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SURVEY OF LAST TEN YEARS OF WORK DONE IN INDIA IN SOME SELECTED AREAS OF FUNCTIONAL ANALYSIS AND OPERATOR THEORY

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Research in functional analysis and operator theory over the last ten years in India has broadly been under the following areas:

1. Hilbert space operator theory and certain special subsets of $\mathbb{C}^2$ and $\mathbb{C}^3$ where strong interaction between complex analysis and operator theory has been developed resulting in important discoveries in complex analysis through Hilbert space methods.

2. Hermitian holomorphic vector bundles and their curvatures. This has intimate connection with Hilbert modules over function algebras. A Hilbert module over a function algebra is a Hilbert space along with a bounded operator T or a commuting tuple of bounded operators $(T_1, T_2, \ldots, T_n)$. That is how operator theory enters this study.

3. Multivariable operator theory involving dilation of commuting contractions of certain special types. This involves the study of the submodule as well as the quotient module of the kind of Hilbert module mentioned above. This also includes important work on weighted shifts.

4. Operator theory motivated by mathematical physics. This includes trace formulae of Krein and Koplienko and their multivariable generalizations. Perturbation of self-adjoint operators is the main theme of study here.

5. From the seminal work from the 60's by S. Kakutani, J. Lindenstrauss, G. Choquet and T. Ando, the study of geometric aspects of Banach spaces (the so called isometric theory) emerged as an important area. Indian researchers contributed both to the abstract study as well as towards understanding the structure of specific function spaces and spaces of operators.

As the descriptions above show, operator theory has a great deal of relationship to several other branches of mathematics - geometry and complex analysis being perhaps the closest. This interaction is one source of pleasure that we shall try to bring out in this article.

There has been a significant amount of excellent work done on operator algebras including Hilbert C^*-modules and in matrix theory in India in recent times. Since there would be separate articles on those, this article does not include them.

Key words : Spectral sets; Hermitian holomorphic vector bundles; submodules; trace formulae; proximinality.

1. Spectral Sets

Ever since von Neumann proved his famous inequality

$$\| f(T) \| \leq \sup\{|f(z)| : z \in \mathbb{C}, |z| \leq 1\} \tag{1}$$

for any contraction T on a Hilbert space and any function f holomorphic in a neighbourhood of $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, generations of analysts have been intrigued. The inequality (1) has spawned research in several directions, the best known among them being the study of spectral and complete spectral sets. Taking cue from (1), a compact subset K of $\mathbb{C}^n$ is called a spectral set for a commuting tuple of bounded operators $(T_1, T_2, \ldots, T_n)$ on a Hilbert space if the Taylor joint spectrum $\sigma(T_1, T_2, \ldots, T_n) \subset K$ and

$$\| f(T_1, T_2, \ldots, T_n) \| \quad \leq \sup\{|f(z_1, z_2, \ldots, z_n)| : (z_1, z_2, \ldots, z_n) \in K\} \tag{2}$$

for any function holomorphic in a neighbourhood of K. If F is a matrix valued holomorphic function, i.e., $F = ((f_{ij}))_{i,j=1}^d$, then define $F(T_1, T_2, \ldots, T_n) = ((f_{ij}(T_1, T_2, \ldots, T_n)))_{i,j=1}^d$, which is a block operator matrix and hence acts as a bounded operator on the direct sum of d copies of H. Call K a complete spectral set if

$$\| F(T_1, T_2, \ldots, T_n) \| \quad \leq \sup\{\| F(z_1, z_2, \ldots, z_n) \|: (z_1, z_2, \ldots, z_n) \in K\} \tag{3}$$

for matrices F of all orders $d = 1, 2, \ldots$. The norm on the left-hand side is the operator norm. The norm on the right-hand side is the norm of the matrix $F(z_1, z_2, \ldots, z_n)$. The question of deciding whether a spectral set is a complete spectral set gained importance because of Arveson's famous result: if K is a spectral set for $(T_1, T_2, \ldots, T_n)$, then K is a complete spectral set for $(T_1, T_2, \ldots, T_n)$ if and only if $(T_1, T_2, \ldots, T_n)$ has a normal ∂K-dilation, i.e., there is an n-tuple of commuting normal operators $(N_1, N_2, \ldots, N_n)$ on a bigger Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that of

$$f(T_1, T_2, \ldots, T_n) = P_{\mathcal{H}} f(N_1, N_2, \ldots, N_n)|_{\mathcal{H}}$$

for every function f holomorphic in a neighbourhood of K.

In general, it is very difficult to decide whether a given spectral set is also a complete spectral set.

In a series of important papers arising out of work done by Indian mathematicians, this issue has been resolved affirmatively for the symmetrized bidisk in $\mathbb{C}^2$.

Let us say that a compact subset K of $\mathbb{C}^n$ has the *dilation property* if

whenever K is a spectral set for $(T_1, T_2, \ldots, T_n)$,
it is also a complete spectral set for $(T_1, T_2, \ldots, T_n)$.

In view of Arveson's theorem mentioned above, the name is self-explanatory. For $n = 1$, the unit disk has this property [111], the annulus has this property [5] and a multiply connected domain does not, see [6, 38]. For $n = 2$, the bidisk

$$\mathbb{D}^2 = \{(z_1, z_2) : |z_1| \leq 1 \text{ and } |z_2| < 1\}$$

has this property [7] and the symmetrized bidisk

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1 \text{ and } |z_2| \leq 1\}$$

has this property. Agler and Young showed it in [12] by rather indirect methods. By Arveson's result, a dilation then exists. This is what was done explicitly by Bhattacharyya, Pal and Shyam Roy in the first paper to come out of India in this topic, see [20]. This paper triggered a spate of research in this area which can be seen in [21-23, 79, 80, 82, 108].

In what follows, we try to bring out the salient features. The object of study is a commuting pair (S, P) of bounded operators that satisfies

$$\| f(S,P) \| \leq \sup\{|f(s,p)| : (s,p) \in \Gamma\} =: \| f \|_{\infty,\Gamma} \tag{4}$$

for any polynomial f. Polynomial convexity of f then takes care of the joint spectral subtleties and Oka's theorem implies that (4) holds for all functions f holomorphic in a neigbourhood of Γ. Using the operator version of Fejer-Riesz theorem, it was shown that the operator equation

$$S - S^*P = (I - P^*P)^{1/2} X (I - P^*P)^{1/2} \tag{5}$$

has a unique solution X (it is automatic that P is a contraction and hence $I - P^*P$ has a unique positive square root). This X has a numerical radius no bigger than 1. The equation (5) is called the fundamental equation of (S, P) and X is called the fundamental operator. This was obtained in

[20], as well as an explicit dilation was constructed. Later, the fundamental operator was also used to characterize all distinguished varieties in Γ by Pal and Shalit. Specifically, every distinguished variety in Γ has the form

$$\{(s,p) \in \Gamma : \det(A + pA^* sI) = 0\}$$

for a matrix A whose numerical radius does not exceed 1. This A arises as the fundamental operator of a natural Γ-contraction associated to the variety. The name Γ-contraction had been coined by Agler and Young for any commuting pair (S, P) satisfying (4).

Bhattacharyya and Pal constructed a functional model for a pure Γ-contraction, i.e., a Γ-contraction (S, P) which satisfies $P^{*n} \to 0$ strongly as $n \to \infty$. Sarkar generalized this and found a functional model for those (S, P) whose contraction part P is completely non-unitary (c.n.u), i.e., P does not have any reducing subspace $\mathcal{M}$ such that $P|_{\mathcal{M}}$ is unitary. A functional model immediately produces a complete unitary invariant. Roughly speaking, the characteristic function θ_P of the contraction P and the fundamental operator together make a complete unitary invariant.

The second geometric object that was treated for function theoretic and operator theoretic purposes is the tetrablock

$$\mathbb{E} = \left\{ (a_{11}, a_{22}, \det A) : \ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ satisfies } \ \| A \| < 1 \right\}.$$

The study of a commuting triple which has $\bar{\mathbb{E}}$ as a spectral set was introduced by Bhattacharyya in [24]. Such a triple was called a tetrablock contraction. The aim was to construct a dilation. The candidate for a dilation (the ∂K dilation defined before with $K = \bar{\mathbb{E}}$ in this case) is the so-called tetrablock unitary whose structure was completely found. It was shown that any tetrablock contraction (A, B, P) satisfies the fundamental equations:

$$A - B^* P = (I - P^* P)^{1/2} F_1 (I - P^* P)^{1/2}$$
$$\text{and } B - A^* P = (I - P^* P)^{1/2} F_2 (I - P^* P)^{1/2}.$$

These two equations have unique solutions F_1 and F_2 in $B(\overline{Ran}(I - P^*P)^{1/2})$. An explicit dilation was constructed for a tetrablock contraction (A, B, P) whose fundamental operators F_1 and F_2 commute and satisfy

$$F_1 F_1^* - F_1^* F_1 = F_2 F_2^* - F_2^* F_2.$$

Sau and Pal later continued work on the tetrablock. Pal also analyzed the structure of distinguished varieties in the tetrablock, found a characterization of them and showed that there is no two-dimensional distinguished variety. See [81] and [113] for interesting recent progress.

Examples of Γ-contractions arise by decomposing weighted Bergman spaces of the bidisk into the direct sum of symmetric functions and anti-symmetric functions and then considering certain natural operators on these spaces. Exploring this idea further, Biswas and Shyam Roy decomposed weighted Bergman spaces on the polydisk into direct summands - each corresponding to an irreducible representation of the symmetric group. They also prove a Beurling-Lax-Halmos type theorem that is relevant to this context, see [27].

The momentum generated by this line of research continues. At the time of writing this article, further work projects in this area by several researchers are in progress.

2. Hermitian Holomorphic Vector Bundles

Cowen and Douglas, in their celebrated paper [33] studied a class of operators, which loosely speaking, have rich spectra as opposed to operators with thin spectra (because specific classes of operators with thin spectra, unitaries for example, were well-understood already). Formally, given an open $\Omega \subset \mathbb{C}$ and a positive integer $n,$ the Cowen-Douglas class $B_n(\Omega)$ consists of all bounded operators T (on a fixed Hilbert space H) such that $\Omega \subset \sigma(T)$ and

1. Ran $(T-\omega) = H$ for all $\omega \in \Omega,$
2. $\dim \ker(T-\omega) = n$ for all $\omega \in \Omega,$
3. $\overline{\text{span}} \left(\bigcup_{\omega \in \Omega} \ker(T-\omega) \right) = H.$

Cowen and Douglas used complex geometry to classify the operators in $B_n(\Omega)$. More specifically, they considered the hermitian holomorphic vector bundle $E_T,$ corresponding to any $T \in B_n(\Omega),$ which is a sub-bundle of the trivial bundle $\Omega \times H$ defined by

$$E_T = \{(\omega, x) \in \Omega \times H : x \in \ker(T-\omega)\},$$

with the natural projection map

$$\begin{aligned} \Pi : E_T &\to \Omega \\ (\omega, x) &\to \omega \end{aligned}$$

and showed that T is unitarily equivalent to T' if and only if E_T and $E_{T'}$ are in the same equivalence class of hermitian holomorphic vector bundles, see [33], Proposition 1.12. The curvature

$$k_T(\omega) = -\frac{\partial^2 \log \parallel \gamma(\omega) \parallel^2}{\partial \omega \partial \bar{\omega}}$$

of the bundle E_T is independent of the choice of the non-zero holomorphic section γ. It is straightforward to see that for a contractive operator T in $B_1(\mathbb{D})$, the curvature of T is bounded above by the curvature of M_z^* on $H^2(\mathbb{D})$, i.e.,

$$k_T(\omega) \leq k_{M_z^*}(\omega) \text{ for all } \omega \in \Omega.$$

In [25], Biswas, Kesari and Misra give a family of counterexamples to show that this inequality does not necessarily imply T to be a contraction. However, if k is a positive definite kernel on $\mathbb{D}$ such that M^* (the adjoint of the coordinate multiplication on H_k) is in $B_1(\mathbb{D})$, then the inequality above ensures that M is an infinitely divisible contraction. If the unit disk $\mathbb{D}$ is replaced by the Euclidean unit ball $\mathbb{B}_m$ and the contraction T by a row contraction $(T_1, T_2, \ldots, T_m)$, i.e.,

$$\sum_{i=1}^{m} T_i T_i^* \leq I \text{ and } T_i T_j = T_j T_i \text{ for all } i \text{ and } j,$$

then an appropriate generalization of the result holds.

The curvature invariant has been a long standing theme of research for Misra and his collaborators. Simultaneous unitary equivalence class of the curvature and the covariant derivatives up to a certain order of E_T determine the unitary equivalence class of T. Cowen and Douglas proved this, but also asked whether the covariant derivatives are necessary. In [78], Misra and Shyam Roy show that they are. Their examples constitute homogeneous operators in $B_n(\mathbb{D})$. An operator T is called homogeneous if $\sigma(T) \subset \bar{\mathbb{D}}$ and $\varphi(T)$ is unitarily equivalent to T for all conformal automorphisms φ of $\mathbb{D}$. Homogeneous operators have the advantage that computations regarding the curvature and the covariant derivatives can be done at the point 0 and the results can be transported to any point $\omega \in \mathbb{D}$.

Thus, efforts to determine the unitary equivalence class of an operator T takes one to the realm of complex geometry (hermitian holomorphic vector bundles) and then to the special operators which are homogeneous. The homogeneous operators have been studied for two decades by Misra - first in a series of papers with Bagchi and now with Koranyi. Misra and Koranyi announced in 2009 and gave detailed proof in 2011 of the complete classification of homogeneous operators in the Cowen-Douglas class, see [69, 70]. This was achieved by doing an explicit construction of E_T for a homogeneous T. Then E_T was broken into its irreducible direct summands. Work in similar spirit has been continued with the disk replaced by bounded symmetric domains in [71].

The recurring theme of Misra and his collaborators has been to identify / characterize the unitary equivalence class of an operator T, which is the same as the equivalence of the associated Hilbert module. Because of the structure of the minimal isometric dilation of a $C_{\cdot 0}$ contraction, it is the quotient module that hogs the limelight. Given a bounded domain Ω in $\mathbb{C}^n$, a hyper surface $Z \subset \Omega$ and a Hilbert module M over $A(\Omega)$ ($A(\Omega)$ is the sup norm closure in $C(\bar{\Omega})$ of functions holomorphic in a neighbourhood of $\bar{\Omega}$ - an obvious generalization of the disk algebra), it is natural that one should first investigate the quotient module that is the orthocomplement in M of M_0, the submodule of functions vanishing up to some order k in Z. Douglas and Misra studied this in [34, 35]. Under a condition called *quasi-freeness*, they find a complete set of unitary invariants for $Q = M \ominus M_0$. In the case $k = 2$, there are three invariants. Two of them (tan and trans) involve the coefficients of the curvature form of a hermitian vector bundle E and Z. The third - angle - can be replaced by the second fundamental form corresponding to the inclusion of the bundle E in the jet bundle $J^{(2)}E$.

Unitary invariant of an operator and hence unitary invariant for a Hilbert module has showed up in the works of Misra and his collaborators consistently over the years. This leads to sheaf models. Biswas, Misra and Putinar in [26] consider a pretty general class of Hilbert modulus, i.e., whose localizations are free and of finite dimension (not necessarily constant). They show that an associated sheaf is coherent. In this case too, if two Hilbert modulus (of the type considered in this paper) are equivalent then the associated hermitian holomorphic bundles one equivalent (Theorem 1.10).

3. Submodules of The Hardy Module of the Polydisk

Work done on quotient modules was explained in the last section. Let us start this section with submodules. Beurling's classical theorem, which has application from dilation theory to interpolation to statistics, says that all submodules of the Hardy module $H^2(\mathbb{D})$ over $A(\mathbb{D})$ are unitarily equivalent to $H^2(\mathbb{D})$. The situation is vastly different in several variables and for the Bergman module. For $H^2(\mathbb{D}^n)$ with $n > 1$, some submodules are unitarily equivalent to $H^2(\mathbb{D}^n)$ and some are not. For the Bergman module over the poly disk $\mathbb{D}^n$ or the Euclidean ball $\mathbb{B}_n$, no proper submodule is unitarily equivalent to the full module. In a series of papers, Douglas and Sarkar studied submodules under certain natural assumptions - purity, quasi-freeness, semi-fredholmness and essential reductivity - see [36, 37]. Among many interesting results that they obtained, here is a sample with a very general flavour. Let Ω be a domain in $\mathbb{C}^n$, let ν be a probability measure on Ω with $\nu(\partial\Omega) < 1$ and $\nu\{f^{-1}(0)\} = 0$ for any f holomorphic on Ω. Let point evaluation at $z \in \Omega$ be bounded for all $z \in \Omega$. Then no two unequal submodules of the Bergman module $L^2_a(\nu)$ can be isometrically isomorphic. This general result subsumes earlier results by Richter [106], Putinar [84] and Guo, Hu and Xu [51].

Beurling's theorem mentioned above, associated to every submodule of $H^2(\mathbb{D})$ an inner function which can be obtained as the characteristic function

$$\Theta_T(z) = \left[-T + \sum_{n=1}^{\infty} z^n D_{T^*} {T^*}^{n-1} D_T \right] \Big|_{\mathcal{D}_T}, \ |z| < 1$$

for a suitable contraction T, where $D_T = (1 - T^*T)^{1/2}$, $D_{T^*} = (1 - TT^*)^{1/2}$ and $\mathcal{D}_{\mathcal{T}} = \overline{\text{Ran}}\mathcal{D}_{\mathcal{T}}$. In a significant piece of work, Foias and Sarkar [49] showed that the characteristic function of a c.n.u. contraction is a polynomial of degree n if and only if the contraction has the form

$$T = \begin{pmatrix} S & * & * \\ 0 & N & * \\ 0 & 0 & C \end{pmatrix},$$

where S and C^* are unilateral shifts (whose multiplicities turn out to be unitary invariants for T) and N is a nilpotent operator of order n.

Since Beurling's theorem does not hold good in $H^2(\mathbb{D}^n)$, it is somewhat natural to ask for a characterization of those submodules $\mathcal{M}$ for which there is an inner function q such that $\mathcal{M} = qH^2(\mathbb{D}^n)$. There is an old paper of Mandrekar [76], where an answer to this question was given for $n = 2$. Generalizing Mandrekar's result to any $n \geq 2$ and also to vector valued $H^2(\mathbb{D}^n)$, Sarkar, Sasane and Wick showed in [110] that of $\mathcal{M}$ is a submodule of $H^2(\mathbb{D}^n) \otimes \mathcal{E}$, then $\mathcal{M}$ is the range of an inner function (suitably operator valued) if and only if the restrictions of the shifts M_{z_i} to $\mathcal{M}$ doubly commute.

Indeed, the Hardy space of the poly disk has been studied for a very long time. Nevertheless, recent work coming out of India has shed a great amount of light on its submodules and quotient modules. Izuchi, Nakazi and Seto in [63] had described doubly commuting quotient modules of $H^2(\mathbb{D}^2)$. A quotient module Q of $H^2(\mathbb{D}^n)$ is called *doubly commuting* if compressions of the shifts doubly commute. Generalizing the results of [63], Sarkar proved in [107] that such a Q is always a tensor product of n quotient modules of $H^2(\mathbb{D})$. This led to further work on tensor products of quotient Hilbert modules by Chattopadhyay, Das and Sarkar in [30]. This time, they considered a reproducing kernel Hilbert space $\mathcal{M}$ (a Hilbert module in their language) on $\mathbb{D}^n$ and obtained a characterization of those quotient modulus Q which can be written as a tensor product of one-variable quotient modules. Again, the result is that Q has to be doubly commuting when $\mathcal{M}$ is a so-called standard reproducing kernel Hilbert module. A quotient module Q is always of the form $H^2(\mathbb{D}^n)/\mathcal{S}$ where $\mathcal{S}$ is a submodule. When $\mathcal{S}$ is so nice that the cross commutators $[P_{\mathcal{S}}M_{z_i}^*|_{\mathcal{S}}, M_{z_j}|_{\mathcal{S}}]$ are compact, then the corresponding one-variable inner functions are finite Blaschke product or $n = 2$ (and vice versa). This was proved by Sarkar in [109] along with a rigidity theorem that two submodules $\mathcal{S}_1$ and $\mathcal{S}_2$ of $H^2(\mathbb{D}^n)$ satisfying $H^2(\mathbb{D}^n)/\mathcal{S}_i$ are doubly commuting for $i = 1, 2$, are unitarily equivalent if and only if they are equal.

Probably, the best work to come out in this theme is [29] by Chattopadhyay, Das and Sarkar. Given a commuting n-tuple of bounded linear operators on a Hilbert space $\mathcal{H}$ and non-empty subset

$\mathcal{S}$ of $\mathcal{H}$, the T-generating hull of $\mathcal{S}$ is defined by

$$[\mathcal{S}]_T := \overline{\text{span}}\,\{p(T_1, T_2, \dots, T_n)h : h \in \mathcal{S} \text{ and } p \text{ is a polynomial}\}.$$

When $[\mathcal{S}]_T = H$, then $\mathcal{S}$ is called a *T-generating set.* The rank of the tuple T is the unique number

$$\text{rank}\, T = \inf\,\{\#\,\mathcal{S} : \mathcal{S} \subset \mathcal{H} \text{ and } [\mathcal{S}]_{\mathcal{T}} = \mathcal{H}\} \in \mathbb{N} \cup \{\alpha\}.$$

Existence of a non-trivial generating set for a given tuple of operators is an open problem. A related question is to determine conditions under which the rank of T is finite. In this paper, the authors investigate such a question for a certain class of quotient modules of $H^2(\mathbb{D}^n)$. Motivated by an example of Rudin, they consider a sequence of Blaschke products $\Phi_i = \{\varphi_{i,k}\}_{k=-\infty}^{\alpha}$ for $i = 1, 2, \dots, n$, such that for every fixed i, the sequence $\{\varphi_{i,k}\}_{k=-\infty}^{\alpha}$ has a least common multiple. Rudin's quotient module Q_Φ is

$$Q_\Phi = \overline{\text{span}}\,\{(H^2(\mathbb{D}) - \varphi_{1,k}H^2(\mathbb{D})) \otimes \cdots \otimes (H^2(\mathbb{D}) - \varphi_{n,k}H^2(\mathbb{D}))\}.$$

They compute the rank of the tuple

$$(Mz_1^*|_{Q_\Phi}, Mz_2^*|_{Q_\Phi}, \dots, Mz_n^*|_{Q_\Phi}).$$

We wish to shift our attention to more dilation related results in several variables. Let D be a strictly pseudo convex domain in $\mathbb{C}^n$ with C^2 boundary. A subnormal tuple $\underline{T} = (T_1, T_2, \dots, T_n)$ is called a *∂D-isometry* if its minimal normal extension has its Taylor spectrum contained in ∂D. Athavale proved a commutant lifting theorem in [8]. He used the fact that there is an open set $\Omega \supset \bar{D}$ such that $\mathcal{O}(\Omega)|_{\bar{D}}$ is dense in the Banach algebra $A(D) = C(\bar{D}) \cap \mathcal{O}(D)$. He then took a bounded operator $X : H \to K$ intertwining two ∂D-isometries $\underline{S}$ and $\underline{T}$ with $\sigma(\underline{S}) \cup \sigma(\underline{T}) \subset \Omega$ and showed that X admits a norm-preserving extension to an intertwiner of the minimal normal extensions of $\underline{S}$ and $\underline{T}$. His theory subsumes the case of the Hardy space of the unit ball, for example.

Since the Taylor joint spectrum of a commuting tuple of operators is defined by generous use of homological algebra, it is significantly more difficult to assert invariance of the Taylor joint spectrum under a unitary or similarity transformation as compared to the single variable case. Two tuples $\underline{S}$ and $\underline{T}$ are called *quasi-similar* if there are operators X and Y such that $XS_i = T_iX$ for all $i = 1, 2, \dots, n$ and both X and Y are injective and have dense ranges. In [9], Athavale showed that for a special class of subnormal tuples, quasi-similarity preserves the Taylor spectrum and the essential Taylor spectrum. This special class includes the tuple of multiplication operators on the Hardy space of the Euclidean unit ball as well as the tuple of multiplication operators on the Bergman space of the Euclidean unit ball.

Using the results mentioned above, Athavale and Patil showed in [10, 11] that certain subnormal tuples and their duals are not quasi-similar. Given a subnormal tuple S and its minimal normal extension N, the dual $\tilde{S}$ of S is the restriction of N^* to the orthocomplement of the space that S acts on. A smart use of the first cohomology groups of the Koszul complexes corresponding to S and $\tilde{S}$ imply that their spectra differ and hence, by what is described above, they cannot be quasi similar. This is done for certain weighted shifts S.

4. Trace Formulae

In 1962, Krein came up with a remarkable trace formula in [72]. He considered two self-adjoint operators H_1 and H_2 such that $H_2 - H_1$ (the perturbation) is trace class and showed that

$$\mathrm{Tr}(\Phi(H_2) - \Phi(H_1)) = \int \xi(t) d\Phi(t)$$

for a real, integrable function ξ that works for a large class of suitably differentiable functions Φ. Also, $\|\xi\|_1 \leq \|H_2 - H_1\|_1$. Voiculescu gave a simple proof of this for polynomial Φ in [112] using a quasi-diagonality argument. Using Voiculescu's idea of finite-dimensional approximations, Chattopadhyay and Sinha obtained a new proof of Koplienko's formula in [32]. This involved the second derivative, so it is in the spirit of Krein's formula, but goes to one more order of derivative and also has the different assumption that $H_2 - H_1$ is Hilbert Schmidt:

$$\mathrm{Tr}(\Phi(H_2) - \Phi(H_1) - D\Phi(H_1)(H_2 - H_1)) = \int \Phi''(t)\eta(t)dt$$

for Schwartz class functions Φ where $D\Phi(H_1)$ denotes the Frechet derivative of Φ at H_1. The function η is integrable.

They later progressed to one more order of derivative, with H_1 bounded through. With this assumption, they established in [31] the existence of an integrable function η such that

$$\mathrm{Tr}\left(\Phi(H_2) - \Phi(H_1) - D^{(1)}\Phi(H_1)(H_2 - H_1) - \frac{1}{2}D^{(2)}\Phi(H_1)(H_2 - H_1, H_2 - H_1)\right) = \int \Phi'''(t)\eta(t)dt$$

for polynomial Φ. If H_1 is bounded below, then the existence of an

$$\eta \in L^1\left(\mathbb{R}, \frac{d\lambda}{(1+\lambda^2)^{1+\varepsilon}}\right)$$

can be concluded for some $\varepsilon > 0$, such that the above holds for all Schwartz class function Φ.

Random operators have played a big role in mathematical physics. A random Schrödinger operator, for example, is an operator of the form $H^\omega = \mathcal{L} + V_\omega$ on the lattice Hilbert space $l^2(\mathbb{Z}^d)$ or on $L^2(\mathbb{R}^d)$. Perturbation of self-adjoint operators has been studied from the beginning of quantum physics and the surge in interest in random operators has, over the years, continued to sustain interest in perturbation. Hislop and Krishna studied the local eigenvalue statistics for H^ω around a certain kind of energy E_0 in [52]. They show that certain random variables can be associated with it. These random variables are distributed according to a compound Poisson distribution. They give examples of random variables with compound Poisson and strictly non-Poisson local statistics.

Let A be a self-adjoint operator and $B \geq 0$. Let $H(t) = A + tB$. Early work had investigated $H(t)$ when B is a rank one projection whereas, over the years, B has been treated in much more generality. Let $\rho^\Phi_{H(t)}$ be the spectral measure for $H(t)$ with respect to the vector Φ in the Hilbert space $\mathcal{H}$ that A and B act on. Krishna and Stollman proved in [73] that when $B \geq 0$ is bounded and $\Phi \in \overline{\text{Range}}\, B$, then the measures

$$\nu = \int \rho^\Phi_{H(t)} h(t) dt$$

are absolutely continuous with respect to Lebesgue measure for any $h \in L'(\mathbb{R})$.

5. Geometry of Banach Spaces

The study of geometric structure of specific function spaces and spaces of operators is the main theme of research. Indian contributors have made deep and substantial contributions in this direction and their results have found applications in areas like convex optimization, proximinality and non-linear approximation theory.

The study of smooth and very smooth points in spaces of operators has led to the solutions to the long standing open question of which operators have the adjoint as a smooth point in the space of operators between dual spaces [99]. These ideas also led to the formulation of analogues of the classical result of Abatzoglou on smooth points in the space of operators on a Hilbert space [100].

In the study of into isometries between spaces of continuous functions on compact sets, the notion of a relative Korovkin closure was introduced and for some specific subspaces, the validity of an analogue of the classical result of Korovkin turns out to be equivalent to the Choquet boundary being the entire space [19]. Abstract Silov boundary was also investigated for subspaces not containing constants but have no common zero in the domain. In the case of a subalgebra one could still identify the Choquet boundary in terms of strong boundary points [95].

Extending the work of Friedman and Russo, the study of closed subspaces of the space of continuous functions on a compact set, that exhibit properties similar to those that are ranges of projections of norm one, were extensively investigated. The property studied (also called coproximinality, in the case of a subspace) was shown to be equivalent to being the range of a projection of norm one, in the case of subspaces of finite codimension [13]. This in particular also gives a reduction theorem, that any such subspace is the intersection of hyperplanes having the same property. Coproximinal subspaces turned out to be C_σ-spaces in the case of space of continuous functions on a compact set [92]. The 3-space problem for these spaces was also settled [98].

The preserver problem for coproximinality in the case of space of Bochner integrable functions was partially solved in the case of coproximinal subspaces that are isometric to a constrained subspace of a weakly compactly generated dual space [105].

Studying of a class of embeddings in terms of trees was initiated leading to an interesting interplay between the tree properties and the Banach space theoretic properties. A stronger version of the Schur property could be described by a uniform norming behavior of a boundary [39].

Several important non-linear optimization problems were investigated with very successful outcome by studying properties of subspaces whose unit ball is a proximinal or densely ball remotal set [16-18]. It turns out that Banach spaces whose dual is isometric to $L^1(\mu)$ for some positive measure μ is a rich domain to solve various deep questions arising here. In such spaces any M-deal is ball proximinal and they are also densely ball remotal in several natural situations. These questions are more amenable when the subspace under consideration is of finite codimension. A deep analysis in terms of norm attaining functionals led to the result that if the bi-annihilator of a subspace of finite codimension is strongly ball proximinal only at the points of the given space, then it is strongly ball proximinal [66].

That the notion of ball proximinality will play a role similar to that of compactness was confirmed by showing subspaces have strong one and half ball property precisely when they are ball proximinal and have the one and half ball property [56, 58]. These ideas lead to partial solutions to Cheney and Wulbert type of lifting theorem for stronger notions of proximinality [104].

The study of the algebraic reflexivity problem for subsets of the group of isometries of the space of operators and the structure of geometric projections like generalized bi-circular projections has led to several deep results in this area [1-4]. The study of the isometry group of operator ideals was initiated by considering weakly compact operators whose adjoint has separable range. For such operators, valued in a continuous function space, under some standard assumptions, the subset of the group of isometries, that preserve constants and are sequentially continuous with respect to the adjoint s.o.t, was shown to be an algebraically reflexive set [28].

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