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SOME RECENT RESULTS IN THE AREA OF NON-COMPLETE ALGEBRAIC VARIETIES

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We will briefly describe the basic theory of non-complete algebraic varieties developed by Japanese algebraic geometers and some of the main contributions of Indian mathematicians in this area during the last ten years.

Key words : Non-complete algebraic varieties; algebraic variety; smooth projective surface.

1. Introduction

The Enriques-Kodaira classification of minimal algebraic surfaces can be briefly described as follows.

Let X be a smooth projective surface which does not contain any exceptional curve of the first kind (i.e. no smooth rational curve with self-intersection -1). We say that X is a relatively minimal surface.

(1) If $\kappa(X) = -\infty$ then either $X \cong \mathbf{P}^2$ or X admits a morphism $X \to B$ onto a smooth projective curve all whose fibers are isomorphic to $\mathbf{P}^1$. In this latter case X is called a relatively minimal ruled surface.

(2) If $\kappa(X) = 0$ then X is either an abelian surface, a K-3 surface, an Enriques surface, or a hyperelliptic surface (i.e. a quotient of a product of two elliptic curves by a finite group of automorphisms, acting freely).

(3) If $\kappa(X) = 1$ then X has an elliptic fibration $\varphi : X \to B$ onto a smooth projective curve. Kodaira [54] gave a rich theory of possible singular fibers of φ, the monodromy action on the first integral homology group of a general fiber in a neighborhood of any singular fiber, a formula for the canonical bundle of X in terms of the canonical bundle of B and the singular fibers, study of

variation of complex structures of the fibers of φ, etc. Elliptic fibrations continue to play a special and important role in surface theory.

(4) If $\kappa(X) = 2$ then X is called a surface of general type. Kodaira and Bombieri [9, 55] proved important results about the pluri-canonical maps given by the linear systems $|nK_X|$. A vanishing theorem (generalising the famous Kodaira vanishing theorem) proved by Ramanujam [77] plays an important role in the study of these maps. Mumford proved that for large n this map is a birational morphism onto a projectively normal surface with at most rational double points. As expected, there are still many mysteries about the nature of possible invariants of a surface of general type like p_g (geometric genus), K_X^2, q_X (irregularity), fundamental group, etc.

It is a great surprise that most of these results have close analogues in the theory of non-complete surface theory. It was S. Iitaka's extraordinary intuition to introduce the notion of logarithmic Kodaira dimension, $\overline{\kappa}(V)$, of a smooth quasi-projective variety V. He proved many basic properties of this invariant. This was followed by a fundamental work of Kawamata [51, 52]. Kawamata [51, 52] proved important structure theorems for smooth surfaces with $\overline{\kappa} \geq 1$. He also made an important study of the log pluricanonical map and the singularities of the image. In 1979, Miyanishi - Sugie - Fujita [61] proved the Cancellation Theorem for $\mathbf{C^2}$. A crucial step in this proof was to prove a general result for varieites with $\overline{\kappa} = -\infty$. Fujita wrote an important paper about topology of non-complete algebraic surfaces which continues to be very useful [14].

One more important contribution in the theory of non-complete algebraic surfaces was a Bogomolov - Miyaoka - Yau type inequality proved by Kobayashi - Nakamura - Sakai. This inequality continues to play a crucial role in many results proved after 1990. The book [63] gives a detailed treatment of the theory of non-complete surface theory. It is safe to say that the Japanese algebraic geometers can take a patent on the theory of non-complete surfaces!

Outside Japan this theory has been extensively used by H. Flenner, R.V. Gurjar, S. Kaliman, M. Koras, S. Kolte, S. Lu. A. Maharana, S. Orevkov, K. Palka, S. Paul, C.R. Pradeep, P. Russell, A.R. Shastri, M. Zaidenberg, D.-Q. Zhang and others to prove important results about affine surfaces.

S. M. Bhatwadekar, A. Dutta, N. Gupta have made important contributions to this area from a more algebraic point of view.

Our aim is to survey these and other results in this area of algebraic geometry and related commutative algebra. We believe that commutative algebraists will benefit by studying this theory. Due to lack of time, we cannot include all the interesting results proved in this area and we will say only a few words about the proofs of the results. At the end we will state some unsolved problems.

2. An Invariant for Open Algebraic Varieties

Let k be an algebraically closed field. Sometimes we may assume $k = \mathbf{C}$ (for example, when we want to study complex analytic or topological properties of varieties).

All varieties will be defined over k.

Logarithmic Kodaira dimension

This theory was initiated by S. litaka. Later Y. Kawamata, T. Fujita, M. Miyanishi, F. Sakai, T. Sugie, R. Kobayashi, T. Tsunoda made important contributions to make the theory rich. It continues to find highly non-trivial applications, as we will mention later.

Let X be a smooth quasi-projective irreducible variety of dimension n. We can embed $X \subset V$, where V is a smooth projective variety such that $D := V - X$ is a divisor with simple normal crossings, i.e. for any $p \in D$, there exist local uniformising parameters $z_1, \cdots, z_n$ for $\mathscr{O}_{V,p}$, such that D is defined by $\{z_1, \cdot z_2 \cdots z_m = 0\}$ for some $m \leq n$. Such an embedding is guaranteed by Hironaka's work on resolution of singularities.

If $H^o(V, n(K_v + D))$ = (0) for n for $n \geq 1$ then we write $\overline{\kappa}(X) = -\infty$.

Otherwise $H^o(V, n(K+D)) \neq (0)$ for some $n \geq 1$. We can prove that $\dim_k H^o(V, n(K+D))$ is $O(n^r)$ for some r with $0 \leq r \leq n$. Then we write $\overline{\kappa}(X) = r.\overline{\kappa}(X)$ is called the logarithmic Kodaira dimension of X.

Iitaka proved some basic results about properties of $\overline{\kappa}$.

Fujita-Miyanishi-Sugie [61] proved the basic result that a smooth affine surface X with $\overline{\kappa}(X) = -\infty$ admits an $\mathbf{A}^1$-fibration. This was strengthened by Russell [79] to the case when X is connected at infinity. Fujita [14] proved some important results when $\overline{\kappa}(X) = 0$. His paper is very influential.

Kawamata [51] proved many basic results when $\overline{\kappa}(X) = 1, 2$.

Finally, Kobayashi-Nakamura-Sakai [53] proved an important inequality generalizing the famous Bogomolov-Miyaoka-Yau [73] inequality involving logarithmic Chern numbers. We refer the reader to the books [61, 63] for the details of this rich theory.

3. Recent Results

We will now describe recent results that deal with $\mathbf{Q}$-homology planes, log del Pezzo surfaces, $\mathbf{A}^1$ and $\mathbf{A}^1$*-fibrations on affine varieties, actions of the additive group $(\mathbf{C}, +)$ on affine varieties, locally nilpotent derivations on commutative rings. For a more exhaustive list of papers in Affine Algebraic Geometry and an overview of the subject, see [64].

A smooth affine surface X is a $\mathbf{Q}$-*homology plane* if $H_i(X;Q) = (0)$ for $i > 0$.

A normal projective surface V is called a *log del Pezzo surface* if V has at most quotient singularities and $-K_V$ is ample.

Several new results about quotients of affine spaces modulo actions of reductive algebraic group were proved.

Gurjar, Koras and Russell [33] proved the following result about 2-dimensional quotients of C^n modulo reductive algebraic groups.

***Theorem* 1** — *Let G be a reductive algebraic group acting regularly on* $\mathbf{C}^n$ *such that* $\mathbf{C}^n//\mathbf{G}$ *is 2-dimensional. Then* $\mathbf{C}^n//\mathbf{G}$ *is isomorphic to* $\mathbf{C}^2/\mathbf{\Gamma}$*, where* Γ *is a finite group of automorphisms of* $\mathbf{C}^n$*.*

Let a connected semisimple algebraic group G act regularly on $\mathbf{C}^n$. It is well-known that the ring of invariants $\mathbf{C}[\mathbf{X}_1..., \mathbf{X}_n]^G$ is finitely generated/$\mathbf{C}$. It corresponds to an affine normal variety X. There is a natural quotient morphism $\pi : \mathbf{C}^n \to \mathbf{X}$. There is a large number of important papers about X and properties of π.

The following very general result was proved in [23].

***Theorem* 2** — *The local first integral homology group at any point in X is trivial. The local second integral homology group at any point in X is finite.*

As a consequence, the completion of the local ring of X at any point is a UFD.

The following two results were conjectured by the second author in 1990. Mathieu gave a proof using infinite dimensional algebraic groups. In [24] his idea was used to give a proof without using infinite dimensional algebraic groups.

***Theorem* 3** — *Let R, S be positively graded domains* $/k$ *such that* $R_0 = S_0 = k$*. If the completions of R, S at their irrelevant maximal ideals are isomorphic then* $R \cong S$ *as affine domains.*

As a consequence, if a reductive algebraic group G *acts regularly on* $\mathbf{C}^n$ *then the logarithmic Kodaira dimension of the smooth locus of* $\mathbf{C}^n//\mathbf{G}$ *is* $-\infty$*.*

Generalized Jacobian Problem

Miyanishi [62] generalized the usual Jacobian Problem as follows.

Generalized Jacobian Problem. Let X be a smooth affine surface which is a $\mathbf{Q}$-homology plane. Is every étale self-morphism $X \to X$ a finite morphism ?

Several positive results about this were proved in [36]. Also, a counterexample was constructed in this paper. In [27, 46] this question was generalized to singular affine surfaces. The following surprising affirmative result was proved in [27, 46].

***Theorem* 4** — *Let X be an affine normal surface which has at least one singular point which is not a quotient singular point. Then any étale self-morphism $X \to X$ is an isomorphism.*

For a related question, in [45] a classification of smooth affine surfaces X which admit a proper self-morphism is given, provided either $\overline{\kappa}(X) \geq 0$, or $\pi_1(X)$ is infinite. The proof uses the theory of open-algebraic surfaces in an essential way.

Additive group actions and locally nilpotent derivations

Let $k^{[r]}$ denote a polynomial algebra in r variables over k. One of the important questions in Affine Geometry is the Zariski Cancellation Problem (ZCP) whjch asks: *Suppose B is an affine domain of dimension n over k such that $B[X]$ is k isomorphic to $k^{[n+1]}$*. Is $B \simeq k^{[n]}$?

Abhyankar-Eakin-Heinzer [1] gave an affirmative answer to ZCP when n = 1 [1], For n = 2, an affirmative answer is due to Miyanishi - Sugie and Fujita when char.(k) = 0 [16, 68]. A surprising result of Gupta [17] says that ZCP does not have an affirmative answer in general for $n \geq 3$ when char.(k) = $p > 0$.

Let $\mathbf{G}_a$ denote the (algebraic) group $(k, +)$. In all these results, study of all possible $\mathbf{G}_a$-actions on an affine variety X = Spec(B) has played very crucial role.

When char. k = 0, giving a $\mathbf{G}_a$-action corresponds to defining a locally nilpotent derivation on B. However such is not the case when char. (k) = $p > 0$. Therefore we first give algebraic formulation of a $\mathbf{G}_a$-action on an affine variety over an algebraically closed field k of arbitrary characteristic and then proceed.

Let B be an affine domain over a field k. A *locally finite iterative higher derivation* on B is a sequence $\{d_i : B \to B \; (0 \leq i < \infty)\}$ of functions satisfying following properties.

1. $d_0 = id$.

2. $d_n(a + b) = d_n(a) + d_n(b) \qquad a, b \in B$.

3. $d_n(a.b) = \sum_{i=0}^{n} d_i(a)d_{n-i}(b)$

4. $d_i d_j = \binom{i+j}{i} d_{i+j}$.

5. For $b \in B \ \exists\, n_0 \in \mathbb{N}$ (depending on b) such that $d_n(b) = 0\ \forall n \geq n_0$.

A ring morphism $\phi : B \to B[T]$ defined by

$$\phi(b) = \sum d_i(b) T^i$$

is called the *exponential map* (associated to a sequence d_i $(0 \leq i < \infty)$). Let $B^\phi = \{a \mid a \in B, \phi(a) = a\}$ (equivalently $B^\phi = \{a \mid \ d_i(a) = 0 \ \forall\ i \geq 1\}$). Then B^ϕ is called the ring of ϕ-invariants. We say ϕ is non-trivial if $B^\phi \neq B$.

It is a well-known result that giving a $\mathbf{G}_a$ -action on an affine variety $X = \operatorname{Spec}(B)$ over k is equivalent to giving an exponential map ϕ on B (associated to a locally finite iterative higher derivation $d_i(0 \leq i < \infty)$).

For some standard properties of these derivations we refer the reader to Miyanishi's book [65, Chapter I].

We associate to an affine k-domain B two invariants related to the set $EXP(B)$ of all **non-trivial** exponential maps on B.

$\operatorname{ML}(B) = \bigcap_{\phi \in EXP(B)} B^\phi$ is called the *Makar-Limanov invariant* (also known as *AK-invariant*).

The *Derksen invariant* $\operatorname{DK}(B)$ is defined to be the subalgebra of B generated by $\{B^\phi | \phi \in EXP(B)\}$.

Remark : Let $B = k[X_1, \cdots, X_n]$ be a polynomial algebra in n variables over k. Then $\operatorname{ML}(B) = k$ and $\operatorname{DK}(B) = B$.

These invariants have played very crucial role in determining whether a given affine domain B over k is $k^{[n]}$. For example, Makar-Limanov has proved that the Russell- Koras threefold $B = k[X, Y, Z, W]/(X^2Y + X + Z^2 + W^3)$ is not $k^{[3]}$ by showing that $\operatorname{ML}(B) = k[X]$. We see some more applications in the rest of this section.

We now turn our attention to results of Neena Gupta.

In what follows, for a ring R, $R^{[n]}$ denotes a polynomial algebra in n variables over R.

We begin with the following well known example of a *non-trivial line* in $k^{[2]}$ in the case the base field k is of positive characteristic.

Example 1 : Let k be an algebraically closed field of characteristic $p > 0$ and let $q = p + 1$. Let $\sigma : k[Z, T] \to k[U]$ be a k-algebra homomorphism defined as:

$\sigma(Z) = U + U^{pq}, \sigma(T) = U^{p^2}$. Let $F = T + T^{pq} - Z^{p^2}$. Then

(1) $\ker(\sigma) = (F)$,

(2) σ is surjective and hence $k[Z,T]/(F) = k[U]$ but

(3) F is not a variable in $k[Z,T]$.

With the help of this example, Asanuma has constructed a threefold which is stably isomorphic to $k^{[4]}$ [2, Theorem 5.1].

Example 2 : Let $F \in k[Z,T]$ be as in Example 1. Let $B = k[U,V,Z,T]/(U^2V - F)$. Then $B^{[1]} = k^{[4]}$.

More precisely, Asanuma showed that in the above example $B^{[1]} = k[U]^{[3]}$ but $B \neq k[U]^{[2]}$. He could not decide whether $B = k^{[3]}$. The following brilliant result of Neena Gupta ended this uncertainty [17].

***Theorem* 5** — *Let k be an algebraically closed field of characteristic $p > 0$, $q = p+1$, $F = T + T^{pq} - Z^{p^2} \in k[Z,T]$ and $B = k[U,V,Z,T]/(U^2V - F)$. Let u,z,t denote images of U,Z,T in B respectively. Then $\mathrm{DK}(B) = k[u,z,t]$ (a proper k-subalgebra of B) and hence $B \neq k^{[3]}$.*

This theorem together with Example 2 shows that ZCP does not have an affirmative answer in general for $n = 3$ and the base field has positive characteristic. In [19], Neena Gupta has extended this result for $n \geq 4$.

In a subsequent paper, Neena Gupta has proved the following interesting generalization of the above result [18, Theorem 3.11].

***Theorem* 6** — *Let k be a field of any characteristic and let B an affine domain defined by*

$$B = k[U,V,Z,T]/(U^mV - F(U,Z,T)), \text{where } m \geq 2.$$

Set $f(Z,T) := F(0,Z,T)$. *Then the following statements are equivalent:*

(i) $f(Z,T)$ is a variable in $k[Z,T]$.

(ii) $B = k[u]^{[2]}$, where u denotes the image of U in B.

(iii) $B = k^{[3]}$.

In the rest of the section we assume that k is an algebraically closed field of characteristic zero.

We now consider locally nilpotent derivations on $k[X_1, \cdots, X_n]$ ($k^{[n]}$) and state some results about structure of their kernels. For interested readers we recommend a monograph by Freudenburg

[15] for extensive literature on this subject.

Let d be a non-zero locally nilpotent derivation on $k[X_1, \cdots, X_n]$ and let $A = \ker(d)$. Then A is factorially closed in $k[X_1, \cdots, X_n]$ and hence A is factorial. Moreover $\mathrm{tr.}_k(A) = n - 1$. It is natural to ask : *Is A affine over k ?*.

We now consider locally nilpotent derivations on $k[X_1, \cdots, X_4]$. Let d be a locally nilpotent derivation and let $A = \ker(d)$. In this case, it is not known whether A is affine over k. However, under the additional assumption that rank of $d \leq 3$ i.e $\ker(d) = A$ contains a variable of $k[X_1, \cdots, X_4]$, it is known that A is affine over k [3].

However, it has been shown in [8] that, if rank of $d \leq 3$ and the affine variety $\mathrm{Spec}(A)$ has only isolated singularities then, as a k-algebra, A is generated by 4 elements. Moreover, if A is regular, then $A = k^{[3]}$. In the same paper, in contrast to a result of Kaliman, an example is given to show that even if $d(X_1) = 0$ and $d(X_2), d(X_3), d(X_4)$ generate the unit ideal, A need not be regular.

We conclude this section with stating the following open question (ZCP):

Question : Let k be an algebraically closed field of characteristic zero and let B be an affine k-domain such that $B^{[1]} \simeq k^{[4]}$. Is $B \simeq k^{[3]}$?

As a subquestion, one can asks whether the Russell-Koras threefold $B = k[X, Y, Z, W]/(X^2Y + X + Z^2 + W^3)$ is stably isomorphic to $k^{[3]}$.

Note that an affirmative answer to the subquestion will show that (ZCP) does not have an affirmative answer in general for $n = 3$ even if the base field k is of characteristic zero.

We will now discuss some other results about open algebraic varieties.

***Theorem* 7** — *An Affine Mumford Theorem. Let X be a normal affine contractible surface such that $X - Sing\ X$ is simply-connected. Then X is smooth.*

The proof of this result, and the next one, depend essentially on the theory of open algebraic surfaces (and some topology) described earlier [31].

Theorem* 8** — *Let X be a normal affine* **Z-homology plane such that $\overline{\kappa}(X - Sing\ X) = 2$. Then X has at most one singular point and it is a cyclic quotient singularity.*

The proof of this result is highly technical and long. It uses almost the whole theory of non-complete algebraic surfaces described earlier [32].

Recently Maharana [66] classified all smooth **Q**-homology planes of the form $\{z^n = f(x, y)\}$.

Gurjar and Paul [40] classified all factorial affine surfaces X such that $\overline{\kappa}(X - Sing\ X) \leq 1$.

These three results make essential use of the theory of open algebraic surfaces.

Log del Pezzo surfaces

The study of normal projective surfaces with at most quotient singulaities arises naturally. Considering the log Kodaira dimension of the smooth locus of such a surface and using the classification due to Kawamata mentioned earlier one can get valuable information about these surfaces. Among them, log del Pezzo surfaces play a special role. If V is a log del Pezzo surface then it is easy to see that V is rational. Using Kawamata-Mori theory one can often reduce the study of these surfaces to log del Pezzo surfaces of rank 1. There are important papers about log del Pezzo surfaces by Belousov, Demazure, Gurjar-Pradeep Zhang, Gurjar-Zhang, Hidaka-Watanabe, Hwang-Keum, Miyanishi-Zhang and Zhang. We will not go into any details of the results proved in these papers.

Complements of plane curves

Let $C \subset \mathbf{P}^2$ be an irreducible curve of degree $d > 1$. It is clear that if C is smooth and $d > 3$ the $\overline{\kappa}(\mathbf{P}^2 \setminus C) = 2$. Hence the classification of C such that $\overline{\kappa}(\mathbf{P}^2 \setminus C) \leq 1$ is a significant problem. There are many papers dealing with this by Iitaka, Kishimoto, Kojima, Matsuoka-Sakai, Orevkov, Tsunoda, Wakabayashi, Yoshihara, and others.

It is easy to construct plane curves of arbitrary genus with maximum (permissible by the genus formula) number of ordinary double points but finding rational curves which have at most unibranch singular points is not an easy task. Here the geometry of $\mathbf{P}^2$ plays an important role. The inequality of Kobayashi-Nakamura-Sakai puts a strong restriction about the number and types of such singular points.

In a recent paper [26] a more general situation of a cuspidal rational curve C in a $\mathbf{Q}$-homology projective plane V (i.e. V has rational homology of $\mathbb{P}^2$) with at worst quotient singular points is considered. It is proved that if C is contained in the smooth locus of V then the singularities of V and the nature of C influence each other. Here again, the case when V is a log del Pezzo surface is quite non-trivial and a complete classification of pairs (V, C) is not yet finished.

$\mathbf{A^1}$ *and* $\mathbf{A^{1*}}$*-fibrations on affine varieties*

In an important paper [5] the structure of a flat fibration $\pi : X \to Y$ over a noetherian domain Y such that π is an $\mathbf{A^1}$-bundle outside a codimension two subvariety of $Spec\, Y$ is described. This work can be seen as a culmination of earlier work of T. Asanuma, Bass-Connell-Wright, Bhatwadekar-Dutta and Dutta.

Similarly, the papers [6] and [7] consider a faithfully flat fibration $X \to Y$ over a noetherian normal domain Y which is locally a Laurent polynomial algebra outside a codimension two subvariety of Y.

Some new results about singular fibers of $\mathbf{A^1}$ and $\mathbf{A^{1*}}$-fibrations on affine varieties have been proved in [28-30]. There is a close connection between $\mathbf{A^1}$-fibrations (resp. $\mathbf{A^{1*}}$-fibrations) and quotients modulo G_a actions (resp. $\mathbf{C}^*$ actions). This relation is explored in [28] and [29]. We state some of the prominent results from these papers.

***Theorem* 9** — *[30]. Let $f : X \to Y$ be a dominant morphism from a smooth affine 3-fold X to a smooth affine surface Y such that a general fiber of f is isomorphic to $\mathbf{A^1}$. Then any 1-dimensional irreducible component of a fiber of f is isomorphic to $\mathbf{A^1}$.*

A Miscellaneous Result

The famous Zariski-Lipman conjecture can be stated as follows.

Conjecture : Let X be a normal quasi-projective variety. If the tangent bundle of the smooth locus of X is trivial then X is smooth.

In [25] the theory of open algebraic surfaces is applied to prove this conjecture for several classes of X. For example, if X is afffine and $\overline{\kappa}(X^0) \leq 1$ then the conjecture is true. Here X^0 denotes the smooth locus of X. Some partial results when X is projective are also proved in this paper (e.g. such a surface X has at most one singular point and $\overline{\kappa}(X^0) \leq 0$). Mysteriously, the case when $\overline{\kappa}(X^0) = -\infty$ is not yet completely settled.

4. Some Open Problems

(1) Let V be a smooth projective rational surface and $C \subset V$ a smooth irreducible curve. Is $\pi_1(V - C)$ finite?

This is true if $\overline{k}(V - C) \leq 1$ [44]. If this result has an affirmative answer then we get a striking consequence.

'Let V be as above. If $\varphi : V \to \mathbf{P}^1$ is a morphism with connected fibers, then φ has at most one multiple fiber'.

(2) Let V be a normal projective, rational surface with a unique singular point p. If p is a quotient singular point, is $\pi_1(V - p)$ finite?

This is true if $\overline{k}(V - p) \leq 1$ [44].

(3) Let $X := \{X^2 + Y^3 + Z^5 = 0\}$. Is every étale map $X \to X$ an isomorphism? Miyanishi - Masuda have proved this for all surfaces $\{X^a + Y^b + Z^c = 0\}$ with a, b, c pairwise coprime integers

> 1, except for $\{a, b, c\} = \{2, 3, 5\}$ [60].

More generally, Miyanishi has raised the question whether any étale self-map of a quotient $\mathbf{C}^2/G$ (G a finite group of automorphisms) is an isomorphism. This problem can be reduced to the case when G is either $\mathbf{Z}/(2)$ or the binary icosahedral group of order 120. The answer is not known for either of these groups.

A tantalising open problem is the following:

Question : Is there a smooth affine surface X with an étale self map whose image misses a non-empty finite subset?

(4) Classify all smooth affine surfaces X with a proper self-map of degree > 1.

Gurjar and Zhang have found the answer if either $\overline{\kappa}(X) \geq 0$, or if $\pi_1(X)$ is infinite.

It is not known if the affine surface $\{x^2 + y^2 + z^2 = 1\}$ has a proper self-morphism of degree > 1.

(5) Let X be a smooth affine 3-fold with a non-trivial G_a action and let $Y := X//G_a$. Are the singularities of Y quotient singularities?

(6) Let X be a smooth affine contractible 3-fold which admits three independent locally nilpotent derivations. Is X isomorphic to $\mathbf{C^3}$?

We have listed only few of the interesting results due to lack of time and space. It is clear that this theory will continue to play an important role for a long time.

Similarly, the study of normal singular points of surfaces (particularly rational singularities) is very useful for surface theory. Because of lack of time we have not touched upon this theory.

We will now give a list of some important papers in affine algebraic geometry in the last fifty years. For a more exhaustive list, see [64].

References

1. S. S. Abhyankar, P. Eakin, and W. Heinzer, On the uniqueness of the coefficient ring in a polynomial ring, *J. Algebra*, **23** (1972), 310-342.
2. T. Asanuma, Polynomial fibre rings of algebras over Noetherian rings, *Invent. Math.*, **87** (1987), 101-127.
3. S. M. Bhatwadekar and D. Daigle, On finite generation of kernels of locally nilpotent R-derivations of $R[X, Y, Z]$, *J. Algebra*, **322** (2009), 2915-2926.

4. S. M. Bhatwadekar and A. K. Dutta, On $\mathbf{A}^*$-fibrations, *J. Pure Appl. Algebra*, **149** (2000), 1-14.

5. S. M. Bhatwadekar, A. K. Dutta, and N. Onoda, On algebras which are locally $\mathbf{A^1}$ in codimension one, *Tran. Amer. Math. Soc*., **365** (2013), 4497-4537.

6. S. M. Bhatwadekar and N. Gupta, On locally quasi A^* algebras in codimension one over a Noetherian normal domain, *J. Pure Appl. Algebra*, **215**(9) (2011), 2242-2256.

7. S. M. Bhatwadekar and N. Gupta, The structure of a Laurent polynomial fibration in n variables, *J. Algebra*, **353** (2012), 142-157.

8. S. M. Bhatwadekar, Neena Gupta, and S. A. Lokhande, Some K-theoretic properties of the kernel of a locally nilpotent derivation on $k[X_1, \cdots, X_4]$, To appear in *Trans. Amer. Math. Soc*.,

9. E. Bombieri, Canonical models of surfaces of general type, *Publ. Math. IHES*, **42** (1973), 171-219.

10. A. J. Crachiola and L. Makar-Limanov, An algebraic proof of a cancellation theorem for surfaces, *J. Algebra*, **320** (2008), 3113-3119.

11. D. Daigle, Affine surfaces with trivial Makar-Limanov invariant, *J. Algebra*, **318**(8) (2008), 3100-3111.

12. D. Daigle and G. Freudenburg, A note on triangular derivations of $k[X_1, X_2, X_3, X_4]$, *Proc. Amer. Math. Soc*., **129** (2001), 657-662.

13. A. Fujiki, R. Kobayashi, and S. Lu, On the fundamental group of certain open normal surfaces, *Saitama Math. J*., **11** (1993), 15-20.

14. T. Fujita, On the topology of noncomplete algebraic surfaces, *J. Fac. Sci. Univ. Tokyo Sec. IA Math*., **29**(3) (1982), 503-566.

15. G. Freudenburg, Algebraic theory of locally nilpotent derivations, *Encyclopaedia of Mathematical Sciences*, **136**: *Invariant Theory and Algebraic Transformation Groups VII*, Springer - Verlag Berlin, 2006.

16. T. Fujita, On Zariski problem, *Proc. Japan Acad*., **55** (1979), 106-110.

17. Neena Gupta, On the cancellation problem for the affine space $\mathbf{A}^3$ in characteristic p, *Invent. Math*., **195** (2014), 279-288.

18. Neena Gupta, On the family of affine threefolds $x^m y = F(x, z, t)$, *Compositio Math*., **150** (2014), 979-998.

19. Neena Gupta, On Zariski's cancellation problem in positive characteristic, *Adv. Math*., **264** (2014), 296-307.

20. R. V. Gurjar, A Topological proof of Cancellation Theorem for $\mathbf{C^2}$, *Math. Zeitschrift*, **240** (2002), 83-94.

21. R. V. Gurjar, A new proof of Suzuki's formula, *Proc. Indian Acad. Sci*., **107** (1997), 237-242.

22. R. V. Gurjar, Two-dimensional quotients of $\mathbf{C}^n$ are isomorphic to $\mathbf{C}^2/\Gamma$, *Transformation Groups*, **12**(1)

(2007), 117-125.

23. R. V. Gurjar, Some topological properties of quotients modulo semisimple algebraic groups, *Comment. Math. Helv*., **84** (2009), 793-806.

24. R. V. Gurjar, *A positively graded domain is determined at its vertex*, Preprint 2015.

25. R. V. Gurjar and Sagar Kolte, On the Zariski-Lipman conjecture for normal algebraic surfaces, *J. London Math. Soc. (2)*, **90**(1) (2014), 270-286.

26. R. V. Gurjar, DongSeon Hwang, and Sagar Kolte, Singular rational homology plane pairs: Part I. Preprint 2016.

27. R. V. Gurjar and M. Miyanishi, Automorphisms of affine surfaces with $\mathbf{A^1}$-fibrations, *Michigan Math. J*., **53** (2005), 33-55.

28. R. V. Gurjar, K. Masuda, and M. Miyanishi, $\mathbf{A^1}$-fibrations on affine threefolds, *Journal of Pure and Applied Algebra*, **216** (2012), 296-313.

29. R. V. Gurjar, M. Koras, K. Masuda, M. Miyanishi, and P. Russell, $\mathbf{A^1_*}$-fibrations on affine threefolds, affine algebraic geometry, *Proceedings of the Conference at Osaka in 2011*, Edited by K. Masuda, H. Kojima, T. Kishimoto, World Scientific 2013.

30. R. V. Gurjar, M. Koras, K. Masuda, M. Miyanishi, and P. Russell, *Affine threefolds admitting G_a-actions*, Preprint 2015.

31. R. V. Gurjar, M. Koras, M. Miyanishi, and P. Russell, Affine normal surfaces with simply-connected smooth locus, *Math. Annalen*., **353** (2012), 127-144.

32. R. V. Gurjar, M. Koras, M. Miyanishi, and P. Russell, A homology plane of general type can have at most a cyclic quotient singularity, *J. of Algebraic Geometry*, **23** (2014), 1-62.

33. R. V. Gurjar, M. Koras, and P. Russell, Two dimensional quotients of $\mathbf{C}^n$ by a reductive group, *Electron. Res. Announc. Math. Sci*., **15** (2008), 62-64.

34. R. V. Gurjar and M. Miyanishi, *Affine surfaces with* $\overline{k} \leq 1$, In: Algebraic Geometry and Commutative Algebra in honour of M. Nagata, (Edited by H. Hijikata, H. Hironaka, M. Maruyama, H. Matsumura, M. Miyanishi, T. Oda, and K. Ueno), Kinokuniya Publ., Tokyo (1988), 99-124.

35. R. V. Gurjar and M. Miyanishi, On contractible curves in the complex affine plane, *Tohoku Math. J*., **48**(2) (1996), 459-469.

36. R. V. Gurjar and M. Miyanishi, On the Jacobian conjecture for $\mathbf{Q}$-homology planes, *J. Reine Angew. Math*., **516** (1999), 115-132.

37. R. V. Gurjar and A. J. Parameswaran, Open surfaces with non-positive Euler characteristic, *Compositio Math*., **99**(3) (1995), 213-229.

38. R. V. Gurjar and A. J. Parameswaran, Affine lines on $\mathbf{Q}-$homology planes, *J. Math. Kyoto Univ*., **35**(1) (1995), 63-77.

39. R. V. Gurjar and C. R. Pradeep, $\mathbf{Q}-$homology planes are rational - III, *Osaka J. Math*., **36** (1999), 259-335.

40. R. V. Gurjar and S. Paul, A classification of factorial surfaces of nongeneral type, *Michigan Math. J*., **61** (2012), 517-529.

41. R. V. Gurjar, C. R. Pradeep, and A. R. Shastri, On rationality of logarithmic $\mathbf{Q}-$homology planes-II, *Osaka J. Math*., **34** (1997), 725-743.

42. R. V. Gurjar, C. R. Pradeep, and D.-Q. Zhang, On Gorenstein surfaces dominated by $\mathbf{P}^2$, *Nagoya J. Math*., **168** (2002), 41-63.

43. R. V. Gurjar and D.-Q. Zhang, π_1 of smooth points of a log del Pezzo surfaces in finite, I, II, *J. Math. Sci. Univ. Tokyo*, **1**(1) (1994), 137-180. **2**(1) (1995), 165-196.

44. R. V. Gurjar and D.-Q. Zhang, On the fundamental group of some open rational surfaces, *Math. Ann*., **306** (1996), 15-30.

45. R. V. Gurjar and D.-Q. Zhang, Non-singular affine surfaces with self-maps, *Affine Algebraic Geometry*, Osaka Univ. Press, 2007.

46. R. V. Gurjar and M. Miyanishi, The Jacobian problem for singular surfaces, *J. Math. Kyoto Univ*., **48**(4) (2008), 757-764.

47. S. Iitaka, *On logarithmic Kodaira dimension of algebraic varieties, Complex analysis and algebraic geometry*, W. L. Baily, Jr. and T. Shioda ed., Iwanami Shoten, Tokyo, Tokyo, 1977, 175-189.

48. S. Kaliman, Polynomials with general $\mathbf{C}^2$-fibers are variables, *Pacific J. Math*., **203**(1) (2002), 161-190.

49. S. Kaliman, Free $(\mathbf{C},+)$- actions on $\mathbf{C}^3$ are translations, *Invent. Math*., **156** (2004), 163-173.

50. Y. Kawamata, *Addition formula of logarithmic Kodaira dimensions for morphisms of relative dimension one*, Proc. International Symposium on Algebraic Geometry (Kyoto, 1977) M. Nagata, ed., Kinokuniya, Tokyo, 1978, 207-217.

51. Y. Kawamata, *On the classification of noncomplete algebraic surfaces*, Algebraic geometry (Copenhagen, 1978) K. Lonsted, ed., Lecture Notes in Math. vol. 732, Springer, Berlin, 1979, 215-232.

52. Y. Kawamata, Characterization of abelian varieties, *Compositio Math*., **43** (1981), 253-276.

53. R. Kobayashi, S. Nakamura, and F. Sakai, A numerical characterization of ball quotients for normal surfaces with branch loci, *Proc. Japan Acad. Ser. A. Math. Sci*., **65**(7) (1989), 238-241.

54. K. Kodaira, On compact analytic surfaces II-III, *Ann. of Math*., **77** (1963), 563-626, 78(1963), 1-40.

55. K. Kodaira, Pluricanonical systems on algebraic surfaces of general type, *J. Math. Soc. Japan*, **20**

(1968), 170-192.

56. S. Keel and J. McKernan, Rational curves on Quasi-projective surfaces, *Mem. A.M.S.*, **669** (1999).

57. M. Koras and P. Russell, $\mathbf{C}^*$-Actions on $\mathbf{C^3}$: The smooth locus of the quotient is not of hyperbolic type, *J. Alg. Geometry*, **8** (1999), 603-694.

58. M. Koras and P. Russell, Contractible affine surfaces with a quotient singularity, *Transformation Groups*, **12**(2) (2007), 293-340.

59. M. Miyanishi, *Normal affine subalgebras of a polynomial ring, Algebraic and topological theories (Kinosaki, 1984)*, 37 -51, Kinokuniya, Tokyo, 1986.

60. K. Masuda and M. Miyanishi, Étale endomorphisms of algebraic surfaces with G_m-actions, *Math. Ann.*, **319**(3) (2001), 493-516.

61. M. Miyanishi, *Non-complete algebraic surfaces*, Lecture Notes in Mathematics 857, Springer-Verlag (1981).

62. M. Miyanishi, Étale endomorphisms of algebraic varieties, *Osaka J. Math.*, **22** (1985), 345-364.

63. M. Miyanishi, *Open algebraic surfaces*, Vol. 12, CRM Monograph Series, Montreal 2001. A.M.S.

64. M. Miyanishi, *Recent developments in affine algebraic geometry*, Affine Algebraic Geometry, Edited by T. Hibi, Osaka University Press, 307-378, 2007

65. M. Miyanishi, *Lectures on curves on rational and unirational surfaces*, Tata Institute of Fundamental Research, 1978.

66. A. Maharana, $\mathbf{Q}$-homology planes as cyclic covers of $\mathbf{A^2}$, *J. Math. Soc. Japan*, **61**(2) (2009), 393-425.

67. L. Makar-Limanov, On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbf{C^4}$, or a $\mathbf{C^3}$-like threefold which is not $\mathbf{C^3}$, *Israel J. Math.*, **96**(part B) (1996), 419-429.

68. M. Miyanishi and T. Sugie, Affine surfaces containing cylinderlike open sets, *J. Math. Kyoto Univ.*, **20** (1980), 11-42.

69. M. Miyanishi and T. Sugie. Homology planes with quotient singularites, *Journal of Math. Kyoto Univ.*, **31**(3) (1991), 755-788.

70. M. Miyanishi and T. Sugie, On Chern numbers of homology planes of certain type, *J. Math. Kyoto Univ.*, **36**(2) (1996), 331-358.

71. M. Miyanishi and S. Tsunoda, Logarithmic del Pezzo surfaces of rank one with non-contractible boundaries, *Japan J. Math.*, **10**(2) (1984), 271-319.

72. M. Miyanishi and S. Tsunoda, Absence of the affine lines on the homology planes of general type, *J. of Math. of Kyoto Univ.*, **32**(3) (1992), 443-450.

73. Y. Miyaoka, On the Chern numbers of the surfaces of general type, *Invent. Math.*, **42** (1977), 225-237.

74. M. Nagata, On automorphism group of $k[X, Y]$, *Kyoto Univ. Lec. Math.*, **5**, Kinokuniya, Tokyo (1972).

75. C. R. Pradeep and A. R. Shastri, On rationality of logarithmic **Q**-homology planes, *I. Osaka J. Math.*, **34**(2) (1997), 429-456.

76. C. P. Ramanujam, A topological characterization of the affine plane as an algebraic variety, *Ann. of Math.*, **94** (1971), 69-88.

77. C. P. Ramanujam, Remarks on the Kodaira vanishing theorems, *J. Indian. Math. Soc.*, **36** (1972), 41-51.

78. R. Rentschler, Operations du groupe additif sur le plan affine, *C.R. Acad. Sc. Paris*, **267** (1968), 384-387.

79. P. Russell, On affine-ruled rational surfaces, *Math. Ann.*, **255** (1981), 287-302.

80. A. Sathaye, Polynomial ring in two variables over a D.V.R. : A criterion, *Invent. Math.*, **74** (1983), 159-168.

81. M. Suzuki, Sur les opérations holomorphes du groupe additif complexe sur l'espace de deux variables complexes, *Ann. Sci. École Norm. Sup.*, **10**(4) (1977), 517-546.

82. M. Zaidenberg, Isotrivial families of curves on affine surfaces and characterizations of the affine plane, *Math. USSR Izvestiya*, **30**(3) (1988), 503-532.