/R is Hausdorff.

PROOF : By Remark 5.7(2), it suffices to prove that R is strongly closed in $T \times T$. Let (t_1, t_2) be any point in the strong closure F of R in $T \times T$. Obviously, F is contained in the Zariski closure of R in $T \times T$, hence, by Lemma 5.8, there exists a non-zero morphism $f : (V(t_1), \rho(t_1)) \to (V(t_2), \rho(t_2))$ of representations of Q over $\mathbf{C}$. Now, for every $a \in Q_0$, the rank function $t \mapsto \dim_{\mathbf{C}}(V_a(t)) : T \to \mathbf{N}$ of the vector bundle V_a is locally constant. Therefore, as θ is continuous, the function $\phi : T \to \mathbf{N}^{Q_0} \times \mathbf{R}^{Q_0}$, which is defined by $\phi(t) = (\dim(V(t), \rho(t)), \theta(t))$, is continuous. Thus, the set

$$G = \left\{(t, t') \in T \times T \mid \phi(t) = \phi(t')\right\}$$

is strongly closed in $T \times T$. As θ is R-invariant, $R \subset G$. Therefore, $F \subset G$, hence $\phi(t_1) = \phi(t_2)$. This implies that $\mu_\theta(V(t_1), \rho(t_1)) = \mu_\theta(V(t_2), \rho(t_2))$, where $\theta = \theta(t_1) = \theta(t_2)$. Since $(V(t), \rho(t))$ is $\theta(t)$-stable for all $t \in T$, by Proposition 2.2(1d), we see that f is an isomorphism. Therefore, $(t_1, t_2) \in R$. This proves that R is strongly closed in $T \times T$. □

6. THE MODULI SPACE OF SCHUR REPRESENTATIONS

6.A. *Quotient premanifolds*

By a *complex premanifold*, we mean a complex manifold without any separation or countability conditions, that is, a topological space with a maximal holomorphic atlas. We use the term *complex manifold* for a complex premanifold whose underlying topological space is Hausdorff.

Let R be an equivalence relation on a complex premanifold X, Y the quotient topological space X/R, and $p : X \to Y$ the canonical projection. It is a theorem of Godement that the following statements are equivalent:

(1) There exists a structure of a complex premanifold on Y with the property that p is a holomorphic submersion.

(2) The relation R is a subpremanifold of $X \times X$, and the restricted projection $\mathrm{pr}_1 : R \to X$ is a submersion.

Moreover, in that case, such a complex premanifold structure on Y is unique [6, Part II, Chapter III, § 12, Theorems 1-2].

We will use the above theorem in the context of group actions. Let X be a topological space, and G a topological group. Suppose that we are given a continuous right action of G on X. Let R denote the equivalence relation on X defined by this action, and $\tau : X \times G \to R$ the map $(x, g) \mapsto (x, xg)$. For any two subsets A and B of X, let

$$P_G(A, B) = \{g \in G \mid Ag \cap B \neq \emptyset\}\,.$$

If the action of G on X is free, then for every $(x, y) \in R$, there exists a unique element $\phi(x, y)$ of G, such that $y = x\phi(x, y)$; we thus get a map $\phi : R \to G$, which is called the *translation* map of the given action. We say that the action of G on X is *principal* if it is free, and its translation map is continuous.

Remark 6.1 : It is easy to verify the following assertions:

(1) The action of G on X is free if and only if the map τ is injective.

(1) The following statements are equivalent:

  (a) The action of G on X is principal.

  (b) The action of G on X is free, and its translation map is continuous at (x, x) for all $x \in X$.

  (c) The action of G on X is free, and for every point $x \in X$, and for every neighbourhood V of the identity element e of G, there exists a neighbourhood U of x in X, such that $P_G(U, U) \subset V$.

  (d) The map τ is a homeomorphism.

Lemma 6.2 — Let X be a complex premanifold, and G a complex Lie group. Suppose that we are given a principal holomorphic right action of G on X. Let R be the equivalence relation on X defined by the action of G, and $\tau : X \times G \to R$ the map $(x, g) \mapsto (x, xg)$. Then, R is a complex subpremanifold of $X \times X$, and τ is a biholomorphism.

PROOF : Let $\sigma : X \times G \to X \times X$ be the map $(x, g) \mapsto (x, xg)$. Thus, $\sigma(X \times G) = R$, and $\tau : X \times G \to R$ is the map induced by σ. Since the action of G on X is principal, by Remark 6.1, the map τ is a homeomorphism. The map σ is obviously holomorphic. As the action of G on X is free, σ is an immersion. Therefore, σ is a holomorphic embedding, its image R is a complex subpremanifold of $X \times X$, and τ is a biholomorphism. □

Lemma 6.3 —Let $p : X \to Y$ be a surjective holomorphic submersion of complex premanifolds, and G a complex Lie group. Suppose that we are given a principal holomorphic right action of G on X, such that $p^{-1}(p(x)) = xG$ for all $x \in X$. Then, this action makes p a holomorphic principal G-bundle.

PROOF : Let R be the equivalence relation on X defined by the action of G, and $\tau : X \times G \to R$ the map $(x, g) \mapsto (x, xg)$. Then, by Lemma 6.2, R is a complex subpremanifold of $X \times X$, and τ is a biholomorphism. Let $b \in Y$. As p is surjective, there exists a point $a \in p^{-1}(b)$. Since p is a submersion at a, there exist an open neighbourhood V of b in Y, and a holomorphic section $s : V \to X$ of p, such that $s(b) = a$. The hypothesis on the fibres of p implies that the map $(c, g) \mapsto s(c)g$ is a G-equivariant holomorphic bijection u from $V \times G$ onto $p^{-1}(V)$. Its inverse is the composite

$$p^{-1}(V) \xrightarrow{\alpha} (p^{-1}(V) \times p^{-1}(V)) \cap R \xrightarrow{\beta} p^{-1}(V) \times G \xrightarrow{\gamma} V \times G,$$

where

$$\alpha(x) = (x, s(p(x))), \quad \beta(y, z) = \tau^{-1}(y, z), \quad \gamma(x, g) = (p(x), g)$$

for all $x \in p^{-1}(V)$, $(y, z) \in (p^{-1}(V) \times p^{-1}(V)) \cap R$, and $g \in G$. Since τ^{-1} is holomorphic, u^{-1} is also holomorphic. By definition, $p(u(c, g)) = c$ for all $c \in V$ and $g \in G$. Thus, u is a local trivialisation of p at b. It follows that p is a holomorphic principal G-bundle. □

Remark 6.4 : The proof of Lemma 6.3 also works to show that if $p : X \to Y$ is a surjective smooth submersion of smooth premanifolds, and G a real Lie group, and if we are given a principal smooth right action of G on X, such that $p^{-1}(p(x)) = xG$ for all $x \in X$, then this action makes p a smooth principal G-bundle.

Proposition 6.5 — Let X be a complex premanifold, and G a complex Lie group. Suppose that we are given a principal holomorphic right action of G on X. Let Y be the quotient topological space X/G, and $p : X \to Y$ the canonical projection. Then, there exists a unique structure of a complex premanifold on Y such that p is a holomorphic submersion. This structure makes p a holomorphic principal G-bundle.

PROOF : Let R be the equivalence relation on X defined by the action of G, and $\tau : X \times G \to R$ the map $(x, g) \mapsto (x, xg)$. Then, by Lemma 6.2, R is a complex subpremanifold of $X \times X$, and τ is a biholomorphism. Since $\mathrm{pr}_1 \circ \tau = \mathrm{pr}_1$, and $\mathrm{pr}_1 : X \times G \to X$ is clearly a submersion, it follows that $\mathrm{pr}_1 : R \to X$ is a submersion. Therefore, by Godement's theorem, there exists a unique structure of a complex premanifold on Y, such that p is a holomorphic submersion. It is obvious that p is surjective, and that $p^{-1}(p(x)) = xG$ for all $x \in X$. Therefore, by Lemma 6.3, p is a holomorphic principal G-bundle. □

Let G be a complex Lie group acting holomorphically on the right of a complex premanifold X, Y the quotient topological space X/G, and $p : X \to Y$ the canonical projection. Let H be a normal complex Lie subgroup of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. If the stabiliser G_x of any point $x \in X$ equals H, then there is an induced holomorphic right action of $\overline{G}$ on X.

Corollary 6.6 — Suppose that the stabiliser G_x of any point $x \in X$ equals H, and that for each $x \in X$, and H-invariant neighbourhood V of e in G, there exists a neighbourhood U of x in X, such that $P_G(U, U) \subset V$. Then, the action of $\overline{G}$ on X is principal, and there exists a unique structure of a complex premanifold on Y, such that p is a holomorphic submersion. Moreover, with the induced action of $\overline{G}$ on X, p is a holomorphic principal $\overline{G}$-bundle.

PROOF : The induced action of $\overline{G}$ on X is free, since $G_x = H$ for all $x \in X$. Let $x \in X$, and let W be a neighbourhood of e in $\overline{G}$. Then, $V = \pi^{-1}(W)$ is an H-invariant neighbourhood of e in G. Therefore, by hypothesis, there exists an open neighbourhood U of x in X, such that $P_G(U, U) \subset V$. Now, $U = U \cap X$ is an open neighbourhood of x in X, and $P_{\overline{G}}(U, U) \subset \pi(P_G(U, U)) \subset \pi(P_G(U, U)) \subset \pi(V) \subset W$. Therefore, by Remark 6.1, the action of $\overline{G}$ on X is principal. It is obvious that the induced action of $\overline{G}$ on X is holomorphic, that $X/\overline{G} = X/G = Y$, and that the canonical projection from X to $X/\overline{G}$ equals p. The Corollary now follows from Proposition 6.5. □

6.B. *The complex premanifold of Schur representations*

Let Q be a non-empty finite quiver. We will consider only complex representations of Q in this subsection. Let $d = (d_a)_{a\in Q_0}$ be a non-zero element of $\mathbf{N}^{Q_0}$, and fix a family $V = (V_a)_{a\in Q_0}$ of $\mathbf{C}$-vector spaces, such that $\dim_{\mathbf{C}}(V_a) = d_a$ for all $a \in Q_0$.

Denote by $\mathcal{A}$ the finite-dimensional $\mathbf{C}$-vector space $\bigoplus_{\alpha\in Q_1}\mathrm{Hom}_{\mathbf{C}}(V_{\mathrm{s}(\alpha)},V_{\mathrm{t}(\alpha)})$. For every element ρ of $\mathcal{A}$, we have a representation (V,ρ) of Q. Moreover, for every representation (W,σ) of Q, such that $\dim(W,\sigma)=d$, there exists an element ρ of $\mathcal{A}$, such that the representations (V,ρ) and (W,σ) are isomorphic.

We give the vector space $\mathcal{A}$ the usual topology, and the usual structure of a complex manifold. For each $a\in Q_0$, denote by E_a the trivial holomorphic vector bundle $\mathcal{A}\times V_a$ on $\mathcal{A}$. Then, for every $\alpha\in Q_1$, we have a morphism $\theta_\alpha : E_{\mathrm{s}(\alpha)}\to E_{\mathrm{t}(\alpha)}$ of holomorphic vector bundles, which is defined by $\theta_\alpha(\rho,v)=(\rho,\rho_\alpha(v))$ for all (ρ,v) in $E_{\mathrm{s}(\alpha)}$. We thus get a family (E,θ) of representations of Q parametrised by $\mathcal{A}$, where $E=(E_a)_{a\in Q_0}$ and $\theta=(\theta_\alpha)_{\alpha\in Q_1}$. By definition, for each point $\rho\in\mathcal{A}$, the fibre representation $E(\rho)$ is precisely (V,ρ).

Let G be the complex Lie group $\prod_{a\in Q_0}\mathrm{Aut}_{\mathbf{C}}(V_a)$. There is a canonical holomorphic linear right action $(\rho,g)\mapsto\rho g$ of G on $\mathcal{A}$, which is defined by

$$(\rho g)_\alpha = g_{\mathrm{t}(\alpha)}^{-1}\circ\rho_\alpha\circ g_{\mathrm{s}(\alpha)}$$

for all $\rho\in\mathcal{A}$, $g\in G$, and $\alpha\in Q_1$. For all $\rho,\sigma\in\mathcal{A}$ and $g\in G$, we have $\sigma=\rho g$ if and only if g is an isomorphism of representations of Q, from (V,σ) to (V,ρ). In other words, two points ρ and σ of $\mathcal{A}$ lie on the same orbit of G if and only if the representations (V,ρ) and (V,σ) of Q are isomorphic. Thus, the map which takes every point ρ of $\mathcal{A}$ to the representation (V,ρ) induces a bijection from the quotient set $\mathcal{A}/G$ onto the set of isomorphism classes of representations (W,σ) of Q, such that $\dim(W,\sigma)=d$.

Denote by H the central complex Lie subgroup of G consisting of all elements of the form ce, as c runs over $\mathbf{C}^\times$, where $e=(\mathbf{1}_{V_a})_{a\in Q_0}$ is the identity element of G. Let $\overline{G}$ denote the complex Lie group $H\backslash G$, $\pi : G\to\overline{G}$ the canonical projection. Define $\mathcal{B}$ to be the set of all points ρ of $\mathcal{A}$, such that the representation (V,ρ) of Q is Schur. It is a G-invariant subset of $\mathcal{A}$. By Proposition 2.2(4), a point ρ of $\mathcal{A}$ lies in $\mathcal{B}$ if and only if its stabiliser G_ρ equals H. Corollary 5.5, applied to the family (E,θ) of representations of Q parametrised by $\mathcal{A}$, implies that $\mathcal{B}$ is Zariski open in $\mathcal{A}$, and is hence an open complex submanifold of $\mathcal{A}$. Let M denote the quotient topological space $\mathcal{B}/G$, and $p:\mathcal{B}\to M$ the canonical projection. By the above observation, there is a canonical bijection from M onto the set of isomorphism classes of Schur representations (W,σ) of Q, such that $\dim(W,\sigma)=d$. We will call M the *moduli space* of Schur representations of Q with dimension vector d. Note that the action of G on $\mathcal{A}$ induces a holomorphic right action of $\overline{G}$ on $\mathcal{B}$.

The Lie algebra $\mathrm{Lie}(G)$ of G is the direct sum Lie algebra $\bigoplus_{a\in Q_0}\mathrm{End}_{\mathbf{C}}(V_a)$, where, for each $a\in Q_0$, the associative $\mathbf{C}$-algebra $\mathrm{End}_{\mathbf{C}}(V_a)$ is given its usual Lie algebra structure. Note that

$\mathrm{Lie}(G)$ has a canonical structure of an associative unital $\mathbf{C}$-algebra, and that G is the group of units of the underlying ring of $\mathrm{Lie}(G)$, and is open in $\mathrm{Lie}(G)$. The Lie algebra of H is the Lie subalgebra of $\mathrm{Lie}(G)$ consisting of all elements of the form ce, as c runs over $\mathbf{C}$. Let $\mathrm{Tr} : \mathrm{Lie(G)} \to \mathbf{C}$ be the $\mathbf{C}$-linear function defined by $\mathrm{Tr}(\xi) = \sum_{a \in Q_0} \mathrm{Tr}(\xi_a)$ for all elements $\xi = (\xi_a)_{a\in Q_0}$ of $\mathrm{Lie}(G)$, and let $\mathrm{Lie}(G)^0$ denote its kernel. Then, as $d \neq 0$, $\mathrm{Tr}(e) = \mathrm{rk}(d) = \sum_{a\in Q_0} d_a$ is a non-zero natural number, and we have a decomposition $\mathrm{Lie}(G) = \mathrm{Lie}(H) \oplus \mathrm{Lie}(G)^0$. As d is a non-zero element of $\mathbf{N}^{Q_0}$, we have $e \neq 0$, hence the map $t \mapsto te : \mathbf{C} \to \mathrm{Lie}(H)$ is a $\mathbf{C}$-isomorphism. For any element ξ of $\mathrm{Lie}(G)$, we define $(c(\xi), \xi^0)$ to be the unique element of $\mathbf{C} \times \mathrm{Lie}(G)^0$, such that $\xi = c(\xi)e + \xi^0$. Then, $c(\xi) = \frac{\mathrm{Tr}(\xi)}{\mathrm{rk}(d)}$, and $\xi^0 = \xi - c(\xi)e$ for all $\xi \in \mathrm{Lie}(G)$.

For every element ρ of $\mathcal{A}$, we denote the orbit map $g \mapsto \rho g : G \to \mathcal{A}$ by μ_ρ, and by D_ρ the $\mathbf{C}$-linear map $\mathrm{T}_e(\mu_\rho) : \mathrm{Lie}(G) \to \mathcal{A}$. Thus,

$$D_\rho(\xi) = (\rho_\alpha \circ \xi_{\mathrm{s}(\alpha)} - \xi_{\mathrm{t}(\alpha)} \circ \rho_\alpha)_{\alpha\in Q_1}$$

for all $\xi \in \mathrm{Lie}(G)$. Therefore, $\mathrm{Ker}(D_\rho) = \mathrm{End}(V, \rho)$. In particular, $\mathrm{Ker}(D_\rho) = \mathrm{Lie}(H)$ if $\rho \in \mathcal{B}$.

It will be convenient to fix a family $h = (h_a)_{a\in Q_0}$ of Hermitian inner products $h_a : V_a \times V_a \to \mathbf{C}$. Thus, for every point $\rho \in \mathcal{A}$, h is a Hermitian metric on the representation (V, ρ) of Q.

For any two finite-dimensional Hermitian inner product spaces V and W, we have a Hermitian inner product $\langle \cdot, \cdot \rangle$ on the $\mathbf{C}$-vector space $\mathrm{Hom}_{\mathbf{C}}(V, W)$, which is defined by $\langle u, v\rangle = \mathrm{Tr}(u \circ v^*)$ for all $u, v \in \mathrm{Hom}_{\mathbf{C}}(V, W)$, where, $v^* : W \to V$ is the adjoint of v. If we denote the norm associated to this Hermitian inner product by $\|\cdot\|$, then $\|u(x)\| \leq \|u\|\|x\|$ for all $u \in \mathrm{Hom}_{\mathbf{C}}(V, W)$ and $x \in V$. Also, $\|u^*\| = \|u\|$, $\|\mathbf{1}_V\| = \sqrt{\dim_{\mathbf{C}}(V)}$, and, for all finite-dimensional Hermitian inner product spaces V, W, and X, and for all $u \in \mathrm{Hom}_{\mathbf{C}}(V, W)$ and $v \in \mathrm{Hom}_{\mathbf{C}}(W, X)$, we have $\|v \circ u\| \leq \|v\|\|u\|$.

Remark 6.7 :Using the above facts, it is easy to verify that for every $u \in \mathrm{Hom}_{\mathbf{C}}(V, W)$, there exists a real number $\theta > 0$, such that $\theta\|x\| \leq \|u(x)\|$ for all $x \in \mathrm{Ker}(u)^\perp$, where $X^\perp$ denotes the orthogonal complement of any subset X of a finite-dimensional Hermitian inner product space.

In particular, the family h induces a Hermitian inner product $\langle \cdot, \cdot \rangle$ on the $\mathbf{C}$-vector space $\mathrm{Hom}_{\mathbf{C}}(V_a, V_b)$ for all $a, b \in Q_0$. We give $\mathrm{Lie}(G)$ the Hermitian inner product $\langle \cdot, \cdot \rangle$ which is the direct sum of the Hermitian inner products $\langle \cdot, \cdot \rangle$ on $\mathrm{End}_{\mathbf{C}}(V_a)$ as a runs over Q_0. Note that $\|e\| = \sqrt{\mathrm{rk}(d)}$ with respect to this Hermitian inner product, and that $\mathrm{Lie}(H)^\perp = \mathrm{Lie}(G)^0$. Let $u : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ be the corresponding orthogonal projection.

Similarly, we give $\mathcal{A}$ the Hermitian inner product $\langle \cdot, \cdot \rangle$ which is the direct sum of the Hermitian inner products $\langle\cdot,\cdot\rangle$ on $\mathrm{Hom}_{\mathbf{C}}(V_{\mathrm{s}(\alpha)}, V_{\mathrm{t}(\alpha)})$ as α runs over Q_1. For every $\rho \in \mathcal{A}$, we have the adjoint $D_\rho^* : \mathcal{A} \to \mathrm{Lie}(G)$ of the **C**-linear map $D_\rho : \mathrm{Lie}(G) \to \mathcal{A}$ that was defined above.

***Theorem* 6.8** — *The action of $\overline{G}$ on $\mathcal{B}$ is principal. In particular, there exists a unique structure of a complex premanifold on the moduli space M of complex Schur representations of Q with dimension vector d, such that $p : \mathcal{B} \to M$ is a holomorphic submersion. Moreover, this structure makes the map p a holomorphic principal $\overline{G}$-bundle.*

PROOF : Let ρ be an arbitrary point of $\mathcal{B}$. Then, as observed above, $\mathcal{B}$ is a G-invariant open complex submanifold of $\mathcal{A}$, so there is an induced holomorphic right action of G on $\mathcal{B}$. Also, $G_\rho = H$ for all $\rho \in \mathcal{B}$. Therefore, by Corollary 6.6, it suffices to prove that for every H-invariant neighbourhood V of e in G, there exists an open neighbourhood U of ρ in $\mathcal{B}$, such that $P_G(U, U) \subset V$.

Let $D_\rho : \mathrm{Lie}(G) \to \mathcal{A}$ be the **C**-linear map defined earlier. Then, as noted above, $\mathrm{Ker}(D_\rho) = \mathrm{Lie}(H)$. Therefore, $\mathrm{Ker}(D_\rho)^\perp = \mathrm{Lie}(H)^\perp = \mathrm{Lie}(G)^0$, hence, by Remark 6.7, there exists a real number $\theta > 0$, such that $\theta\|f\| \le \|D_\rho(f)\|$ for all $f \in \mathrm{Lie}(G)^0$. Let $q_1 = \mathrm{card}(Q_1)$. Then, the continuity of the norm function on $\mathcal{A}$ implies that the set X of all $(\sigma, \tau) \in \mathcal{A} \times \mathcal{A}$, such that $q_1(\|\sigma - \rho\| + \|\tau - \rho\|) < \theta$, is an open neighbourhood of (ρ, ρ) in $\mathcal{A} \times \mathcal{A}$.

Consider a point $(\sigma, \tau) \in X$, let $g \in G$, and suppose $\tau = \sigma g$. Then, by the above paragraph,

$$\theta\|g^0\| \le \|D_\rho(g^0)\|.$$

Let $\sigma' = \sigma - \rho$ and $\tau' = \tau - \rho$. The relation $\tau = \sigma g$ implies that

$$D_\rho(g) = \left(g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}\right)_{\alpha \in Q_1}.$$

Therefore, as $c(g)e \in \mathrm{Ker}(D_\rho)$, we have

$$\begin{aligned}\|D_\rho(g^0)\| = \|D_\rho(g)\| &\le \sum_{\alpha \in Q_1} \|g_{\mathrm{t}(\alpha)} \circ \tau'_\alpha - \sigma'_\alpha \circ g_{\mathrm{s}(\alpha)}\| \\ &\le q_1\big(\mid c(g) \mid \sqrt{\mathrm{rk}(d)} + \|g^0\|\big)\big(\|\tau'\| + \|\sigma'\|\big).\end{aligned}$$

Thus,

$$\theta\|g^0\| \le q_1\big(\mid c(g) \mid \sqrt{\mathrm{rk}(d)} + \|g^0\|\big)\big(\|\sigma'\| + \|\tau'\|\big).$$

As $(\sigma, \tau) \in X$, we have $q_1(\|\sigma'\| + \|\tau')\|) < \theta$, hence this implies that

$$\|g^0\| \le \frac{q_1 \mid c(g) \mid \sqrt{\mathrm{rk}(d)}(\|\sigma'\| + \|\tau'\|)}{\theta - q_1(\|\sigma'\| + \|\tau'\|)}.$$

In particular, $c(g) \neq 0$; for, if $c(g) = 0$, then, by the above inequality, we get $g^0 = 0$, hence $g = c(g)e + g^0 = 0$; therefore, as $g_a \in \mathrm{Aut}_{\mathbf{C}}(V_a)$, we have $V_a = 0$ for every $a \in Q_0$, a contradiction, since d is a non-zero element of $\mathbf{N}^{Q_0}$. It follows that

$$\left\|\frac{1}{c(g)}g - e\right\| = \left\|\frac{1}{c(g)}g^0\right\| = \frac{1}{\mid c(g) \mid}\|g^0\| \leq \frac{q_1\sqrt{\mathrm{rk}(d)}(\|\sigma'\| + \|\tau'\|)}{\theta - q_1(\|\sigma'\| + \|\tau'\|)}.$$

We have thus shown that for all $(\sigma, \tau) \in X$ and $g \in G$, such that $\tau = \sigma g$, we have $c(g) \neq 0$, and

$$\left\|\frac{1}{c(g)}g - e\right\| \leq \delta(\sigma, \tau),$$

where $\delta : X \to [0, +\infty)$ is the function defined by

$$\delta(\sigma, \tau) = \frac{q_1\sqrt{\mathrm{rk}(d)}(\|\sigma - \rho\| + \|\tau - \rho\|)}{\theta - q_1(\|\sigma - \rho\| + \|\tau - \rho\|)}.$$

Now, let V be an H-invariant open neighbourhood of e in G. Then, as G is open in $\mathrm{Lie}(G)$, there exists $\epsilon > 0$, such that the open ball $B(e, \epsilon)$ in $\mathrm{Lie}(G)$, with radius ϵ and centre e, is contained in V. As X is an open neighbourhood of (ρ, ρ) in $\mathcal{A} \times \mathcal{A}$, and the function δ is continuous, there exists an open neighbourhood U of ρ in $\mathcal{B}$, such that $U \times U \subset X$, and $\delta(\sigma, \tau) < \epsilon$ for all $(\sigma, \tau) \in U \times U$. We claim that $P_G(U, U) \subset V$. Let $g \in P_G(U, U)$. Then, there exists a point (σ, τ) of $U \times U$, such that $\tau = \sigma g$. As $(\sigma, \tau) \in X$, by the above paragraph, $c(g) \neq 0$, and $\|\frac{1}{c(g)}g - e\| \leq \delta(\sigma, \tau) < \epsilon$, hence $\frac{1}{c(g)}g \in B(e, \epsilon) \subset V$. As V is H-invariant, $g = \big(c(g)e\big)\big(\frac{1}{c(g)}g\big) \in HV \subset V$. Thus, $P_G(U, U) \subset V$. $\square$

7. The Kähler Metric on Moduli of Stable Representations

7.A. *Moment maps*

Let (X, Ω) be a smooth symplectic manifold. Recall that a smooth vector field ξ on X is called *symplectic* if $L_\xi(\Omega) = 0$, where L_ξ is the Lie derivative with respect to ξ. The set $\mathrm{V}(X, \Omega)$ of symplectic vector fields on X is a Lie subalgebra of the real Lie algebra $\mathrm{V}(X)$ of smooth vector fields on X. For any smooth real function f on X, we denote by $H(f)$ the Hamiltonian vector field of f, that is, the unique smooth vector field on X, such that $\Omega(x)(H(f)(x), w) = w(f)$ for all $x \in X$ and $w \in \mathrm{T}_x(X)$, where $\mathrm{T}_x(X)$ denotes the tangent space of X at x; it is a symplectic vector field. We define the Poisson bracket of any two smooth real functions f and g on X, by $\{f, g\} = \Omega(H(g), H(f))$. This makes the $\mathbf{R}$-vector space $S(X)$ of smooth real functions on X a Lie algebra, and the map $f \mapsto H(f)$ is a homomorphism of real Lie algebras $H : S(X) \to \mathrm{V}(X, \Omega)$.

Let K be a real Lie group. Suppose we are given a smooth right action of K on X, which is symplectic, by which we mean that it preserves the symplectic form Ω on X. Thus, Ω is K-invariant, that is, $\rho_g^*(\Omega) = \Omega$ for all $g \in K$, where ρ_g denotes the translation by g on every right K-space. Then, for every element ξ of the Lie algebra $\mathrm{Lie}(K)$ of K, the induced vector field $\xi^\sharp$ on X is symplectic. The map $\xi \mapsto \xi^\sharp$ is a homomorphism of real Lie algebras from $\mathrm{Lie}(K)$ to $\mathrm{V}(X, \Omega)$, which is K-equivariant for the adjoint action of K on $\mathrm{Lie}(K)$, and the canonical action of K on $\mathrm{V}(X, \Omega)$.

A *moment map* for the action of K on X is a smooth map $\Phi : X \to \mathrm{Lie}(K)^*$, which is K-invariant for the coadjoint action of K on $\mathrm{Lie}(K)^*$, and has the property that $H(\Phi^\xi) = \xi^\sharp$ for all $\xi \in \mathrm{Lie}(K)$, where Φ^ξ is the smooth real function $x \mapsto \Phi(x)(\xi)$ on X. If Φ is a moment map, then, for every $x \in X$, the $\mathbf{R}$-linear map $\mathrm{T}_x(\Phi) : \mathrm{T}_x(X) \to \mathrm{Lie}(K)^*$ is given by $\mathrm{T}_x(\Phi)(v)(\xi) = \Omega(x)(\xi^\sharp(x), v)$ for all $v \in \mathrm{T}_x(X)$ and $\xi \in \mathrm{Lie}(K)$. Thus, $\mathrm{Ker}(\mathrm{T}_x(\Phi)) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)}$, where $\nu_x : K \to X$ is the orbit map of x, $\mathrm{T}_e(\nu_x) : \mathrm{Lie}(K) \to \mathrm{T}_x(X)$ is the induced $\mathbf{R}$-linear map, and $S^{\perp(\Omega)}$ denotes the set of elements of $\mathrm{T}_x(X)$ that are $\Omega(x)$-orthogonal to any subset S of $\mathrm{T}_x(X)$. This implies that $\mathrm{Ker}(\mathrm{T}_x(\Phi))^{\perp(\Omega)} = \mathrm{Im}(\mathrm{T}_e(\nu_x))$. Also, $\mathrm{Im}(\mathrm{T}_x(\Phi)) = \mathrm{Ann}(\mathrm{Lie}(K_x))$, where K_x is the stabiliser of x in K, and $\mathrm{Ann}(M)$ denotes the annihilator in $\mathrm{Lie}(K)^*$ of any subset M of $\mathrm{Lie}(K)$. In particular, Φ is a submersion at x if and only if the subgroup K_x of K is discrete.

For later use, we state here a simple fact about moment maps for linear actions. Let V be a finite-dimensional $\mathbf{R}$-vector space, and Ω a symplectic form on V. Since the tangent space of V at any point is canonically isomorphic to V itself, Ω defines a smooth 2-form on V. By abuse of notation, we will denote this smooth 2-form also by Ω. In any linear coordinate system on V, we can express Ω as a form with constant coefficients, so $\mathrm{d}\,\Omega = 0$. Therefore, (V, Ω) is a symplectic manifold.

Let K be a real Lie group, and suppose that we are given a smooth linear right action of K on V, which preserves the symplectic form Ω on V. For any element ξ of $\mathrm{Lie}(K)$, let $\xi^\sharp$ be the vector field on V defined by ξ; then $\xi^\sharp$ is an $\mathbf{R}$-endomorphism of V and $\Omega(\xi^\sharp(x), y) + \Omega(x, \xi^\sharp(y)) = 0$ for all $x, y \in V$. For each element α of $\mathrm{Lie}(K)^*$, define a map $\Phi_\alpha : V \to \mathrm{Lie}(K)^*$ by

$$\Phi_\alpha(x)(\xi) = \frac{1}{2}\Omega(\xi^\sharp(x), x) + \alpha(\xi)$$

for all $x \in V$ and $\xi \in \mathrm{Lie}(K)$.

Lemma 7.1 — The map $\alpha \mapsto \Phi_\alpha$ is a bijection from the set of K-invariant elements of $\mathrm{Lie}(K)^*$ onto the set of moment maps for the action of K on V.

PROOF : It is easy to see that Φ_0 is a moment map for the action of K on V. Let $\alpha \in \mathrm{Lie}(K)^*$. Then, $\Phi_\alpha = \Phi_0 + \alpha$. Therefore, $H(\Phi_\alpha^\xi) = H(\Phi_0) = \xi^\sharp$ for all $\xi \in \mathrm{Lie}(K)$; moreover, since Φ_0

is K-equivariant, Φ_α is K-equivariant if and only if α is K-invariant, that is, $\alpha(\mathrm{Ad}(g)\xi) = \alpha(\xi)$ for all $g \in K$ and $\xi \in \mathrm{Lie}(K)$. Thus, Φ_α is a moment map for the action of K if and only if α is K-invariant.

Now, the map $\alpha \mapsto \Phi_\alpha$ is clearly injective. Suppose $\Psi : V \to \mathrm{Lie}(K)^*$ is a moment map for the action of K on V. Then, for all $\xi \in \mathrm{Lie}(K)$, we have $H(\Psi^\xi) = H(\Phi_0^\xi) = \xi^\sharp$, hence $\mathrm{T}_x(\Psi^\xi) = \mathrm{T}_x(\Phi_0^\xi)$ for all $x \in X$; as V is connected, this implies that $\Psi^\xi - \Phi_0^\xi = \alpha(\xi)$ for some $\alpha(\xi) \in \mathbf{R}$. The function $\alpha : \xi \mapsto \alpha(\xi)$ is obviously $\mathbf{R}$-linear, that is, an element of $\mathrm{Lie}(K)^*$. As we have seen above, it is necessarily K-invariant. Thus, the given map is surjective too. □

7.B. *Kähler quotients*

For any complex premanifold X and $x \in X$, we will identify the tangent space at x of the underlying smooth manifold of X, with the holomorphic tangent space $\mathrm{T}_x(X)$ of X at x, using the canonical $\mathbf{R}$-isomorphism between them. With this identification, for any holomorphic map $f : X \to Y$ of complex premanifolds, and $x \in X$, the real differential of f at x is equal to the $\mathbf{C}$-linear map $\mathrm{T}_x(f) : \mathrm{T}_x(X) \to \mathrm{T}_{f(x)}(Y)$, considered as an $\mathbf{R}$-linear map.

Let X be a complex manifold, g a Kähler metric on X, and Ω its Kähler form, that is, the closed real 2-form on X defined by $\Omega(x)(v,w) = -2\Im(g(x)(v,w))$ for all $x \in X$, and $v, w \in \mathrm{T}_x(X)$, where $\Im(t)$ denotes the imaginary part of a complex number t. Since Ω is positive, (X, Ω) is a smooth symplectic manifold. The following Lemma follows directly from the definition.

Lemma 7.2 — Let B be the smooth Riemannian metric on X defined by $B(x)(v,w) = 2\Re(g(x)(v,w))$, where $\Re(t)$ denotes the real part of a complex number t. Then, for every point $x \in X$, and $\mathbf{R}$-subspace W of $\mathrm{T}_x(X)$, we have $W^{\perp(\Omega)} = (\sqrt{-1}W)^{\perp(B)}$, where $S^{\perp(\Omega)}$ (respectively, $S^{\perp(B)}$) is the set of all elements of $\mathrm{T}_x(X)$ that are $\Omega(x)$-orthogonal (respectively, $B(x)$-orthogonal) to any subset S of $\mathrm{T}_x(X)$. In particular, we have an $\mathbf{R}$-vector space decomposition $\mathrm{T}_x(X) = W^{\perp(\Omega)} \oplus (\sqrt{-1}W)$.

Let G be a complex Lie group, and K a compact subgroup of G; in particular, K is a real Lie subgroup of G. Suppose that we are given a holomorphic right action of G on X, such that the induced action of K on X preserves the Kähler metric g on X. Then, the Kähler form Ω on X is K-invariant, that is, the action of K on X is symplectic. Let $\Phi : X \to \mathrm{Lie}(K)^*$ be a moment map for the action of K on X, X_{m} the closed subset $\Phi^{-1}(0)$ of X, and $X_{\mathrm{ms}} = X_{\mathrm{m}}G$. Then, X_{ms} is G-invariant, and, since Φ is K-equivariant, X_{m} is a K-invariant subset of X_{ms}. Denote by Y the quotient topological space X/G, and let $p : X \to Y$ be the canonical projection. Let $Y_{\mathrm{ms}} = p(X_{\mathrm{ms}})$, $p_{\mathrm{ms}} : X_{\mathrm{ms}} \to Y_{\mathrm{ms}}$ the map induced by p, and $p_{\mathrm{m}} = p_{\mathrm{ms}}|X_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$.

Proposition 7.3 — Suppose that the action of G on X is principal, and that

$$P_G(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset K.$$

Then:

(1) The set X_{m} is a closed smooth submanifold of X, X_{ms} is open in X, Y_{ms} is open in Y, the action of K on X_{m} is principal, and $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is a smooth principal K-bundle.

(2) The action of G on X_{ms} is proper, Y_{ms} is a Hausdorff open subspace of Y, the action of G on X_{ms} is principal, and $p_{\mathrm{ms}} : X_{\mathrm{ms}} \to Y_{\mathrm{ms}}$ is a holomorphic principal G-bundle.

(3) There exists a unique Kähler metric h_{ms} on Y_{ms}, such that ${p_{\mathrm{m}}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$, where Θ_{ms} is the Kähler form of h_{ms}, $\Omega_{\mathrm{m}} = i_{\mathrm{m}}^*(\Omega)$, and $i_{\mathrm{m}} : X_{\mathrm{m}} \to X$ is the inclusion map.

PROOF : (1) We will use the remarks and the notation in Section 7.A. Since the action of G on X is free, we have $K_x = \{e\}$ for all $x \in X$, hence the moment map $\Phi : X \to \mathrm{Lie}(K)^*$ is a submersion. Therefore, $X_{\mathrm{m}} = \Phi^{-1}(0)$ is a closed submanifold of X, and, for all $x \in X_{\mathrm{m}}$, we have $\mathrm{T}_x(X_{\mathrm{m}}) = \mathrm{Ker}(\mathrm{T}_x(\Phi)) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)}$.

We will next check that X_{ms} is open in X. Let $\mu_{\mathrm{m}} : X_{\mathrm{m}} \times G \to X$ be the restriction of the action map $\mu : X \times G \to X$. We claim that the smooth map μ_{m} is a submersion. For all $g \in G$, we have $\mu_{\mathrm{m}} \circ (\mathbf{1}_{X_{\mathrm{m}}} \times \rho_g) = \rho_g \circ \mu_{\mathrm{m}}$, where ρ_g denotes the translation by g on any right G-space. Therefore, it suffices to check that the $\mathbf{R}$-linear map

$$\mathrm{T}_{(x,e)}(\mu_{\mathrm{m}}) : \mathrm{T}_x(X_{\mathrm{m}}) \oplus \mathrm{Lie}(G) \to \mathrm{T}_x(X)$$

is surjective for every $x \in X_{\mathrm{m}}$. For all $w \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi \in \mathrm{Lie}(G)$, we have

$$\mathrm{T}_{(x,e)}(\mu_{\mathrm{m}})(w, \xi) = \mathrm{T}_{(x,e)}(\mu)(w, \xi) = w + \xi^\sharp(x) = w + \mathrm{T}_e(\mu_x)(\xi),$$

where $\mu_x : G \to X$ is the orbit map of X. Now, putting $W = \mathrm{Im}(\mathrm{T}_e(\nu_x))$ in Lemma 7.2, we get

$$\mathrm{T}_x(X) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)} \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x))) = \mathrm{T}_x(X_{\mathrm{m}}) \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x))).$$

Therefore, for each $u \in \mathrm{T}_x(X)$, there exist $w \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\eta \in \mathrm{Lie}(K)$, such that $u = w + \sqrt{-1}\mathrm{T}_e(\nu_x)(\eta)$. But, since $\nu_x : K \to X$ is the restriction of $\mu_x : G \to X$, we have $\mathrm{T}_e(\nu_x)(\eta) = \mathrm{T}_e(\mu_x)(\eta)$. Also, since μ_x is holomorphic, the map $\mathrm{T}_e(\mu_x) : \mathrm{Lie}(G) \to \mathrm{T}_x(X)$ is $\mathbf{C}$-linear. Therefore,

$$u = w + \sqrt{-1}\mathrm{T}_e(\mu_x)(\eta) = w + \mathrm{T}_e(\mu_x)(\sqrt{-1}\eta) = \mathrm{T}_{(x,e)}(\mu_{\mathrm{m}})(w, \sqrt{-1}\eta).$$

This proves that $\mathrm{T}_{(x,e)}(\mu_{\mathrm{m}})$ is surjective for all $x \in X_{\mathrm{m}}$, hence μ_{m} is a submersion. Therefore, it is an open map. In particular, $X_{\mathrm{ms}} = X_{\mathrm{m}}G = \mu_{\mathrm{m}}(X_{\mathrm{m}} \times G)$ is open in X.

The map $p : X \to Y$ is the quotient map for a continuous action of a topological group, and is hence an open map. Therefore, as X_{ms} is open in X, $Y_{\mathrm{ms}} = p(X_{\mathrm{ms}})$ is open in Y. Moreover, as X_{ms} is G-invariant, we have $X_{\mathrm{ms}} = p^{-1}(p(X_{\mathrm{ms}})) = p^{-1}(Y_{\mathrm{ms}})$. Now, by Proposition 6.5, there exists a unique structure of a complex premanifold on Y, such that p is a holomorphic submersion; moreover, this structure makes p a holomorphic principal G-bundle. Therefore, p_{ms} is also a holomorphic principal G-bundle.

The map $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is obviously smooth. It is surjective, because $Y_{\mathrm{ms}} = p(X_{\mathrm{ms}}) = p(X_{\mathrm{m}}G) = p(X_{\mathrm{m}}) = p_{\mathrm{m}}(X_{\mathrm{m}})$. We will now check that it is a submersion. Let $x \in X_{\mathrm{m}}$, and $w \in \mathrm{T}_{p(x)}(Y)$. Then, since $p : X \to Y$ is a holomorphic submersion, there exists $v \in \mathrm{T}_x(X)$, such that $\mathrm{T}_x(p)(v) = w$. As $\mathrm{T}_x(X) = \mathrm{T}_x(X_{\mathrm{m}}) \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x)))$, there exist $v_{\mathrm{m}} \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi \in \mathrm{Lie}(K)$, such that $v = v_{\mathrm{m}} + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)$. Now, $\mathrm{T}_e(\nu_x)(\xi) = \mathrm{T}_e(\mu_x)(\xi)$ belongs to the $\mathbf{C}$-subspace $\mathrm{Im}(\mathrm{T}_e(\mu_x)) = \mathrm{T}_x(xG) = \mathrm{Ker}(\mathrm{T}_x(p))$ of $\mathrm{T}_x(X)$, hence $\mathrm{T}_x(p)(\sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)) = 0$. Therefore, $w = \mathrm{T}_x(p)(v_{\mathrm{m}}) = \mathrm{T}_x(p_{\mathrm{m}})(v_{\mathrm{m}})$. This proves that p_{m} is a submersion. The condition $P_G(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset K$, and the K-invariance of p_{m}, imply that ${p_{\mathrm{m}}}^{-1}(p_{\mathrm{m}}(x)) = xK$ for all $x \in X_{\mathrm{m}}$. Lastly, if R is the graph of the action of G on X, R_{m} that of the action of K on X_{m}, and $\phi : R \to G$ and $\phi_{\mathrm{m}} : R_{\mathrm{m}} \to K$ the translation maps, then $R_{\mathrm{m}} \subset R$, and ϕ_{m} is induced by ϕ. As the action of G on X is principal, ϕ is continuous, hence so is ϕ_{m}, so the action of K on X_{m} is also principal. Now, by Remark 6.4, p_{m} is a smooth principal K-bundle.

(2) As $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is a smooth principal G-bundle, it is an open map. Therefore, $p_{\mathrm{m}} \times p_{\mathrm{m}} : X_{\mathrm{m}} \times X_{\mathrm{m}} \to Y_{\mathrm{ms}} \times Y_{\mathrm{ms}}$ is also an open map. Since it is also a continuous surjection, it is a quotient map. Now, $R_{\mathrm{m}} = (p_{\mathrm{m}} \times p_{\mathrm{m}})^{-1}(\Delta_{\mathrm{ms}})$, where R_{m} is the graph of the action of K on X_{m}, and Δ_{ms} is the diagonal of Y_{ms}. Since K is compact and X_{m} is Hausdorff, the action of K on X_{m} is proper, hence R_{m} is closed in $X_{\mathrm{ms}} \times X_{\mathrm{m}}$. Therefore, Δ_{ms} is closed in $Y_{\mathrm{ms}} \times Y_{\mathrm{ms}}$, so Y_{ms} is Hausdorff. The graph R_{ms} of the action of G on X_{ms} equals $(p_{\mathrm{ms}} \times p_{\mathrm{ms}})^{-1}(\Delta_{\mathrm{ms}})$, and is hence closed in $X_{\mathrm{ms}} \times X_{\mathrm{ms}}$. Moreover, R_{m} is contained in the graph R of the action of G on X, and the translation map $\phi_{\mathrm{ms}} : R_{\mathrm{ms}} \to G$ is the restriction of the translation map $\phi : R \to G$. As the action of G on X is principal, this implies that the action of G on X_{ms} is also principal. Let $\sigma_{\mathrm{ms}} : X_{\mathrm{ms}} \times G \to X_{\mathrm{ms}} \times X_{\mathrm{ms}}$ be the map $(x, g) \mapsto (x, xg)$, and let $\tau_{\mathrm{ms}} : X_{\mathrm{ms}} \times G \to R_{\mathrm{ms}}$ be the map induced by σ_{ms}. Then, by Remark (6.1), τ_{ms} is a homeomorphism. Since R_{ms} is closed in $X_{\mathrm{ms}} \times X_{\mathrm{ms}}$, it follows that the map σ_{ms} is proper. In other words, the action of G on X_{ms} is proper. It has been proved above that Y_{ms} is open in Y, and p_{ms} is a holomorphic principal G-bundle.

(3) By hypothesis, the Kähler metric g on X is K-invariant, hence its Kähler form Ω is K-invariant. Therefore, its restriction Ω_{m} to the K-invariant smooth submanifold X_{m} of X is also K-invariant. Let $x \in X_{\mathrm{m}}$, $v \in \mathrm{T}_x(X_{\mathrm{m}})$, and $\xi \in \mathrm{Lie}(K)$. Then, since $\mathrm{T}_x(X_{\mathrm{m}}) = \mathrm{Im}(\mathrm{T}_e(\nu_x))^{\perp(\Omega)}$, we have $\Omega_{\mathrm{m}}(x)(v, \mathrm{T}_e(\nu_x)(\xi)) = \Omega(x)(v, \mathrm{T}_e(\nu_x)(\xi)) = 0$. Therefore, $\Omega(x)(v, w) = 0$ if either v or w is a vertical tangent vector at x for the principal K-bundle $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$. It follows that there exists a unique smooth 2-form Θ_{ms} on Y_{ms}, such that ${p_{\mathrm{m}}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$. As Ω is closed, so is Ω_{m}, and hence so is Θ_{ms}.

We claim that Θ_{ms} is positive. Let $y \in Y_{\mathrm{ms}}$, and $w, w' \in \mathrm{T}_y(Y)$. Let $x \in {p_{\mathrm{m}}}^{-1}(y)$. Then, there exists $v, v' \in \mathrm{T}_x(X_{\mathrm{m}})$, such that $w = \mathrm{T}_x(p_{\mathrm{m}})(v)$ and $w' = \mathrm{T}_x(p_{\mathrm{m}})(v')$. Since $\mathrm{T}_x(X) = \mathrm{T}_x(X_{\mathrm{m}}) \oplus (\sqrt{-1}\mathrm{Im}(\mathrm{T}_e(\nu_x)))$, there exist $a, a' \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi, \xi' \in \mathrm{Lie}(K)$, such that $\sqrt{-1}v = a + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)$ and $\sqrt{-1}v' = a' + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi')$. As $\mathrm{T}_x(p)$ is **C**-linear,

$$\begin{aligned}\sqrt{-1}w &= \sqrt{-1}\mathrm{T}_x(p_{\mathrm{m}})(v) = \sqrt{-1}\mathrm{T}_x(p)(v) = \mathrm{T}_x(p)(\sqrt{-1}v)\\ &= \mathrm{T}_x(p)(a + \sqrt{-1}\mathrm{T}_e(\nu_x)(\xi)) = \mathrm{T}_x(p)(a) + \sqrt{-1}\mathrm{T}_x(p)(\mathrm{T}_e(\nu_x)(\xi))\\ &= \mathrm{T}_x(p_{\mathrm{m}})(a) + \sqrt{-1}\mathrm{T}_x(p)(\mathrm{T}_e(\mu_x)(\xi)) = \mathrm{T}_x(p_{\mathrm{m}})(a).\end{aligned}$$

Similarly, $\sqrt{-1}w' = \mathrm{T}_x(p_{\mathrm{m}})(a')$. Therefore, as $\Omega(x)$ vanishes on vertical tangent vectors for p_{m},

$$\begin{aligned}\Theta_{\mathrm{ms}}(y)(\sqrt{-1}w, \sqrt{-1}w') &= \Theta_{\mathrm{ms}}(y)(\mathrm{T}_x(p_{\mathrm{m}})(a), \mathrm{T}_x(p_{\mathrm{m}})(a')) = \Omega_{\mathrm{m}}(x)(a, a') = \Omega(x)(a, a')\\ &= \Omega(x)(\sqrt{-1}(v - \mathrm{T}_e(\nu_x)(\xi)), \sqrt{-1}(v' - \mathrm{T}_e(\nu_x)(\xi')))\\ &= \Omega(x)(v - \mathrm{T}_e(\nu_x)(\xi), v' - \mathrm{T}_e(\nu_x)(\xi'))\\ &= \Omega_{\mathrm{m}}(x)(v - \mathrm{T}_e(\nu_x)(\xi), v' - \mathrm{T}_e(\nu_x)(\xi'))\\ &= \Omega_{\mathrm{m}}(x)(v, v') = \Theta_{\mathrm{ms}}(y)(w, w').\end{aligned}$$

Similarly, it can be checked that

$$\Theta_{\mathrm{ms}}(y)(w, \sqrt{-1}w) = \Omega(x)(v - \mathrm{T}_e(\nu_x)(\xi), \sqrt{-1}(v - \mathrm{T}_e(\nu_x)(\xi))).$$

Now, if $w = \mathrm{T}_x(p_{\mathrm{m}})(v)$ is non-zero, then $v \neq \mathrm{T}_e(\nu_x)(\xi)$, hence, as $\Omega(x)$ is positive, $\Omega(x)(v - \mathrm{T}_e(\nu_x)(\xi), \sqrt{-1}(v - \mathrm{T}_e(\nu_x)(\xi))) > 0$. Therefore, $\Theta_{\mathrm{ms}}(y)(w, \sqrt{-1}w) > 0$. It follows that Θ_{ms} is positive. Thus, the rule

$$h_{\mathrm{ms}}(y)(w, w') = \frac{1}{2}\big(\Theta_{\mathrm{ms}}(y)(w, \sqrt{-1}w') - \sqrt{-1}\Theta_{\mathrm{ms}}(y)(w, w')\big),$$

for all $y \in Y_{\mathrm{ms}}$ and $w, w' \in \mathrm{T}_y(Y)$, defines a Kähler metric on Y_{ms}, whose Kähler form equals Θ_{ms}. If h is another Kähler metric on Y_{ms}, whose Kähler form Θ satisfies the condition $p_{\mathrm{m}}^*(\Theta) = \Omega_{\mathrm{m}}$, then, since Θ_{ms} is the unique smooth 2-form on Y_{ms} such that $p_{\mathrm{m}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$, we get $\Theta = \Theta_{\mathrm{ms}}$. Therefore,

$$h(y)(w, w') = \frac{1}{2}\big(\Theta(y)(w, \sqrt{-1}w') - \sqrt{-1}\Theta(y)(w, w')\big) = h_{\mathrm{ms}}(y)(w, w')$$

for all $y \in Y_{\mathrm{ms}}$ and $w, w' \in \mathrm{T}_y(Y)$. This establishes the uniqueness of h_{ms}. □

In addition to the notations used above, let H be a normal complex Lie subgroup of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. Let $\overline{K}$ be the compact subgroup $\pi(K)$ of $\overline{G}$, and $\pi_K : K \to \overline{K}$ the homomorphism of real Lie groups induced by π. The subset $H \cap K$ of G is a real Lie subgroup of G, and $\mathrm{Lie}(H \cap K)$ equals the real Lie subalgebra $\mathrm{Lie}(H) \cap \mathrm{Lie}(K)$ of $\mathrm{Lie}(G)$. The map $\mathrm{T}_e(\pi) : \mathrm{Lie}(G) \to \mathrm{Lie}(\overline{G})$ is a surjective homomorphism of complex Lie algebras with kernel $\mathrm{Lie}(H)$, and $\mathrm{T}_e(\pi_K) : \mathrm{Lie}(K) \to \mathrm{Lie}(\overline{K})$ is a surjective homomorphism of real Lie algebras with kernel $\mathrm{Lie}(H \cap K)$.

Corollary 7.4 — Suppose that $G_x = H$ for all $x \in X$, that the induced action of $\overline{G}$ on X is principal, and that

$$\Phi(X) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K)), \quad P_G(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset HK.$$

Then:

(1) The set X_{m} is a closed smooth submanifold of X, X_{ms} is open in X, Y_{ms} is open in Y, the action of $\overline{K}$ on X_{m} is principal, and $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$ is a smooth principal $\overline{K}$-bundle.

(2) The action of $\overline{G}$ on X_{ms} is proper, Y_{ms} is a Hausdorff open subspace of Y, the action of $\overline{G}$ on X_{ms} is principal, and $p_{\mathrm{ms}} : X_{\mathrm{ms}} \to Y_{\mathrm{ms}}$ is a holomorphic principal $\overline{G}$-bundle.

(3) There exists a unique Kähler metric h_{ms} on Y_{ms}, such that $p_{\mathrm{m}}^*(\Theta_{\mathrm{ms}}) = \Omega_{\mathrm{m}}$, where Θ_{ms} is the Kähler form of h_{ms}, $\Omega_{\mathrm{m}} = i_{\mathrm{m}}^*(\Omega)$, and $i_{\mathrm{m}} : X_{\mathrm{m}} \to X$ is the inclusion map.

Proof : The action of $\overline{K}$ on X induced by that of $\overline{G}$ on X preserves the Kähler metric g on X. Since $\Phi(X) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K))$ and $\mathrm{Ker}(\mathrm{T}_e(\pi_K)) = \mathrm{Lie}(H \cap K)$, there exists a unique map $\overline{\Phi} : X \to \mathrm{Lie}(\overline{K})^*$, such that $\Phi(x) = \overline{\Phi}(x) \circ \mathrm{T}_e(\pi_K)$ for all $x \in X$. It is easy to see that $\overline{\Phi}$ is a moment map for the action of $\overline{K}$ on X. Moreover, $\overline{\Phi}^{-1}(0) = \Phi^{-1}(0) = X_{\mathrm{m}}$, $\overline{\Phi}^{-1}(0)\overline{G} = X_{\mathrm{m}}G = X_{\mathrm{ms}}$, and $P_{\overline{G}}(X_{\mathrm{m}}, X_{\mathrm{m}}) \subset \pi(P_G(X_{\mathrm{m}}, X_{\mathrm{m}})) \subset \pi(HK) \subset \pi(K) = \overline{K}$. It is obvious that

$X/\overline{G} = X/G = Y$, and the canonical projection from X to $X/\overline{G}$ equals p. Therefore, the Corollary follows from Proposition 7.3. □

7.C. *The Kähler metric on the moduli of stable representations*

We will follow the notation of Section 6.B. Thus, Q is a non-empty finite quiver, $d = (d_a)_{a\in Q_0}$ a non-zero element of $\mathbf{N}^{Q_0}$, and $V = (V_a)_{a\in Q_0}$ a family of $\mathbf{C}$-vector spaces, such that $\dim_{\mathbf{C}}(V_a) = d_a$ for all $a \in Q_0$. Fix a family $h = (h_a)_{a\in Q_0}$ of Hermitian inner products $h_a : V_a \times V_a \to \mathbf{C}$. In addition, we also fix now a rational weight $\theta \in \mathbf{Q}^{Q_0}$ of Q.

Denote by $\mathcal{A}$ the finite-dimensional $\mathbf{C}$-vector space $\bigoplus_{\alpha\in Q_1} \mathrm{Hom}_{\mathbf{C}}(V_{\mathrm{s}(\alpha)}, V_{\mathrm{t}(\alpha)})$. For any subset X of $\mathcal{A}$, let X_{schur} (respectively, X_{s}) denote the set of all points ρ in X, such that the representation (V,ρ) of Q is Schur (respectively, θ-stable). Also, denote by X_{eh} (respectively, X_{irr}) the set of all $\rho \in X$, such that the Hermitian metric h on (V,ρ) is Einstein-Hermitian with respect to θ (respectively, irreducible).

Recall that G is the complex Lie group $\prod_{a\in Q_0} \mathrm{Aut}_{\mathbf{C}}(V_a)$, with its canonical holomorphic linear right action on $\mathcal{A}$. Denote by H the central complex Lie subgroup $\mathbf{C}^{\times}e$ of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. Let K denote the compact subgroup $\prod_{a\in Q_0} \mathrm{Aut}(V_a, h_a)$, where, for each $a \in Q_0$, $\mathrm{Aut}(V_a, h_a)$ is the subgroup of $\mathrm{Aut}_{\mathbf{C}}(V_a)$ consisting of $\mathbf{C}$-automorphisms of V_a which preserve the Hermitian inner product h_a on V_a. Let $\overline{K}$ be the compact subgroup $\pi(K)$ of $\overline{G}$, and $\pi_K : K \to \overline{K}$ the homomorphism of real Lie groups induced by π.

Let $\mathcal{B} = \mathcal{A}_{\mathrm{schur}}$. Then, by Proposition 2.3(3), $\mathcal{B}_{\mathrm{s}} = \mathcal{A}_{\mathrm{s}}$. On the other hand, by Proposition 4.6, $\mathcal{B}_{\mathrm{eh}} = \mathcal{A}_{\mathrm{eh}} \cap \mathcal{A}_{\mathrm{irr}} = \mathcal{A}_{\mathrm{eh}} \cap \mathcal{A}_{\mathrm{s}} = \mathcal{A}_{\mathrm{eh}} \cap \mathcal{B}_{\mathrm{s}}$. As noted in Section 6.B, $\mathcal{B}$ is a G-invariant open complex submanifold of $\mathcal{A}$, and, by Proposition 2.2(4), $G_\rho = H$ for all $\rho \in \mathcal{B}$. Let M denote the moduli space $\mathcal{B}/G$ of Schur representations of Q with dimension vector d, and $p : \mathcal{B} \to M$ the canonical projection. It was proved in Theorem 6.8 that the action of $\overline{G}$ on $\mathcal{B}$ is principal, that there exists a unique structure of a complex premanifold on M such that p is a holomorphic submersion, and that this structure in fact makes p a holomorphic principal $\overline{G}$-bundle. Let $M_{\mathrm{s}} = p(\mathcal{B}_{\mathrm{s}})$, $p_{\mathrm{s}} : \mathcal{B}_{\mathrm{s}} \to M_{\mathrm{s}}$ the map induced by p, and $p_{\mathrm{eh}} = p_{\mathrm{s}}|\mathcal{B}_{\mathrm{eh}} : \mathcal{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$. Recall that for any two subsets A and B of $\mathcal{A}$, $P_G(A, B)$ denotes the set of all $g \in G$, such that $Ag \cap B \neq \emptyset$.

Lemma 7.5 — We have $\mathcal{B}_{\mathrm{eh}}G = \mathcal{B}_{\mathrm{s}}$ and $P_G(\mathcal{B}_{\mathrm{eh}}, \mathcal{B}_{\mathrm{eh}}) \subset HK$.

PROOF : It is obvious that the subset $\mathcal{B}_{\mathrm{s}}$ of $\mathcal{B}$ is G-invariant. By the above paragraph, $\mathcal{B}_{\mathrm{eh}} \subset \mathcal{B}_{\mathrm{s}}$. Therefore, $\mathcal{B}_{\mathrm{eh}}G \subset \mathcal{B}_{\mathrm{s}}$. Conversely, if $\sigma \in \mathcal{B}_{\mathrm{s}}$, then, by Proposition 4.3, (V,σ) has an Einstein-

Hermitian metric k with respect to θ. For each $a \in Q_0$, h_a and k_a are two Hermitian inner products on V_a, hence there exists a $\mathbf{C}$-automorphism g_a of V_a, such that $h_a(g_a(x), g_a(y)) = k_a(x, y)$ for all $x, y \in V_a$. We thus get an element $g = (g_a)_{a \in Q_0}$ of G. Let $\rho = \sigma g^{-1}$. Then, for all $\alpha \in Q_1$, we have $\rho_\alpha^{*(h)} = g_{\mathrm{s}(\alpha)} \circ \sigma_\alpha^{*(k)} \circ g_{\mathrm{t}(\alpha)}^{-1}$, where $\rho_\alpha^{*(h)}$ is the adjoint of ρ_α with respect to $h_{\mathrm{s}(\alpha)}$ and $h_{\mathrm{t}(\alpha)}$, and $\rho_\alpha^{*(k)}$ the adjoint of ρ_α with respect to $k_{\mathrm{s}(\alpha)}$ and $k_{\mathrm{t}(\alpha)}$. Therefore, for every $a \in Q_0$, we get

$$K_\theta(V, \rho, h)_a = g_a \circ K_\theta(V, \sigma, k) \circ g_a^{-1} = \mu_\theta(d)\mathbf{1}_{V_a}.$$

Thus, the Hermitian metric h on (V, ρ) is Einstein-Hermitian, so $\rho \in \mathcal{B}_{\mathrm{eh}}$, and $\sigma = \rho g$ belongs to $\mathcal{B}_{\mathrm{eh}} G$. This proves that $\mathcal{B}_{\mathrm{eh}} G = \mathcal{B}_{\mathrm{s}}$.

Next, let $g \in P_G(\mathcal{B}_{\mathrm{eh}}, \mathcal{B}_{\mathrm{eh}})$. Then, there exist $\rho, \sigma \in \mathcal{B}_{\mathrm{eh}}$, such that $\sigma = \rho g$. Then, g is an isomorphism from (V, σ) to (V, ρ). For every $a \in Q_0$, define a Hermitian inner product k_a on V_a by $k_a(x, y) = h_a(g_a(x), g_a(y))$ for all $x, y \in V_a$. Then, as observed above, since $\sigma \in \mathcal{B}_{\mathrm{eh}}$, we have

$$K_\theta(V, \rho, k)_a = g_a^{-1} \circ K_\theta(V, \sigma, h) \circ g_a = \mu_\theta(d)\mathbf{1}_{V_a}$$

for all $a \in Q_0$, hence the Hermitian metric $k = (k_a)_{a \in Q_0}$ on (V, ρ) is Einstein-Hermitian with respect to θ. As $\rho \in \mathcal{B}_{\mathrm{eh}}$, the Hermitian metric h on (V, ρ) is also Einstein-Hermitian with respect to θ. Therefore, by Proposition 4.3, there exists an automorphism f of (V, ρ), such that $k_a(x, y) = h_a(f_a(x), f_a(y))$ for all $a \in Q_0$, and $x, y \in V_a$. Now, since $\rho \in \mathcal{B}$, $f = ce$ for some $c \in \mathbf{C}$. As the dimension vector d is non-zero, we have $c \neq 0$, and

$$h_a\Big(\frac{1}{c} g_a(x), \frac{1}{c} g_a(y)\Big) = k_a\Big(\frac{1}{c} x, \frac{1}{c} y\Big) = h_a\Big(\frac{1}{c} f_a(x), \frac{1}{c} f_a(y)\Big) = h_a(x, y)$$

for all $a \in Q_0$, and $x, y \in V_a$. Therefore, $\frac{1}{c} g_a \in \mathrm{Aut}(V_a, h_a)$ for all $a \in Q_0$, so $\frac{1}{c} g \in K$. Thus, $g = (ce)\big(\frac{1}{c} g\big)$ belongs to HK. It follows that $P_G(\mathcal{B}_{\mathrm{eh}}, \mathcal{B}_{\mathrm{eh}}) \subset HK$. □

The family h induces a Hermitian inner product $\langle \cdot, \cdot \rangle$ on the $\mathbf{C}$-vector space $\mathcal{A}$, and the complex Lie algebra $\mathrm{Lie}(G)$. For every point ρ of $\mathcal{A}$, the $\mathbf{C}$-vector space $\mathrm{T}_\rho(\mathcal{A})$ is canonically isomorphic to $\mathcal{A}$. Therefore, the Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{A}$ defines a Hermitian metric g on the complex manifold $\mathcal{A}$, namely $g(\rho)(\sigma, \tau) = \langle \sigma, \tau \rangle$ for all $\rho, \sigma, \tau \in \mathcal{A}$. The fundamental 2-form Ω of g is given by $\Omega(\rho)(\sigma, \tau) = -2\Im(\langle \sigma, \tau \rangle)$ for all $\rho, \sigma, \tau \in \mathcal{A}$. Since $\Omega(\rho)$ is independent of ρ, we have $\mathrm{d}\,\Omega = 0$, hence the Hermitian metric g on $\mathcal{A}$ is Kähler. The action of K on $\mathcal{A}$ induced by that of G preserves the Hermitian inner product $\langle \cdot, \cdot \rangle$, and hence the Kähler metric g on $\mathcal{A}$. Similarly, the action of K on $\mathrm{Lie}(G)$ induced by that of G preserves the Hermitian inner product on $\mathrm{Lie}(G)$.

For each $a \in Q_0$, let $\mathrm{End}(V_a, h_a)$ denote the real Lie subalgebra of $\mathrm{End}_{\mathbf{C}}(V_a)$ consisting of $\mathbf{C}$-endomorphisms u of V_a that are skew-Hermitian with respect to h_a, that is,

$$h_a(u(x), y) + h_a(x, u(y)) = 0$$

for all $x, y \in V_a$. Then, $\mathrm{Lie}(K)$ equals the real Lie subalgebra $\bigoplus_{a \in Q_0} \mathrm{End}(V_a, h_a)$ of $\mathrm{Lie}(G)$. The Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{Lie}(G)$ restricts to a real inner product on $\mathrm{Lie}(K)$, which is given by $\langle \xi, \eta \rangle = -\sum_{a \in Q_0} \mathrm{Tr}(\xi_a \circ \eta_a)$ for all $\xi, \eta \in \mathrm{Lie}(K)$.

For any point ρ in $\mathcal{A}$, let $\nu_\rho : K \to \mathcal{A}$ be the orbit map of ρ, and denote by D_ρ the $\mathbf{R}$-linear map $\mathrm{T}_e(\nu_\rho) : \mathrm{Lie}(K) \to \mathcal{A}$. Then, as in Section 6.B., we have

$$D_\rho(\xi) = (\rho_\alpha \circ \xi_{\mathrm{s}(\alpha)} - \xi_{\mathrm{t}(\alpha)} \circ \rho_\alpha)_{\alpha \in Q_1}.$$

For every element ξ of $\mathrm{Lie}(K)$, the vector field $\xi^\sharp$ on $\mathcal{A}$ induced by ξ is the $\mathbf{C}$-endomorphism of $\mathcal{A}$ given by $\xi^\sharp(\rho) = \mathrm{T}_e(\nu_\rho)(\xi) = D_\rho(\xi)$ for all $\rho \in \mathcal{A}$.

Recall the notation

$$\deg_\theta(d) = \sum_{a \in Q_0} \theta_a d_a, \quad \mathrm{rk}(d) = \sum_{a \in Q_0} d_a, \quad \mu_\theta(d) = \frac{\deg_\theta(d)}{\mathrm{rk}(d)},$$

where $\theta \in \mathbf{R}^{Q_0}$ is the rational weight of Q that we have fixed. If $a, b \in Q_0$, and $f \in \mathrm{Hom}_{\mathbf{C}}(V_a, V_b)$, let $f^* \in \mathrm{Hom}_{\mathbf{C}}(V_b, V_a)$ be the adjoint of f with respect to the Hermitian inner products h_a and h_b on V_a and V_b, respectively. For every point ρ of $\mathcal{A}$, and $a \in Q_0$, define an element $L_\theta(\rho)_a$ of $\mathrm{End}(V_a, h_a)$ by

$$L_\theta(\rho)_a = \sqrt{-1}\Big((\theta_a - \mu_\theta(d))\mathbf{1}_{V_a} + \sum_{\alpha \in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha\Big),$$

and let $L_\theta(\rho)$ be the element $(L_\theta(\rho)_a)_{a \in Q_0}$ of $\mathrm{Lie}(K)$. Define a map $\Phi_\theta : \mathcal{A} \to \mathrm{Lie}(K)^*$ by

$$\Phi_\theta(\rho)(\xi) = \langle \xi, L_\theta(\rho) \rangle$$

for all $\rho \in \mathcal{A}$ and $\xi \in \mathrm{Lie}(K)$, where $\langle \cdot, \cdot \rangle$ is the real inner product on $\mathrm{Lie}(K)$.

Lemma 7.6 — Let η denote the element $\big(\sqrt{-1}(\theta_a - \mu_\theta(d))\mathbf{1}_{V_a}\big)_{a \in Q_0}$ of $\mathrm{Lie}(K)$, and α the element of $\mathrm{Lie}(K)^*$, which is defined by $\alpha(\xi) = \langle \xi, \eta \rangle$ for all $\xi \in \mathrm{Lie}(K)$. Then,

$$\Phi_\theta(\rho)(\xi) = \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) + \alpha(\xi)$$

for all $\rho \in \mathcal{A}$ and $\xi \in \mathrm{Lie}(K)$. In particular, Φ_θ is a moment map for the action of K on $\mathcal{A}$.

PROOF : Let $\rho \in \mathcal{A}$ and $\xi \in \mathrm{Lie}(K)$. For every $a \in Q_0$, define an element $A(\rho)_a$ of $\mathrm{End}(V_a, h_a)$, by

$$A(\rho)_a = \sqrt{-1}\Big(\sum_{\alpha \in \mathrm{t}^{-1}(a)} \rho_\alpha \circ \rho_\alpha^* - \sum_{\alpha \in \mathrm{s}^{-1}(a)} \rho_\alpha^* \circ \rho_\alpha\Big),$$

and let $A(\rho)$ denote the element $(A(\rho)_a)_{a\in Q_0}$ of $\mathrm{Lie}(K)$. Then,

$$L_\theta(\rho) = A(\rho) + \eta, \quad \Phi_\theta(\rho)(\xi) = \langle \xi, A(\rho)\rangle + \alpha(\xi).$$

We claim that

$$\langle \xi, A(\rho)\rangle = \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho).$$

By the definition of Ω,

$$\frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) = -\Im(\langle \xi^\sharp(\rho), \rho\rangle) = \frac{\sqrt{-1}}{2}(\langle \xi^\sharp(\rho), \rho\rangle - \langle \rho, \xi^\sharp(\rho)\rangle).$$

Since K preserves the Hermitian inner product on $\mathcal{A}$, the $\mathbf{C}$-endomorphism $\xi^\sharp$ of $\mathcal{A}$ is skew-Hermitian, that is,

$$\langle \xi^\sharp(\rho), \rho\rangle + \langle \rho, \xi^\sharp(\rho)\rangle = 0.$$

Therefore,

$$\frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) = \sqrt{-1}\langle \xi^\sharp(\rho), \rho\rangle = \sqrt{-1}\langle D_\rho(\xi), \rho\rangle.$$

It is easy to see that

$$\langle D_\rho(\xi), \rho\rangle = \sqrt{-1}\sum_{a\in Q_0} \mathrm{Tr}(\xi_a \circ A(\rho)_a).$$

Thus,

$$\frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) = -\sum_{a\in Q_0} \mathrm{Tr}(\xi_a \circ A(\rho)_a) = \langle \xi, A(\rho)\rangle,$$

which proves the above claim, and gives the relation

$$\Phi_\theta(\rho)(\xi) = \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) + \alpha(\xi)$$

for all $\rho \in \mathcal{A}$ and $\xi \in \mathrm{Lie}(K)$. Now, the inner product on $\mathrm{Lie}(K)$ is K-invariant, and for all $g \in K$ and $\xi \in \mathrm{Lie}(K)$, we have $\mathrm{Ad}(g)^{-1}\eta = (g_a^{-1} \circ \eta_a \circ g_a)_{a\in Q_0} = \eta$, hence

$$\alpha(\mathrm{Ad}(g)\xi) = \langle \mathrm{Ad}(g)\xi, \eta\rangle = \langle \xi, \mathrm{Ad}(g)^{-1}\eta\rangle = \langle \xi, \eta\rangle = \alpha(\xi).$$

Therefore, the element α of $\mathrm{Lie}(K)^*$ is K-invariant. It follows from Lemma 7.1 that Φ_θ is a moment map for the action of K on $\mathcal{A}$. □

Lemma 7.7 — We have $\Phi_\theta(\mathcal{A}) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K))$, and $\Phi_\theta^{-1}(0) = \mathcal{A}_{\mathrm{eh}}$.

PROOF : Let $\rho \in \mathcal{A}$, and $\xi \in \mathrm{Lie}(H \cap K) = \mathrm{Lie}(H) \cap \mathrm{Lie}(K)$. Then, there exists a real number c, such that $\xi = \sqrt{-1}ce$, where $e = (\mathbf{1}_{V_a})_{a\in Q_0}$ is the identity element of $G \subset \mathrm{Lie}(G)$. Therefore,

$$\Phi_\theta(\rho)(\xi) = \langle \xi, L_\theta(\rho)\rangle = -\sum_{a\in Q_0} \mathrm{Tr}(\xi_a \circ L_\theta(\rho)_a) = -\sqrt{-1}c \sum_{a\in Q_0} \mathrm{Tr}(L_\theta(\rho)_a).$$

But, with $A(\rho)$ as in the proof of Lemma 7.6, we have

$$\sum_{a\in Q_0} \mathrm{Tr}(L_\theta(\rho)_a) = \sum_{a\in Q_0} \big(\mathrm{Tr}(A(\rho)_a) + \sqrt{-1}(\theta_a - \mu_\theta(d))d_a\big) = \sum_{a\in Q_0} \mathrm{Tr}(A(\rho)_a) = 0.$$

Therefore, $\Phi_\theta(\rho)(\xi) = 0$, hence $\Phi_\theta(\mathcal{A}) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K))$.

Lastly, in the notation of Section 4.C., we have $L_\theta(\rho) = \sqrt{-1}(K_\theta(V,\rho) - \mu_\theta(d)e)$ for all $\rho \in \mathcal{A}$. As $L_\theta(\rho) \in \mathrm{Lie}(K)$, and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathrm{Lie}(K)$, we have

$$\Phi_\theta(\rho) = 0 \Leftrightarrow L_\theta(\rho) = 0 \Leftrightarrow K_\theta(V,\rho) = \mu_\theta(d)e.$$

Therefore, $\Phi_\theta^{-1}(0) = \mathcal{A}_{\mathrm{eh}}$. □

***Theorem* 7.8** — *With notation as above, the following statements are true:*

(1) The set $\mathcal{B}_{\mathrm{eh}}$ is a closed smooth submanifold of $\mathcal{B}$, $\mathcal{B}_{\mathrm{s}}$ is open in $\mathcal{B}$, M_{s} is open in M, the action of $\overline{K}$ on $\mathcal{B}_{\mathrm{eh}}$ is principal, and $p_{\mathrm{eh}} : \mathcal{B}_{\mathrm{eh}} \to M_{\mathrm{s}}$ is a smooth principal $\overline{K}$-bundle.

(2) The action of $\overline{G}$ on $\mathcal{B}_{\mathrm{s}}$ is proper, M_{s} is a Hausdorff open subspace of M, the action of $\overline{G}$ on $\mathcal{B}_{\mathrm{s}}$ is principal, and $p_{\mathrm{s}} : \mathcal{B}_{\mathrm{s}} \to M_{\mathrm{s}}$ is a holomorphic principal $\overline{G}$-bundle.

(3) There exists a unique Kähler metric h_{s} on M_{s}, such that $p_{\mathrm{eh}}^*(\Theta_{\mathrm{s}}) = \Omega_{\mathrm{eh}}$, where Θ_{s} is the Kähler form of h_{s}, $\Omega_{\mathrm{eh}} = i_{\mathrm{eh}}^*(\Omega_s)$, $i_{\mathrm{eh}} : \mathcal{B}_{\mathrm{eh}} \to \mathcal{B}$ is the inclusion map, and Ω is the Kähler form on $\mathcal{B}$.

PROOF : The stabiliser G_ρ of any point ρ of $\mathcal{B}$ equals H, and, by Theorem 6.8, the induced action of $\overline{G}$ on $\mathcal{B}$ is principal. Let $\Psi_\theta : \mathcal{B} \to \mathrm{Lie}(K)^*$ be the restriction of Φ_θ. By Lemma 7.7, Φ_θ is a moment map for the action of K on $\mathcal{A}$, hence Ψ_θ is a moment map for the action of K on $\mathcal{B}$. Moreover, $\Psi_\theta(\mathcal{B}) \subset \Phi_\theta(\mathcal{A}) \subset \mathrm{Ann}(\mathrm{Lie}(H\cap K))$, and $\mathcal{B}_{\mathrm{m}} := \Psi_\theta^{-1}(0) = \Phi_\theta^{-1}(0) \cap \mathcal{B} = \mathcal{A}_{\mathrm{eh}} \cap \mathcal{B} = \mathcal{B}_{\mathrm{eh}}$. Finally, by Lemma 7.5, we have $\mathcal{B}_{\mathrm{ms}} := \mathcal{B}_{\mathrm{m}}G = \mathcal{B}_{\mathrm{eh}}G = \mathcal{B}_{\mathrm{s}}$, and $P_G(\mathcal{B}_{\mathrm{m}}, \mathcal{B}_{\mathrm{m}}) = P_G(\mathcal{B}_{\mathrm{eh}}, \mathcal{B}_{\mathrm{eh}}) \subset HK$. Therefore, the Theorem follows from Corollary 7.4. □

8. The Line Bundle on the Moduli of Stable Representations

8.A. *Line bundles on quotients of vector spaces*

Let V be a finite-dimensional $\mathbf{C}$-vector space, $\langle \cdot, \cdot \rangle$ a Hermitian inner product on V, and $\Omega = -2\Im(\langle \cdot, \cdot \rangle)$ its fundamental 2-form. We will consider V to be a Kähler manifold in the usual way. Let G be a complex Lie group, and K a real Lie subgroup of G. Suppose that we are given a holomorphic linear right action of G, and that the induced action of K on V preserves the Hermitian inner product $\langle \cdot, \cdot \rangle$ on V.

Let $\chi : G \to \mathbf{C}^\times$ be a character of G, and suppose that $\chi(K) \subset \mathrm{U}(1)$. Then, $\mathrm{T}_e(\chi)(\mathrm{Lie}(K))$ is contained in the $\mathbf{R}$-subspace $\mathrm{Lie}(\mathrm{U}(1)) = \sqrt{-1}\mathbf{R}$ of $\mathrm{Lie}(\mathbf{C}^\times) = \mathbf{C}$. Fix a non-zero real number λ. Let α be the element of $\mathrm{Lie}(K)^*$ defined by $\alpha(\xi) = -\frac{\sqrt{-1}}{\lambda}\mathrm{T}_e(\chi)(\xi)$ for all $\xi \in \mathrm{Lie}(K)$. Since $\chi(gag^{-1}) = \chi(a)$ for all $a, g \in G$, α is K-invariant. Therefore, by Lemma 7.1, the map $\Phi_\alpha : V \to \mathrm{Lie}(K)^*$, which is defined by

$$\Phi_\alpha(x)(\xi) = \frac{1}{2}\Omega(\xi^\sharp(x), x) + \alpha(\xi)$$

for all $x \in V$ and $\xi \in \mathrm{Lie}(K)$, is a moment map for the action of K on V.

Let E denote the trivial holomorphic line bundle $V \times \mathbf{C}$ on V. Define a right action of G on E by setting $(x, a)g = (xg, \chi(g)^{-1}a)$ for all $(x, a) \in E$ and $g \in G$. Let $\Gamma(E)$ denote the $\mathbf{C}$-vector space of smooth sections of E on V. For each $\xi \in \mathrm{Lie}(G)$ and $s \in \Gamma(E)$, define another section $\xi s \in \Gamma(E)$ by

$$(\xi s)(x) = \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=0}\big(s(x\exp(t\xi))\exp(-t\xi)\big)$$

for all $x \in V$. For every $x \in V$, define a Hermitian inner product $h(x) : E(x) \times E(x) \to \mathbf{C}$ by

$$h(x)((x, a), (x, b)) = \exp(\lambda\|x\|^2)a\bar{b}$$

for all $a, b \in \mathbf{C}$. This gives a smooth Hermitian metric h on E.

Lemma 8.1 — Let ∇ be the canonical connection of the Hermitian holomorphic line bundle (E, h) on V. Then:

(1) For all $\xi \in \mathrm{Lie}(K)$ and $s \in \Gamma(E)$, we have $\nabla_{\xi^\sharp}(s) = \xi s - \lambda\sqrt{-1}\Phi_\alpha^\xi s$.

(2) The first Chern form $c_1(E, h)$ of ∇ equals $-\frac{\lambda}{2\pi}\Omega$.

Proof : Let $\xi \in \mathrm{Lie}(K)$. Define a map $s_0 : V \to E$ by $s_0(x) = (x, 1)$ for all $x \in V$. It is a holomorphic frame of E on V. We have

$$\nabla_v(s_0) = \lambda\langle v, x\rangle s_0(x)$$

for all $x \in V$ and $v \in \mathrm{T}_x(V) = V$. Therefore,

$$\nabla_{\xi^\sharp}(s_0)(x) = \lambda\langle \xi^\sharp(x), x\rangle s_0(x) = -\frac{\lambda\sqrt{-1}}{2}\Omega(\xi^\sharp(x), x)s_0(x).$$

On the other hand,

$$(\xi s_0)(x) = \mathrm{T}_e(\chi)(\xi)s_0(x) = \lambda\sqrt{-1}\alpha(\xi)s_0(x).$$

Thus,

$$(\xi s_0)(x) - \nabla_{\xi^\sharp}(s_0)(x) = \lambda\sqrt{-1}\big(\alpha(\xi) + \frac{1}{2}\Omega(\xi^\sharp(x), x)\big)s_0(x) = \lambda\sqrt{-1}\Phi_\alpha(x)(\xi)s_0(x).$$

It follows that

$$\xi s_0 - \nabla_{\xi^\sharp}(s_0) = \lambda\sqrt{-1}\Phi_\alpha^\xi s_0.$$

Now, let s be an arbitrary element of $\Gamma(E)$. Then, there exists a smooth complex function f on V, such that $s = fs_0$. It is easy to see that

$$\xi(fs_0) = \xi^\sharp(f)s_0 + f(\xi s_0).$$

Therefore,

$$\begin{aligned}\xi s - \nabla_{\xi^\sharp}(s) &= \big(\xi^\sharp(f)s_0 + f(\xi s_0)\big) - \big(\xi^\sharp(f)s_0 + f\nabla_{\xi^\sharp}(s_0)\big)\\ &= f(\xi s_0 - \nabla_{\xi^\sharp}(s_0)) = f\lambda\sqrt{-1}\Phi_\alpha^\xi s_0 = \lambda\sqrt{-1}\Phi_\alpha^\xi s.\end{aligned}$$

This proves (1).

Let ω be the connection form of ∇ with respect to the holomorphic frame s_0 of E on V, and R the curvature form of ∇. Then, $\omega = \lambda\partial N$, and $R = \overline{\partial}\omega$, where $N : V \to \mathbf{R}$ is the smooth function $x \mapsto \|x\|^2$. Therefore,

$$c_1(E, h) = \frac{\sqrt{-1}}{2\pi}R = -\frac{\lambda\sqrt{-1}}{2\pi}\partial\overline{\partial}N.$$

It is easy to see that $\sqrt{-1}\partial\overline{\partial}N = \Omega$. Thus, $c_1(E, h) = -\frac{\lambda}{2\pi}\Omega$, as stated in (2). □

Let H be a normal complex Lie subgroup of G, $\overline{G}$ the complex Lie group $H\backslash G$, and $\pi : G \to \overline{G}$ the canonical projection. Let $\overline{K}$ be the compact subgroup $\pi(K)$ of $\overline{G}$, and $\pi_K : K \to \overline{K}$ the homomorphism of real Lie groups induced by π.

Let X be a G-invariant open subset of V, X_{m} the closed subset $\Phi_\alpha^{-1}(0) \cap X$ of X, and $X_{\mathrm{ms}} = X_{\mathrm{m}}G$. Denote by Y the quotient topological space X/G, and let $p : X \to Y$ be the canonical projection. Let $Y_{\rm ms} = p(X_{\rm ms})$, $p_{\rm ms} : X_{\rm ms} \to Y_{\rm ms}$ the map induced by p, and $p_{\rm m} = p_{\rm ms}|_{X_{\rm m}} : X_{\rm m} \to Y_{\rm ms}$.

The subset $E_X = X \times \mathbf{C}$ is a G-invariant open subset of E. Let F denote the quotient topological space E_X/G, and $q : E_X \to F$ the canonical projection. There is a canonical continuous surjection from F to Y, and every fibre of this map has a canonical structure of a 1-dimensional $\mathbf{C}$-vector space. Thus, F is a family of 1-dimensional $\mathbf{C}$-vector spaces on Y. Let $F_{\rm m}$ (respectively, $F_{\rm ms}$) denote the restriction of this family to the subspace $Y_{\rm m}$ (respectively, $Y_{\rm ms}$) of Y. For every $x \in X$, the map $q : E_X \to F$ restricts to a $\mathbf{C}$-isomorphism $q(x) : E(x) \to F(p(x))$.

Note that if H is contained in the kernel of the character $\chi : G \to \mathbf{C}^\times$, then we have an induced action of $\overline{G}$ on E, and hence on E_X. If, moreover, the action of $\overline{G}$ on X is principal, then so is its action on E_X. Thus, in that case, there is a unique structure of a complex premanifold on F, such that q is a holomorphic submersion. With this structure, the family F of 1-dimensional $\mathbf{C}$-vector spaces is a holomorphic line bundle on Y(It is the holomorphic line bundle associated with the holomorphic principal $\overline{G}$-bundle $p : X \to Y$, and the character of $\overline{G}$ induced by $\chi : G \to \mathbf{C}^\times$.). For every holomorphic (respectively, smooth) section t of F on any open subset V of Y, there exists a unique holomorphic (respectively, smooth) section s of E_X on $p^{-1}(V)$, which is $\overline{G}$-invariant (that is, $s(xa) = s(x)a$ for all $x \in p^{-1}(V)$ and $a \in \overline{G}$), such that $q(s(x)) = t(p(x))$ for all $x \in p^{-1}(V)$.

Proposition 8.2 — Consider the context of Corollary 7.4. Suppose that $G_x = H$ for all $x \in X$, the induced action of $\overline{G}$ on X is principal, $H \subset \mathrm{Ker}(\chi)$, and

$$\Phi_\alpha(X) \subset \mathrm{Ann}(\mathrm{Lie}(H \cap K)), \quad P_G(X_{\rm m}, X_{\rm m}) \subset HK.$$

Then, there exists a unique smooth Hermitian metric $k_{\rm ms}$ on the holomorphic line bundle $F_{\rm ms}$ on $Y_{\rm ms}$, such that $c_1(F_{\rm ms}, k_{\rm ms}) = -\frac{\lambda}{2\pi}\Theta_{\rm ms}$, where $\Theta_{\rm ms}$ is the Kähler form on the open complex submanifold $Y_{\rm ms}$ of Y.

PROOF : For every point $y \in Y_{\rm ms}$, define $k_{\rm ms}(y) : F(y) \times F(y) \to \mathbf{C}$ by $k_{\rm ms}(y)(a, b) = h(x)(a', b')$, where x is any point of $p_{\rm m}^{-1}(y)$ and $a', b' \in E(x)$ are such that $q(a') = a$ and $q(b') = b$. Then, since $p_{\rm m} : X_{\rm m} \to Y_{\rm ms}$ is a smooth principal $\overline{K}$-bundle, and the metric h is K-invariant, the above rule gives a well-defined smooth Hermitian metric $k_{\rm ms}$ on $F_{\rm ms}$.

Suppose t is a smooth section of $F_{\rm ms}$ on an open subset V of $Y_{\rm ms}$, $y \in Y_{\rm ms}$, and $w \in \mathrm{T}_y(Y)$. We will define an element $\nabla'_w(t)$ of $F(y)$ as follows. Let $x \in p_{\rm m}^{-1}(y)$, and choose $v \in \mathrm{T}_x(X_{\rm m})$, such that $\mathrm{T}_x(p_{\rm m})(v) = w$. Let s be the unique K-invariant section of E on $p_{\rm m}^{-1}(V)$ which projects to t. Define $\nabla'_w(t) = q(\nabla_v(s))$. If $x' \in p_{\rm m}^{-1}(y)$ and $v' \in \mathrm{T}_{x'}(X_{\rm m})$ are two other choices, such that

$\mathrm{T}_{x'}(p_{\mathrm{m}})(v') = w$, then there exists a unique $g \in K$, such that $x' = xg$. Now, $v' - \mathrm{T}_x(\rho_g)(v)$ belongs to $\mathrm{Ker}(\mathrm{T}_{x'}(p_{\mathrm{m}}))$, and is hence of the form $\xi^\sharp(x')$ for some $\xi \in \mathrm{Lie}(K)$. Thus, by Lemma 8.1,

$$\nabla_{v'}(s) = \nabla_{\xi^\sharp(x')}(s) + \nabla_{\mathrm{T}_x(\rho_g)(v)}(s) = \big((\xi s)(x') - \lambda\sqrt{-1}\Phi^\xi_\alpha(x')s(x')\big) + \big(\nabla_v(s)\big)g,$$

since the action of K preserves the metric h on E, and hence its canonical connection ∇ also. Now, since s is K-invariant, we have $\xi s = 0$, and since $x' \in X_{\mathrm{m}}$, we have $\Phi^\xi_\alpha(x') = 0$. Therefore, $\nabla_{v'}(s) = \big(\nabla_v(s)\big)g$, hence $q(\nabla_{v'}(s)) = q(\big(\nabla_v(s)\big))$. It follows that $\nabla'_w(t)$ is well-defined. Since q_{m} is a smooth principal $\overline{K}$-bundle, this rule defines a smooth connection ∇' on F_{ms}.

We claim that ∇' is the canonical connection of the Hermitian holomorphic line bundle $(F_{\mathrm{ms}}, k_{\mathrm{ms}})$ on Y_{ms}. As ∇ is compatible with the metric h on E, and K preserves h, ∇' is compatible with the metric k_{ms} on F_{ms}. Therefore, we only need to check that ∇' is compatible with the holomorphic structure on F_{ms}. Let t be a holomorphic section of F_{ms} on an open subset V of Y_{ms}, $y \in V$, and $w \in \mathrm{T}_y(Y)$. We have to check that $\nabla'_{\sqrt{-1}w}(t) = \sqrt{-1}\nabla'_w(t)$. Let s be the G-invariant holomorphic section of E on $p_{\mathrm{ms}}^{-1}(V)$ corresponding to t. Let $x \in p_{\mathrm{m}}^{-1}(y)$, and choose $v \in \mathrm{T}_x(X_{\mathrm{m}})$, such that $\mathrm{T}_x(p_{\mathrm{m}})(v) = w$. Then, by Lemma 7.2, $\sqrt{-1}v = v' + \sqrt{-1}\xi^\sharp(x)$, where $v' \in \mathrm{T}_x(X_{\mathrm{m}})$ and $\xi \in \mathrm{Lie}(K)$. By definition, $\nabla'_w(t) = \nabla_v(s)$. Similarly, since $\mathrm{T}_x(p_{\mathrm{m}})(v') = \mathrm{T}_x(p)(\sqrt{-1}(v - \xi^\sharp(x))) = \sqrt{-1}w$, we have $\nabla'_{\sqrt{-1}w}(t) = \nabla_{v'}(s)$. Now, since ∇ is compatible with the holomorphic structure on E, we get

$$\nabla_{v'}(s) = \nabla_{\sqrt{-1}(v-\xi^\sharp(x))}(s) = \sqrt{-1}(\nabla_v(s) - \nabla_{\xi^\sharp(x)}(s))$$

But, as we saw above, $\nabla_{\xi^\sharp(x)}(s) = 0$. It follows that $\nabla'_{\sqrt{-1}w}(t) = \sqrt{-1}\nabla'_w(t)$. This proves the above claim.

Thus, the canonical connection ∇' on $(F_{\mathrm{ms}}, k_{\mathrm{ms}})$ is the descent of ∇ through $p_{\mathrm{m}} : X_{\mathrm{m}} \to Y_{\mathrm{ms}}$. Therefore,

$$p_{\mathrm{m}}^* c_1(F_{\mathrm{ms}}, k_{\mathrm{ms}}) = i_{\mathrm{m}}^* c_1(E, h),$$

where $i_{\mathrm{m}} : X_{\mathrm{m}} \to X$ is the inclusion. But, by Lemma 8.1, $c_1(E, h) = -\frac{\lambda}{2\pi}\Omega$, hence

$$p_{\mathrm{m}}^* c_1(F_{\mathrm{ms}}, k_{\mathrm{ms}}) = -\frac{\lambda}{2\pi} i_{\mathrm{m}}^* c_1(E, h) = -\frac{\lambda}{2\pi} p_{\mathrm{m}}^* \Theta_{\mathrm{ms}}.$$

As p_{m} is a smooth submersion, it follows that $c_1(F_{\mathrm{ms}}, k_{\mathrm{ms}}) = -\frac{\lambda}{2\pi}\Theta_{\mathrm{ms}}$. □

8.B. *The line bundle on the moduli space*

We will follow the notation of Section 7.C. Recall that θ is a rational weight of Q. Let n be an integer

> 0, such that $n(\theta_a - \mu_\theta(d)) \in \mathbf{Z}$ for all $a \in Q_0$. Let $\lambda = -n$. Let $\chi : G \to \mathbf{C}^\times$ be the character

$$\chi(g) = \prod_{a \in Q_0} \det(g_a)^{n(\mu_\theta(d) - \theta_a)}.$$

Then, $\chi(K) \subset \mathrm{U}(1)$, and $H \subset \mathrm{Ker}(\chi)$, since $\sum_{a \in Q_0}(\mu_\theta(d) - \theta_a)d_a = 0$. Let $\alpha = -\frac{\sqrt{-1}}{\lambda}\mathrm{T}_e(\chi)$. Then,

$$\alpha(\xi) = \frac{\sqrt{-1}}{n}\mathrm{T}_e(\chi)(\xi) = \frac{\sqrt{-1}}{n}\sum_{a \in Q_0} n(\mu_\theta(d) - \theta_a)\mathrm{Tr}(\xi_a) = \langle \xi, \eta \rangle,$$

where $\eta = \left(\sqrt{-1}(\theta_a - \mu_\theta(d))\mathbf{1}_{V_a}\right)_{a \in Q_0}$. Thus,

$$\Phi_\alpha(\rho)(\xi) = \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) + \alpha(\xi) = \frac{1}{2}\Omega(\xi^\sharp(\rho), \rho) + \langle \xi, \eta \rangle = \Phi_\theta(\rho)(\xi)$$

for all $\rho \in \mathcal{A}$ and $\xi \in \mathrm{Lie}(K)$.

Let E be the trivial line bundle on $\mathcal{A}$ with the action of G defined by χ as above. Let F_s be its quotient by G on M_s. As above, F_s is a holomorphic line bundle on M_s. Now, the following result is an immediate consequence of Proposition 8.2.

***Theorem* 8.3** — *Let n be any positive integer, such that $n(\theta_a - \mu_\theta(d)) \in \mathbf{Z}$ for all $a \in Q_0$. There exists a unique smooth Hermitian metric k_s on the holomorphic line bundle F_s on M_s, such that $c_1(F_\mathrm{s}, k_\mathrm{s}) = \frac{n}{2\pi}\Theta_\mathrm{s}$, where Θ_s is the Kähler form on M_s.*

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