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RECENT RESULTS ON VECTOR BUNDLES, PRINCIPAL BUNDLES AND RELATED TOPICS[1]

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This is a short survey of the interesting work in the theory of vector bundles (with various augmenting structures), principal bundles and related topics by Indian mathematicians in the last decade. The focus is on topics in which Indian mathematicians have made major contributions.

Key words : Vector bundles; moduli spaces; principal bundles.

1. INTRODUCTION

The theory of vector bundles and more generally principal bundles has strong ties with Physics (Conformal field theory, String theory) and has connection with many other branches of mathematics including Number theory, Topology, Representation theory. It is one of the most studied branches of Algebraic geometry. This is an attempt to give a short survey of the interesting work in the theory of vector bundles (with various augmenting structures), principal bundles and related topics by Indian mathematicians in the last decade. It is impossible to describe in detail their vast contributions in a few pages, so this has been a steep task. I have mainly chosen topics in which Indian mathematicians have made major contributions.

2. HIGGS BUNDLES

In the last three decades, the theory of Hitchin pairs (or Higgs bundles) on a smooth curve has grown in leaps and bounds, there have been generalisations to Higgs G-bundles and Higgs bundles on higher dimensional varieties. Generalisation of this very rich theory to singular curves Y was initiated by indian mathematicians in recent works [3, 6, 22, 24, 32].

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Let L be a fixed line bundle on the singular curve Y. Let $p : X \to Y$ be the normalisation of Y and $L_0 = p^*L$ the pull back of L to the normalisation X.

Definition 2.1 — A (L-twisted) Hitchin pair (E, ϕ) on Y is a coherent torsion free $\mathcal{O}_Y$-module E together with a homomorphism $\phi\colon E \to E \otimes L$.

A Hitchin Bundle (or Higgs bundle) (E, ϕ) is a Hitchin pair with E locally free.

There exits moduli spaces of semistable Hitchin pairs on a (possibly reducible) projective curve Y [22, 24].

2.1 *Generalized parabolic Hitchin pairs*

In [22], Hitchin pairs on any singular integral curve Y were studied by Bhosle by introducing the notion of (semi)stable generalized parabolic Hitchin pairs (GPH) on the normalisation X. A GPH is a Hitchin pair together with a generalized parabolic structure on finitely many divisors D_j on X, it is a good GPH if the Hitchin field preserves the generalized parabolic structure. This was generalised to GPH on a disjoint union X of integral smooth curves and their moduli spaces were constructed [24]. Denote by $\mathcal{M}$ the moduli space of good GPH on X, it is a closed subscheme of the moduli space of GPH. Let $\mathcal{H}$ be the moduli space of Hitchin pairs on the singular curve (possibly reducible) Y [24].

***Theorem* 2.2** — *[22, 24].*

(1) *There is an analogue of Hitchin map on the moduli space of GPH, it is a proper map and induces a proper map*

$$P_{\mathcal{M}}\colon \mathcal{M} \longrightarrow A\,;\; A = \bigoplus_{i=1}^{n} H^0(X, L_0^i) \;=\; \bigoplus_{i=1}^{n} H^0(Y, L^i \otimes p_*\mathcal{O}_X)\,.$$

(2) *There is a birational morphism $f\colon \mathcal{M} \longrightarrow \mathcal{H}$, with $f(M)$ containing the subset of $\mathcal{H}$ corresponding to Hitchin bundles.*

(3) *The Hitchin map on $\mathcal{M}$ induces a proper Hitchin map on $f(\mathcal{M})$*

$$P_{\mathcal{H}}\colon f(M) \longrightarrow A\,.$$

Let $A' = \bigoplus_{i=1}^{n} H^0(Y, L^i) \subset A$. Then $P_{\mathcal{H}}$ maps the Hitchin bundles to A'.

(4) *The Hitchin map is surjective.*

(5) *If all the irreducible components of Y are smooth, then the Hitchin map is defined on the entire moduli space $\mathcal{H}$.*

The general fibres of the Hitchin map are determined.

When the base field is the field of complex numbers, one can associate Higgs bundles to representations of the (topological) fundamental group $\pi_1(Y)$ of Y, these bundles are semistable or stable under suitable conditions on the representations [24, 32]. However, there exit stable Higgs bundles on Y which are not associated to any representations of $\pi_1(Y)$.

2.2 *Degenerations of smooth curves and Gieseker-Hitchin pairs*

Another approach to study Hitchin pairs on a singular curve is to consider degeneration of a smooth curve to a singular curve as done by Balaji, Barik and Nagaraj [6]. They define Gieseker-Hitchin pairs, these pairs are Gieseker bundles, with Higgs fields, defined on cycles $X^{(m)}$ of curves whose one component is the normalisation X of Y and the rest of components form a chain of projective lines. Their main result is the following.

***Theorem* 2.3** — *Let $X_S \to S$ be a proper and flat fibered surface over $S = SpecR$ such that X_S is a regular k-scheme, the general fibre is a smooth curve and the special fibre is nodal. Let L be a line bundle on X_S such that for $t \in SpecR$, $L|_{X_t} = \omega_X$ or $deg(L|_{X_t}) > deg(\omega_{X_t})$, where ω_X and ω_{X_t} are dualising sheaves.*

Let n and d be coprime integers.

(1) *There is a quasi-projective S-scheme $G_S^H(n,d)$ of Gieseker-Hitchin pairs which is a regular k-scheme flat over S. The generic fibre is the moduli space of Hitchin pairs and the closed fibre is a reduced divisor with normal crossing singularities.*

(2) *There is a proper Hitchin map $g_S : G_S^H(n,d) \to A_S$ to an affine space over S.*

As fibre of Hitchin map over the closed point, they get a new compactification of the Picard variety which has normal crossing singularities (toric blow ups of Oda-Seshadri compactifications).

2.3 *Tensor products of Higgs bundles in arbitrary characteristics*

Illangovan, Parameswaran and Mehta had given an algebraic proof of the semistability of tensor product of vector bundles in any characteristic (with an obvious bound on the characteristic) [63]. This was done by showing that under a low hight representation of a reductive group, semistable G-bundles induce semistable vector bundles. The idea was to show that if the induced bundle is unstable, then one can construct a Kempf parabolic and a reduction that becomes unstable by suitably applying the work of Ramanan and Ramanathan [84].

Balaji and Parameswaran generalised these results to tensor products of Hitchin Pairs (on a smooth curve) in any characteristic (with an obvious bound on the prime characteristic) [11]. In the presence of Higgs structures, the problem is to prove that the bundle constructed using the above technique naturally carries a Higgs structure i.e. the reduction obtained is a Higgs reduction. To achieve this they introduced the concept of Higgs schemes and used functorial approach to produce the unstable Higgs reduction.

Bhosle, Parameswaran and Singh further extended these results to the case of singular integral curves [32]. Here it is a careful blend of the construction of Holonomy group scheme for singular curves (which will be described in the next section) and constructions described in the previous paragraph on Higgs schemes done over singular curves.

3. Tannakian Group Schemes and Vector Bundles

The celebrated Narasimhan-Seshadri theorem gives a bijective correspondence between irreducible unitary representations of the topological fundamental group of a smooth curve over complex numbers and stable vector bundles on the curve. The quest for a generalisation of the theorem to positive characteristic has lead to certain Tannakian subcategories of the category of vector bundles (as well as of principal G-bundles) and group schemes associated with them called the holonomy group schemes (or monodromy group schemes). The objects in the Tannakian subcategory are in bijective correspondence with the representations of the holonomy group scheme (or monodromy group scheme). For generalities on Tannakian categories, the reader may refer to [54, 86].

3.1 *Fundamental group scheme*

The fundamental group scheme $\pi(X, x_0)$ of a connected and reduced scheme defined by Nori [76, 77] is the group scheme associated to the category of essentially finite vector bundles. The fundamental group scheme is an invariant finer than the étale fundamental group. The étale fundamental group is a quotient of the fundamental group scheme, the two coincide when characteristic is 0.

Let X be a projective variety defined over an algebraically closed field k. A vector bundle E on X is defined to be essentially finite if there is a principal G-bundle P on X, with G a finite group, such that the pull back of E to P is trivial. Mehta and Antei showed that a vector bundle E is essentially finite if and only if there is a projective variety Z defined over k and a finite surjective morphism $f : Z \to X$ such that $f^*(E)$ is trivial [2].

If X is a normal projective variety defined over an algebraically closed field k and $D \subset X$ a reduced ample hypersurface with a k-rational point x_0, then the homomorphism of étale fundamental groups $\pi^{et}(D, x_0) \rightarrow \pi^{et}(X, x_0)$ is surjective if dim. $X \geq 2$ and isomorphism if dim. $X \geq 3$. Biswas and Holla have the following results for the fundamental group scheme [40].

***Theorem* 3.1** — *Let X be a normal projective variety defined over an algebraically closed field k and L an ample line bundle on X. Let $D \in |L^{\otimes d}|$ be a smooth divisor with a k-rational point x_0. Then there exists an integer $d_1 = d_1(X, L)$ such that for $d \geq d_1$ the homomorphism of fundamental group schemes*

$$h : \pi(D, x_0) \rightarrow \pi(X, x_0)$$

is surjective and faithfully flat if dim. $X \geq 2$. There exists an integer $d_2 = d_2(X, L)$ such that for $d \geq d_2$ the homomorphism h is a closed immersion if dim. $X \geq 3$ and hence h is an isomorphism for $d \geq d_1, d_2$ and dim.$X \geq 3$.

They also give an example of a pair (X, D) such that the homomorphism h is not surjective.

Mehta and Subramanian studied the fundamental group scheme of a smooth projective variety X over the ring of Witt vectors $W(k)$ over an algebraically closed field k of positive characteristic [70]. Let $\bar{W}$ denote the integral closure of $W(k)$ and $X_{\bar{W}}$ the base change of X to $\bar{W}$. They prove that a representation of the fundamental group scheme of $\bar{W}$ can be identified to a vector bundle on $X_{\bar{W}}$ which is essentially finite on the general fibre as well as the special fibre. They also show that if X and Y are smooth projective varieties over the ring of Witt vectors $W(k)$, then the fundamental group scheme of $X \times_{W(k)} Y$ is the fibre product of the fundamental group schemes of X and Y.

3.2 *Gieseker's conjectures*

Let X be a smooth projective scheme over a perfect field k of characteristic $p > 0$. A stratified bundle E on X is an abelian sheaf endowed with a module structure over the sheaf of differential operators $\mathcal{D}_X$, so that it is coherent over both $\mathcal{O}_X$ and $\mathcal{D}_X$.

D. Gieseker's conjecture (1975)

• If there are no nontrivial étale coverings over $X \times_k \bar{k}$, then there are no stratified bundles over X [58].

The conjecture can be restated in terms of Tannakian group schemes as follows:

• The vanishing of the étale fundamental group π_1^{et} (a profinite group) implies the vanishing of the proalgebraic group π_1^{strat}.

The conjecture was proved by Mehta and Esnault [56]. In characteristic 0, the result goes back to Maĺtsev (1940) and Grothendieck (1970).

3.3 *S-fundamental group scheme*

The S-fundamental group scheme $\pi_1^S(X, x_0)$ of a projective variety X is the group scheme associated to the Tannakian category of numerically flat vector bundles on X, i.e., vector bundles E such that both E and its dual E^* are nef [68]. Mehta and Hogadi prove that if $\phi : X \dashrightarrow Y$ is a birational rational map of smooth projective varieties defined over an algebraically closed field of positive characteristic and x_0 a point where ϕ is defined, then $\pi_1^S(X, x_0)$ and $\pi_1^S(Y, \phi(x_0))$ are isomorphic [61].

3.4 *Holonomy group schemes for curves and monodromy group schemes*

Let X be a geometrically irreducible smooth projective curve defined over any field k. Assume that X has a k-rational point, fix a k-rational point x_0. A vector bundle E over X is called (Frobenius) strongly semistable if $F^{n*}E$ is semistable for all $n \geq 1$, where F^n denotes n-times iterated composition of the Frobenius F. Biswas, Parameswaran and Subramanian [46, 47] built a neutral Tannakian category of all (Frobenius) strongly semistable vector bundles on X. By a theorem of Saavedra Rivano [86], there is an affine group scheme $\mathcal{G}_X$ associated to the neutral Tannakian category which we call the holonomy group scheme of X.

Let G be a linear algebraic group, defined over k which does not admit any non-trivial character which is trivial on the connected component, containing the identity element, of the reduced centre of the group G. Let E_G be a (Frobenius) strongly semistable principal G-bundle on X. Using Tannakian category approach, Biswas, Parameswaran and Subramanian [47] associate a group scheme M_E to E_G called the ”monodromy group scheme” of E_G, it is a quotient of $\mathcal{G}_X$. It is canonically embedded in the fibre $Ad((E_G)_x)$ over x of the adjoint bundle of E_G, the bundle associated to E_G by Adjoint representation of G.

Bhosle and Parameswaran [31] introduced a new notion of strong semistability for vector bundles and principal G-bundles over a singular integral curve Y. A vector bundle E over Y is called strongly semistable if all its tensor powers are semistable. This coincides with Frobenius strong semistability in characteristic $p > 0$. However, in characteristic 0, the bundle E is strongly semistable if and only if its pull back to the normalisation is semistable. A neutral Tannakian category of strongly semistable vector bundles on Y was constructed, the associated affine group scheme was called holonomy group scheme of Y. For a strongly semistable G-bundle E on Y, a neutral Tannakian subcategory was constructed, the associated group scheme being called the monodromy group scheme for the G-bundle. In case Y is an integral complex nodal curve, there is a representation ρ of the fundamental group $\pi_1(Y)$ of Y such that E is associated to ρ and the monodromy group scheme is the Zariski closure of $\rho(\pi_1(Y))$ in $GL(n, \mathbb{C})$.

3.5 *Holonomy group schemes for higher dimensional varieties*

For generalisation to arbitrary smooth projective varieties, Balaji and Parameswaran introduced the notion of lf-graded (locally free graded) vector bundles [10]. They showed that the Tannakian category of lf-graded bundles is the smallest Tannakian category containing all stable bundles. As an application they also constructed stable bundles on any smooth surface over an uncountable field.

Balaji and Kollar [9] defined a holonomy group for a stable vector bundle F on a variety in terms of the Narasimhan-Seshadri unitary representation associated to its restriction to curves of a sufficiently high degree. They related this group to the minimal structure group and decomposition of tensor powers of F.

3.6 *Hitchin holonomy group schemes*

The notion of strong semistability for vector bundles and principal G-bundles over a (possibly) singular integral curve Y was generalised to Hitchin bundles and Hitchin G-bundles on Y in [32]. A neutral Tannakian category of strongly semistable Hitchin bundles on Y was constructed, the associated affine group scheme was called Hitchin holonomy group scheme of Y. For a strongly semistable Hitchin G-bundle E on Y, the monodromy group scheme was constructed.

4. BRILL-NOETHER THEORY AND COHERENT SYSTEMS ON CURVES

4.1 *Butler's Conjecture*

Let X be an integral smooth projective curve of genus g. Let L be a line bundle on X and let $V \subset H^0(X, L)$ be a subspace of the space of sections of L. Define evaluation map

$$ev_V : X \times V \to L$$

by

$$(x, s) \mapsto s(x), x \in X, s \in V.$$

If this map is surjective, then L is said to be generated by V.

Definition 4.1 — Suppose that the line bundle L is generated by a sub space V of its sections. The kernel of the map ev_V, denoted by $M_{L,V}$, is a vector bundle, called a Syzygy bundle (or a kernel bundle or Lazarsfeld bundle).

It fits in an exact sequence

$$0 \to M_{L,V} \to X \times V \to L \to 0\,.$$

The bundle $M_{L,V}$ has been extensively studied for many years because of its several applications. As its name suggests, it has applications to Syzygy problems - Green's Conjectures, Minimal resolution conjecture, Ideal generation conjecture, Theta divisors, Picard bundles, Brill-Noether Theory and Coherent systems. It can be interpreted as restriction of the cotangent bundle of $\mathbb{P}(V)$.

• **Butler's Conjecture** : For a general curve X of genus $g \geq 1$ and a general choice of (L, V) the vector bundle $M_{L,V}$ is semistable [50].

Here a general curve means a curve belonging to a non-specified Zariski open subset of the moduli space of curves.

A lot of work had been done on these conjectures for eighteen years by several authors. Different techniques have been employed to solve them, including deformations of curves, classical Brill-Noether theory and Coherent systems for higher ranks. Many cases have been proved. Some of the earliest results are due to Ballico and Hein [8, Proposition 1.6, Theorem 1.7 and Remark 1.10]. Recent results of Aprodu M., Farkas G., and Ortega A. paved way to solving the conjecture. Finally the conjecture was proved by Bhosle, Newstead and Brambila-Paz [15] using coherent systems. Some results on related conjectures including that on stability of $M_{L,V}$ were also proved. For a detailed survey on Butler's conjecture and syzygy bundles, see [23].

A curve X is called a Petri curve if for any line bundle N on X, the multiplication map

$$H^0(X, N) \otimes H^0(X, K_X \otimes N^*) \to H^0(X, K_X)$$

is injective. Here, N^* denotes the dual of N and K_X is the canonical bundle (dual of tangent bundle) of X. Petri curves form an open subset in the moduli space of curves.

***Theorem* 4.2** — *Let X be a (smooth) Petri curve of genus $g \geq 3$ and (L, V) is a general linear series of type $(d, n+1)$, i.e., L has degree d and dim. $V = n + 1$. Then $M_{L,V}$ is stable for*

(1) $n \leq 4$ *[14, Theorems 7.1, 7.2, 7.3].*

(2) $n \geq 5$ *and* $g \geq 2n - 4$ *[15, Theorem 6.1].*

***Theorem* 4.3** — *[15, Theorem 6.12]. Let n be a prime number, X a general curve of genus $g \geq 3$ and (L, V) a general linear series of type $(d, n+1)$. Then $M_{L,V}$ is stable except possibly in the following cases:*

- $n = 11$, $g = 13, 14$, $d = 33$.
- $n = 13$, $g = 15, 16$, $d = 39$.

- $n \geq 11$, $n + 2 \leq g \leq \min\{n + 4, g_n\}$, $d = 3n$
- $n \geq 17$, $n + 5 \leq g \leq g_n$, $d = 3n, 4n$, *where* $g_n = \frac{4(n-1)^2}{3n-5}$.

Proofs of all these results use coherent systems. What is a coherent system?

Definition 4.4 — A coherent system (E, V) of type (r, d, k) on X is a pair consisting of a torsionfree sheaf E of rank r, degree d and a linear subspace $V \subset H^0(X, E)$ of dimension k.

The coherent system (E, V) is said to be *generated* if the evaluation map $X \times V \rightarrow E$ is surjective. Note that the pair (L, V) in Butler's conjecture is a generated coherent system of type $(1, d, n + 1)$.

For any real number α, there is a notion of α-stability of a coherent system.

Definition 4.5 — For any real number α, a coherent system (E, V) of type (r, d, k) is α-*stable* if, for any proper coherent subsystem (E', V') of type (r', d', k') of (E, V),

$$\frac{d' + \alpha k'}{r'} < \frac{d + \alpha k}{r}.$$

4.2 *Moduli spaces of coherent systems*

There exist moduli spaces $G(\alpha; r, d, k)$ of α-stable coherent systems of type (r, d, k). For $G(\alpha; r, d, k), k > 0$ to be nonempty, we must have $\alpha > 0$.

There are finitely many critical values $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L$ of α; as α varies, the concept of α-stability remains constant between two consecutive critical values. Denote by $G_0(r, d, k)$ (resp. $G_L(r, d, k)$) the moduli spaces corresponding to $0 < \alpha < \alpha_1$ (resp. $\alpha > \alpha_L$).

$$\begin{array}{ccccccccc} & G_0 & & G_1 & & & & G_L & \\ 0 & ---- & \alpha_1 & ---- & \alpha_2 & ------ & \alpha_L & ----- & \end{array}$$

For $(E, V) \in G_0$, the vector bundle E is semistable.

Moreover, if E is stable, then $(E, V) \in G_0(r, d, k)$ for any V. A detailed study of G_L was done, this was used to prove Butler's conjecture.

4.3 *Coherent systems and Brill Noether Loci on nodal curves*

Coherent systems and Brill-Noether loci have been studied extensively for smooth curves. Extension of these theories to singular curves was initiated at the beginning of last decade [18, 19].

Let Y be an integral projective curve with at most ordinary nodes and cusps as singularities defined over an algebraically closed field k. Let g denote the arithmetic genus of Y, $g =$ dim.

$H^1(Y, \mathcal{O}_Y)$. Let $U(n,d)$ denote the moduli space of semistable torsionfree sheaves of rank n and degree d on Y. The varieties $U(n,d)$ are not normal for all $n \geq 1$. The Brill-Noether scheme

$$B(n,d,k) \subset U(n,d)$$

is defined as the locus of stable torsionfree sheaves with at least k independent sections. Denote by $\tilde{B}(n,d,k) \subset U(n,d)$ the Brill-Noether locus of semistable torsionfree sheaves E with $h^0(grE) \geq k$, where grE denotes the associated graded of E for stable filtration.

Poincare formula and Riemann Singularity Theorem

Over smooth curves, for $n = 1$, the Brill-Noether theory is the classical one. The Poincare formula and Riemann singularity theorem are important classical results on the Jacobian of a smooth curve. The variety $J^d(Y)$ (called generalised Jacobian) of line bundles of degree d on a singular curve Y is not compact, it has a natural compactification $\bar{J}^d(Y) := U(1,d)$ called compactified Jacobian of Y. There is a poincare sheaf $\mathcal{P}$ on $\bar{J}^d(Y) \times Y$. For $d \geq 2g-1$, the direct image E_d of $\mathcal{P}$ on $\bar{J}^d(Y)$ is a vector bundle called a Picard bundle.

Bhosle and Parameswaran [29] proved analogues of classical Poincare Formula and Riemann singularity theorem for the compactified Jacobian of a nodal curve. Fixing a nonsingular point $y_0 \in Y$, one has an isomorphism $t_i : \bar{J}^0(Y) \to \bar{J}^i(Y)$ defined by $L \mapsto L \otimes \mathcal{O}_Y(iy_0)$. For $1 \leq i \leq g-1$, define $W_i \subset \bar{J}^0(Y)$ by $W_i := t_i^{-1} B(1, g-i, 1)$. Thus W_1 is the theta divisor.

Theorem* 4.6 — *[30]. For $0 \leq i \leq g-1$ we have the generalized Poincaré formula

$$W_i = \frac{W_1^i}{i!} \tag{4.1}$$

as cycles modulo numerical equivalence.

We note that the Poincaré formula is false in the Chow ring of the Jacobian even for a smooth curve.

Let $\tilde{W}_i$ be the inverse image of W_i in the natural normalisation $\tilde{J}(Y)$ of $\bar{J}(Y)$, $\tilde{W}_i$ is normal for all i [29]. The classical Riemann singularity theorem says that if X is a smooth projective curve and $i < g(X)$, then a point w of $W_i \subset J(X)$ is nonsingular if and only if the line bundle over X corresponding to w has a unique section. The same is true for $\tilde{W}_i$ [29, Theorem 5.4, Remark 5.5]. This is the analogue of Riemann singularity theorem over nodal curves.

These results have several applications including applications to Brill-Noether theory, theta divisors and Picard bundles on the compactified Jacobian [30], maps of the nodal curve into a projective space [21]. Unlike in the smooth case, Picard bundles on the compactified Jacobian are stable but not ample.

If Y has only singularity one node y, then $P := \tilde{J}(Y)$ is a $\mathbb{P}^1$-bundle on the Jacobian $J(X)$ of X. Let $\mathcal{O}_P(1)$ be the relatively ample line bundle on $\tilde{J}(Y)$. There is a canonical section S_y of $\tilde{J}(Y)$. For each integer n, there is a morphism $n : P \to P$ ("multiplication by n"). The tensor product on line bundles can be used to define Pontryagin product $*$ on the rational Chow ring $A^*(P)_Q$ of P [62]. Define the Tautological ring $\tilde{R}$ as the smallest Q-subalgebra of $A^*(P)_Q$ containing all $\tilde{W}_i, S_y, c_1(\mathcal{O}_P(1))$ and closed under pull back maps n^*, push forward maps n_*, Pontryagin product $*$. Iyer [62] proved that the Tautological ring $\tilde{R}$ is generated by the pull backs of $W_i, 1 \leq i \leq g$ and $c_1(\mathcal{O}_P(1))$, generalising the result of Beauville in smooth case.

4.4 *Syzygy bundles and Brill-Noether theory on Nodal Curves*

Assume that the base field k has characteristic 0. Let ω_Y denote the dualising sheaf of Y. Since Y is a Gorenstein curve, ω_Y is locally free.

Definition 4.7 — For a torsion free sheaf E on Y generated by global sections, define a torsion free sheaf $D(E)$ on Y by

$$0 \to D(E)^* \to H^0(Y, E) \otimes \mathcal{O}_Y \to E \to 0. \tag{4.2}$$

We note that $D(E)$ is locally free if and only if E is locally free, $D(D(E)) = E$.

The following generalisation of a result by Paranjape and Ramanan [78, Corollary 3.5] holds over Y.

***Theorem* 4.8** — *Let $g \geq 2$.*

(1) The vector bundle $D(\omega_Y)$ is semistable.

(2) If Y is nonhyperelliptic, then $D(\omega_Y)$ is stable.

The proof of Theorem 4.8 by Paranjape and Ramanan [78] in case Y is a smooth curve (of genus at least 2) crucially uses the fact that if W is a vector bundle of rank r generated by sections on Y, then W is generated by $r + 1$ independent sections. In case Y is nodal or cuspidal, W could be non locally free and may not be generated by $r + 1$ sections. Hence the proof of [78] fails to generalise to the singular case.

Unlike in the smooth case, a semistable torsionfree sheaf E on Y with $\mu(E) \geq 2g$ may not be globally generated (i.e., generated by global sections). We have the following analogue of a result of Butler [49, Theorem 1.2].

***Theorem* 4.9** — *Let E be a torsionfree sheaf on Y with $\mu(E) \geq 2g$ such that E is generated by global sections.*

(1) *If E is semistable, then the torsionfree sheaf $D(E)$ is semistable.*

(2) *Suppose that E is stable. Then $D(E)$ is stable if $\mu(E) > 2g$.*
For $\mu(E) = 2g$, $D(E)$ is stable except when Y is hyperelliptic or $E \supset \omega_Y$.

One needed a new strategy to prove Theorems 4.8 and 4.9.

Conditions for non-emptiness and irreducibility of Brill-Noether loci in $U(n,d)$ for $\mu = \frac{d}{n} \leq 2$ were determined [18, 33]. A complete classification of coherent systems of genus 1 on nodal curves is done [20].

5. The Harder-Narasimhan Trace and Hitchin Connection

5.1 *The Harder-Narasimhan trace*

Let $(\hat{R}, \hat{\theta})$ be a polarised projective variety with the action of a reductive group G. The non-semistable locus in $\hat{R}$ for the action of G is, by definition, the locus where all the G-invariant sections of $\hat{\theta}$ vanish. The locus has a stratification by Harder-Narasimhan strata [81]. Ramadas [81] defined a map, called Harder-Narasimhan trace, from the space $H^0(\hat{R}, \hat{\theta})^G$ of G-invariant sections of θ to the space $H^0(S_\mu, V_\mu)^G$ of G-invariant sections of a G-vector bundle V_μ on a G-variety S_μ. This is done by fixing a Harder-Narasimhan stratum and attaching to each section an appropriate normal derivative on the stratum.

5.2 *Moduli of parabolic bundles*

A parabolic vector bundle on a smooth curve X with n marked points is a vector bundle E together with a flag of subspaces of the fibre E_x at each marked point x and fixed real numbers called weights attached to the flag. There is a semistability notion for the parabolic bundles. The moduli space of semistable parabolic vector bundles of fixed rank and degree on X is a geometric invariant theoretic quotient of a polarised variety $(\hat{R}, \hat{\theta})$ by a reductive group G [17, 69], hence the construction of the previous section applies to it. The G-invariant sections of $\hat{\theta}$ can be identified to sections of a theta line bundle on the moduli space called generalised theta functions (conformal blocks). Over the moduli space of n-poined curves, there is a vector bundle with fibres generalised theta functions, it has a flat connection called (parabolic) Hitchin connection. Ramdas describes the Harder-Narasimhan trace on generalised theta functions in rank 2 case in geometric terms.

5.3 *Case of a Projective line with marked points (Genus 0 case)*

Specialise now to the case when X is the projective line with n distinct points. In this case, locally over the configuration space of n distinct points, the Hitchin connection can be identified with the K-Z connection. Let $Z_n = \mathbb{C}^n - \Delta$, where Δ is the generalised diagonal. Ramadas [81] introduces hypergeometric local systems (depending on parabolic structure and choice of Harder-Narasimhan stratum) on Z_n with fibres the (top) cohomology with values in a flat unitary line bundle. By explicit computation of the Harder-Narasimhan trace, unitarity of the K-Z / Hitchin connection is proved by showing that the Harder-Narasimhan trace lands injectively in a unitary factor of the hypergeometric local system. The injectivity is proved by Hodge-theoretic results of Deligne.

6. Parahoric $\mathcal{G}$–Torsors

Let X denote a smooth integral complex algebraic curve of genus $g \geq 2$. Fix points $x_1, \cdots, x_m$ of X and integers n_i attached to each x_i. Let G be a semisimple simply connected algebraic group. Fix a maximal torus T of G. Let $Y(T)$ denote the group of 1-parameter subgroups of the torus T.

Definition 6.1 — A parahoric bundle or parahoric G-torsor on X is a pair (E, θ) where E is a torsor (i.e. a principal homogeneous space) on X under a parahoric Bruhat-Tits group scheme $\mathcal{G}$ and weights $\theta \in Y(T) \otimes \mathbb{Q}$.

Parahoric torsors (without weights) were studied by Pappas and Rapoport (see [79, 80]), many results on moduli stacks of these torsors were proved by Heinloth [60] over arbitrary ground fields.

6.1 *Work of Seshadri and Balaji*

To a set τ of conjugacy classes of G, one can associate weight $\theta_\tau = \{\theta_i\} \in (Y(T) \otimes \mathbb{Q})^m$ and a parahoric group scheme $G_{\theta_\tau, X}$ with only ramification points x_i. Balaji and Seshadri [13] introduced notions of stability and semistability for parahoric G-torsors. They constructed the moduli spaces $M_X(G_{\theta_\tau, X})$ of semistable parahoric G-torsors and showed that they are irreducible, normal, projective varieties.

Let K be a maximal compact subgroup of G. Balaji and Seshadri [13] showed that there is a Fuchsian group π and a homeomorphism of the space of conjugacy classes of representations of π in K of local type τ onto the moduli space $M_X(G_{\theta_\tau, X})$. This homeomorphism identifies the subset corresponding to irreducible representations with the subset of $M_X(G_{\theta_\tau, X})$ corresponding to isomorphism classes of stable $G_{\theta_\tau, X}$-torsors. This generalises the work of Seshadri and Mehta [69] on parabolic vector bundles to parahoric G-torsors.

6.2 *The differential geometric analogue*

Let $H^{\mathbb{C}}$ be a complex reductive group. Fix a maximal compact subgroup $H \subset H^{\mathbb{C}}$, a maximal torus

$T \subset H$ and an alcove $\mathcal{A}$ in the Lie algebra of T. Let $\bar{\mathcal{A}}$ be the closure of $\mathcal{A}$. Let $\alpha = (\alpha_1, \cdots, \alpha_m)$ be a collection of elements of $\iota\bar{\mathcal{A}}$, ι being the root of (-1). A parabolic principal $H^{\mathbb{C}}$- bundle of weight α is a (holomorphic) principal $H^{\mathbb{C}}$-bundle together with a choice of weight α_i at each x_i. The weight α_i determines a parabolic subgroup P_{α_i} in the fibre E_{x_i} of E. Let D denote the set of points $x_i, i = 1, \cdots, m$.

Let $\mathcal{G}_\alpha^{hol}$ be the sheaf of groups on X such that for an open subset $U \subset X$, $\mathcal{G}_\alpha^{hol}(U)$ is the group of holomorphic maps $g : U \to H^{\mathbb{C}}$ with $g(x_i) \in P_{\alpha_i}$ if $x_i \in U$. Define $\mathcal{G}_\alpha^{mer}$ to be the sheaf on X such that for an open subset $U \subset X - D$, $\mathcal{G}_\alpha^{mer}(U)$ is the group of holomorphic maps $g : U \to H^{\mathbb{C}}$ and for small discs Δ_i centred at x_i and disjoint from other x_j, $\mathcal{G}_\alpha^{mer}(\Delta_i)$ is the group of meromorphic maps $U \to H^{\mathbb{C}}$ of the form $g(z_i)exp(n_i/z_i)$ where $g : \Delta_i \to H^{\mathbb{C}}$ is a holomorphic map with $g(0) \in n_i$ where n_i is a certain subgroup of P_{α_i} and z_i is a local coordinate in Δ_i, $z_i(x_i) = 0$.

Given a holomorphic principal $H^{\mathbb{C}}$-bundle E, one can associate a $\mathcal{G}_\alpha^{hol}$-module $\mathcal{E}$ to it [51, Remark 2.1]. Define $\mathcal{E}^{mer} = \mathcal{E} \times_{\mathcal{G}_\alpha^{hol}} \mathcal{G}_\alpha^{mer}$ (extension of structure group). The sheaf $\mathcal{G}_\alpha^{mer}$ is the holomorphic analogue of the Bruhat-Tits group scheme $G_{\theta_\tau, X}$ and $\mathcal{E}^{mer}$ is the holomorphic analogue of parahoric bundles.

7. Rationality and Brauer groups of moduli spaces

We start with a few definitions.

Definition 7.1 — (1) A variety Z is called unirational if there is a dominant rational map $\mathbb{P}^m \dashrightarrow Z$ for some positive integer m.

(2) A variety Z is called rational if it is birational to $\mathbb{P}^m$ for some positive integer m, i.e. there is an isomorphism from an open subset of $\mathbb{P}^m$ onto an open subset of Z.

(3) A variety Z is called *stably rational* if $X \times \mathbb{P}^m$ is rational for some positive integer m.

Rationality implies stable rationality and stable rationality implies unirationality, the reverse implications are not true.

Definition 7.2 — The (cohomological) Brauer group of a quasi compact scheme is defined to be the torsion subgroup of the second étale cohomology group $H^2_{et}(X, G_m)$.

Gabber had shown that for any scheme with an ample line bundle (in particular for a quasi projective variety over a commutative ring), the cohomological Brauer group coincides $H^2_{et}(X, G_m)$.

7.1 *The Brauer groups of Moduli spaces over a Smooth Curve*

Let X be an irreducible projective smooth curve of genus $g \geq 2$ over an algebraically closed field of characteristic 0. Fix a line bundle L of degree d on X. Let $U_X(n, L)$ denote the moduli space of semistable vector bundles of rank r and determinant $\wedge^r E$ isomorphic to L. Denote by $U_X^s(n, L)$ its subset corresponding to stable vector bundles. The rationality and Brauer group of moduli spaces have been investigated over years ([7, 52, 67, 73] to name a few). The moduli space is simply connected and it is unirational [73]. For n, d mutually coprime [67] and for $g = n = d = 2$, the moduli space is known to be rational. In rest of the cases, it is not even known if it is stably rational.

For a variety to be rational (or even stably rational), it is necessary that its desingularisation has Brauer group trivial.

***Theorem* 7.3** — *[42]. Let X be an irreducible projective smooth curve of genus $g \geq 2$ over an algebraically closed field of characteristic 0. The Brauer group of any desingularisation of $U_X(n, L)$ is trivial.*

This generalises the result for rank 2 in [4] and [74]. The proofs in [4] and [74] use the explicit description of the desingularisation of $U_X(2, L)$. The proof in [42] is quite different, it uses the restriction of the projective universal bundle over $U_X(n, L) \times X$ to $U_X(n, L) \times x$, x being a point of X.

The Brauer group of $U_X^s(n, L)$ is isomorphic to $\mathbb{Z}/h\mathbb{Z}$ where h is the greatest common divisor of n and d.

Let G be a semisimple algebraic group and Z_G^v the group of characters of the centre of G. Let $M_X(G)$ denote the moduli stack of principal G-bundles on the smooth curve X. Let $M_X^{ss}(G)$ be the moduli space of semistable principal G-bundles on X and $M_X^{rs}(G)$ its regular locus. One has $\pi_0(M_X^{ss}(G)) = \pi_1(G)$, in particular if G is simply connected, then $M_X^{ss}(G)$ is connected. For $\delta \in \pi_1(G)$, let $M_X^{\delta}(G)$ denote the connected component of $M_X(G)$ corresponding to δ.

***Theorem* 7.4** — *[41]. Assume that genus $g \geq 3$.*

(1) If G is simply connected, then $Br(M_X(G)) = 0\,; Br(M_X^{rs}(G)) = Z_G^v$.

(2) There is a surjective morphism $f : Br(M_X^{\delta, rs}(G)) \to Br(M_X^{\delta}(G))$.

Biswas and Holla [41] describe the kernel of f. They also do explicit computations for some classical groups including $SL_n, PSL_n, Sp_{2n}, PSp_{2n}$. The case of PGL_n was dealt with in [39].

Let X be a smooth projective curve over $\mathbb{C}$. Let $Q(r, d)$ be the Quot-scheme parametrising all coherent subsheaves of $\mathcal{O}_X^r$ with quotient a torsion sheaf of degree d. It is a projective $\mathbb{C}$- scheme with natural homomorphisms to the symmetric power $Sym^d(X)$ of X and $Q(1, d)$. Biswas, Dhillon and Hurtubuise [34] show that the induced homomorphisms $Br(sym^d(X)) \to Br(Q(r,d))$ and $Br(Q(1,d)) \to Br(Q(r,d))$ are isomorphisms.

7.2 *The Brauer groups and Rationality of Moduli spaces over a Nodal Curve* [28]

Let Y be an irreducible reduced curve of arithmetic genus $g \geq 2$, defined over an algebraically closed field of characteristic 0, with at most ordinary nodes as singularities. Let $U_Y(n,L)$ (resp. $U_Y'^s(n,L)$) be the moduli space of vector bundles (respectively stable vector bundles) over Y of rank n with determinant L of degree d. Let g_X denote the genus of the normalisation X.

***Theorem* 7.5** — *[27]. Assume that $g_X \geq 2$; also, if $g_X = 2 = n$, then assume that d is odd. Then:*

(1) the Brauer group

$$Br(U_Y'^s(n,L)) \cong \mathbb{Z}/h\mathbb{Z}\,,\ h = g.c.d.(n,d).$$

(2) $U_Y'^s(n,L)$ is simply connected.

***Theorem* 7.6** — *[27]. Assume that $g \geq 2$.*

(1) $U_Y(n,d)$ is birational to $U_Y(h,0) \times \mathbb{A}^{(n^2-h^2)(g-1)}$.

(2) If $h = 1$, then $U_Y(n,L)$ is rational.

(3) If $g_Y = 2 = n$, then $U_Y(n,L)$ is rational.

The following results support the conjecture that moduli spaces over rational varieties are rational.

***Theorem* 7.7** — *[27]. Let Y be a rational nodal curve.*

(1) The compactified Jacobian $\overline{J}(Y)$ of Y is rational.

(2) If $g = 1$, then $U_Y(n,d)$ is rational.

(3) If $g = 2$, and $h \leq 4$, then $U_Y(n,d)$ is rational.

(4) Assume that $g \geq 2$, $n \geq 2$, and n is coprime to d. Then $U_Y(n,d)$ is rational.

(5) For h dividing $420 = 3.4.5.7$, the moduli space $U_Y(n,d)$ is stably rational.

If the following well-known unsolved problem has an affirmative answer, then rationality (respectively stable rationality) in Theorem 7.7 in fact holds for arbitrary h.

Question : For $\mathrm{PGL}_n(k)$ acting diagonally on $\mathrm{GL}_n(k) \times \cdots \times \mathrm{GL}_n(k)$ by the conjugation action, is the quotient $(\mathrm{GL}_n(k) \times \cdots \times \mathrm{GL}_n(k))/\!\!/\mathrm{PGL}_n(k)$ rational (respectively, stably rational)?

Barik, Dey and Suhas [16] showed that for a nodal curve with two irreducible smooth components defined over an algebrically closed field of characteristic 0, each of the two components of the moduli space (Seshadri-Nagaraj moduli space) of stable torsion free sheaves of rank 2 with Euler characteristic χ odd is rational. In particular, the moduli space is rationally connected.

Let Y be a connected projective nodal curve with n components C_i of genus $g_i \geq 2$ and $n-1$ nodes P_i such that $C_i \cap C_{i+1} = P_i$. Let $U_Y(2, \chi, L)$ be the closure of the subset corresponding to vector bundles with fixed determinant L in the moduli space of torsion free sheaves of rank 2 and Euler characteristic χ, which are semistable with respect to a fixed polarisation on Y. Dey and Suhas [55] showed that every component of $U_Y(2, \chi, L)$ is rational.

8. Real Algebraic Vector Bundles and Principal Bundles

A real curve is a geometrically connected smooth projective curve X defined over the field of real numbers or equivalently, a complex curve $X_{\mathbb{C}}$ ($X_{\mathbb{C}} := X \times_{\mathbb{R}} \mathbb{C}$) equipped with an anti–holomorphic involution σ. For a holomorphic vector bundle $V_{\mathbb{C}}$ over $X_{\mathbb{C}}$, let $\overline{V_{\mathbb{C}}}$ be the C^{∞} complex vector bundle over $X_{\mathbb{C}}$ whose underlying real vector bundle is $V_{\mathbb{C}}$, and the complex structure of each fibre is the conjugate of the complex structure of the fibres of $V_{\mathbb{C}}$.

Definition 8.1 — A pseudo-real algebraic vector bundle V over X is a complex algebraic vector bundle $V_{\mathbb{C}}$ over $X_{\mathbb{C}}$ together with an anti holomorphic involution $\delta : V_{\mathbb{C}} \rightarrow \sigma^* \overline{V_{\mathbb{C}}}$ lifting σ such that the composition

$$V_{\mathbb{C}} \xrightarrow{\delta} \sigma^* \overline{V_{\mathbb{C}}} \xrightarrow{\sigma^* \overline{\delta}} \sigma^* \overline{\sigma^* \overline{V_{\mathbb{C}}}} = V_{\mathbb{C}}$$

is the identity map of $V_{\mathbb{C}}$ or negative of the identity map of $V_{\mathbb{C}}$.

If the above composition equals the identity map of $V_{\mathbb{C}}$, then V is called a real algebraic vector bundle.

Let G be a connected complex reductive affine algebraic group. A pseudo-real algebraic principal G- bundle E over X is similarly defined. One fixes an element c of the centre of the group G. If the composite $\sigma^* \overline{\delta} \circ \delta$ equals cI_E where I_E is the identity of E, then E is called pseudo-real (more precisely c-pseudo-real), if the composite is identity, then E is called real.

8.1 *Real algebraic vector bundles*

Bhosle and Biswas [25] classified the isomorphism classes of all stable real algebraic vector bundles over a Klein bottle (i.e. a geometrically connected smooth projective real curve of genus one, having no real points).

Let X be a complex elliptic curve with an antiholomorphic involution σ. Biswas and Schaffhauser [48] determined isomorphism classes of the moduli space $M_X(r,d)$ of semistable vector bundles of rank r and degree d on X as a real variety. As a complex variety, $M_X(r,d)$ is isomorphic to the symmetric power $S^h(X)$, $h=(r,d)$. They also determined components of $M_X(r,d)^\sigma$ in the coprime case. They showed that the set of isomorphism classes of indecomposable vector bundles of rank r and degree d is isomorphic as real variety to $M_X(r/h, d/h)$.

Let X be a nodal curve of arithmetic genus one defined over $\mathbb{R}$, with exactly one node, such that X does not have any real points apart from the node. Bhosle and Biswas [26] classified all isomorphism classes of stable real algebraic torsion free sheaves over X. There are no real vector bundles of odd rank on such a curve.

Bhosle and Biswas [28] determined geometric properties of moduli spaces $M_X(r,c_1,c_2)$ parametrizing slope semistable vector bundles of rank r and fixed Chern classes c_1, c_2 on a singular ruled surface whose base is a rational nodal curve. They showed that under certain conditions, these moduli spaces are irreducible, smooth and rational (when non-empty), they proved non-emptiness in some cases.

They showed that for a rational ruled surface defined over real numbers, the moduli space $M_X(r,c_1,c_2)$ is rational as a variety defined over $\mathbb{R}$, giving first non-trivial example of a singular rational variety with rational moduli spaces of vector bundles.

8.2 *Pseudo-real principal G- bundles and Pseudo-real principal Higgs bundles*

Let X be a compact complex Kahler manifold with an anti-holomorphic involution σ compatible with the Kahler structure. Biswas, García-Prada and Hurtubise [37] defined notions of stability, semistability and polystability for pseudo real principal G-bundles on X. A pseudo-real principal G-bundle (E,δ) is semistable (resp. polystable) if and only if the underlying holomorphic vector bundle E is semistable (resp. polystable). They proved that a pseudo-real principal G-bundle (E,δ) is polystable if and only if it admits a compatible Hermitian-Einstein connection.

The extended fundamental group $\Gamma(X,x_0)$ of X is the group of homotopy classes of paths from x_0 which end either in x_0 or $\sigma(x_0)$. There is a bijective correspondence between the following two sets [37]:

(1) The set of isomorphism classes of polystable pseudo-real principal G-bundles such that all the rational characteristic classes of the underlying topological G- bundles vanish.

(2) The equivalences classes of twisted representations (defined using the element c) of the extended fundamental group in a δ- invariant maximal compact subgroup of G.

All these results were also extended to pseudo-real principal Higgs bundles [37].

Pseudo real principal G-bundles on real curves were studied in [38].

9. Some Interesting Results on Vector bundles and Principal G-bundles

Let G be a connected reductive algebraic group over an algebraically closed field k.

9.1 *Donaldson-Uhlenbeck compactification*

A quasibundle is a generalisation of a principal bundle. Roughly speaking, it is a scheme $P \to X$ such that on an open subset it is a principal G-bundle (see [5] for precise definitions). For a semisimple algebraic group G, Balaji [5] proved the following semistable reduction theorem for slope-semistable quasi-bundles on a smooth complex projective variety X .

***Theorem* 9.1** — *[5, Theorem 1.1]. Let A be a discrete valuation ring with quotient field K and residue field $k = \mathbb{C}$. Let P_K be a family of G-quasibundles on $X_K := X \times SpecK$ or equivalently a G_K-quasibundle on X_K. There exists a finite extension L over K with the integral closure B of A in L, such that P_K after base change extends to a semistable H_B-quasibundle P_B on X_B.*

In case G is simple, he constructed algebraic geometric Donaldson-Uhlenbeck compactification of the moduli space of slope-semistable principal G-bundles with fixed characteristic classes c on a smooth projective surface [5, Theorem 1.2]. The boundary points in the compactification belong to $M_G^{\mu-polyst}(c-\ell)\times S^{\ell}(X)$, where $M_G^{\mu-polyst}(c-\ell)$ is the moduli space of slope semistable G-bundles with characteristic classes $c-\ell\,, \ell > 0$ and $S^{\ell}(X)$ is the $\ell-$fold symmetric power of X.

For large characteristic classes, he showed that the moduli space of semistable G -bundles is nonempty.

9.2 *Extension of structure groups of principal bundles*

Let X be a smooth projective variety over k with a fixed polarisation H. Let E be a rational G-bundle over X, i.e. a principal G-bundle defined over an open subset of X whose complement has codimension at least 2. Given a representation $\rho : G \to GL(V)$, let $E(V) = E \times_{\rho} V$ be the rational vector bundle associated to ρ. If E is semistable, then $E(V)$ is known to be semistable if characteristic is 0 [84], it may not be semistable in characteristic $p > 0$.

A vector bundle F has a unique filtration (called Harder-Narasimhan filtration)

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{r-1} \subset F_r = F\,; \mu(F_i/F_{i-1}) > \mu(F_{i+1}/F_i)\,, i = 1, \cdots, r-1. \tag{9.1}$$

Here $\mu(N) = (c_1(N).H)/rank(N)$ for any vector bundle N on X. Define $\mu_{max(F)} = \mu(F_1)\,$, $\mu_{min}(F) = \mu(F_r/F_{r-1})$.

Coiai and Holla [53] obtain the following bounds in arbitrary characteristics.

***Theorem* 9.2** — *[53]. Let the characteristic p of k be arbitrary.*

(1) There exists a constant $C(X,\rho)$ such that for every semistable rational G-bundle E on X,

$$\mu_{max}(E(V)) - \mu_{min}(E(V)) \leq C(X,\rho)\,.$$

(2) There exist constants $C(X,G)$ and $N(G)$ such that for every rational G-bundle E,

$$Ideg(F^*E) \geq pN(G)Ideg(E) + C(X,G)\,,$$

where $Ideg$ denotes an invariant of E called Instability degree and F^* denotes the Frobenius pull back.

As an application of this theorem, Coiai and Holla [53] showed that the set of isomorphism classes of semistable G-bundles with fixed Chern classes is bounded.

9.3 *Schematic Harder-Narasimhan Stratification*

The analogue of the Harder-Narasimhan filtration (see equation 9.1) for a principal G-bundle E on a smooth projective curve X (char. $k = 0$) is the canonical reduction (P,σ) where P is a parabolic subgroup of G and $\sigma : X \to E/P$ is a section [82, 83]. The section σ determines a P-bundle on X and an associated L-bundle, where L is a Levi subgroup of P. It satisfies the condition that the L-bundle E_L is semistable and for every dominant character χ on L, the associated line bundle $E_L(\chi)$ has positive degree.

Now, let $X \to S$ be a (flat) family of smooth projective curves over a Noetherian scheme S and E a principal G-bundle on X. Gurjar and Nitsure [59] showed that for each Harder-Narasimhan type τ, there is a locally closed subscheme $S^\tau \subset S$ with the universal property that a morphism of k-schemes $f : T \to S$ factors through S^τ if and only if f^*E admits a global relative canonical reduction of Harder-Narasimhan type τ. If S is reduced and Harder-Narasimhan type is constant over S then there exits a global canonical reduction over S. They deduce that the G-bundles of fixed Harder-Narasimhan type form an Artin stack.

All these results are also proved for flat families of pure sheaves of Λ-modules.

For $\mathcal{O}$-modules these results were proved by Nitsure [74].

9.4 *de Rham bundles*

The classical Chern-Cheeger-Simons theorey for bundles with flat connections has motivated questions for algebraic connections in other cohomology theories, work of Jaya Iyer, Indranil Biswas and

C. T. Simpson addressed these questions [44, 65, 66]. Let $\pi : X_U \to U$ be a smooth projective morphism of relative dimension n between nonsingular varieties. The i^{th} hypercohomology bundle of the complex $\Omega^\bullet(X_U/U)$, denoted by $\mathcal{H}^i := R^i\pi_*\Omega^\bullet(X_U/U)$, is equipped with the flat Gauss-Manin connection ∇. The pair $(\mathcal{H}^i, \nabla)$ is called the de Rham bundle or Gauss-Manin bundle of weight i. Let S denote a nonsingular compactification of U such that $D = S - U$ is a normal crossing divisor. Parabolic bundles $\overline{\mathcal{H}}^i(X_U/U))$ on S associated to the logarithmic connections were considered.

Iyer and Simpson [65] showed that the alternating sum of Chern characters (in the rational Chow group) of the parabolic de Rham bundles lies in the 0th Chow group. Similar results were proved for locally Abelian parabolic bundles. They defined Chern-Simon classes for the Deligne's canonical extension of a flat bundle on a smooth quasi-projective variety with unipotent monodromy around a smooth divisor and showed that these classes in $H^{2p-1}(X, \mathbf{R}/\mathbf{Z})$ are torsion for $p > 1$. Further, if X is projective, the Chern classes of the extension in degree ≥ 1 are torsion in any cohomology of X.

9.5 *Bundles with theta divisors*

A Castelnuovo curve is an integral (rational) nodal curve of arithmetic genus $g \geq 2$ with g nodes.

Definition 9.3 — A vector bundle E of degree 0 on an integral curve of arithmetic genus $g \geq 2$ is said to have a theta divisor if there exits a line bundle L of degree $g - 1$ such that $H^0(E \otimes L) = 0$.

A vector bundle having a theta divisor is semistable, the converse is not true.

***Theorem* 9.4** — *(Joshi and Mehta [64]).*

(1) Every semistable vector bundle on a Castelnuovo curve has a theta divisor.

(2) Let $X \to Spec(k[[t]])$ be a flat proper family of curves with the generic fibre X_η a smooth curve and the special fibre X_0 a Castelnuovo curve. Suppose that a semistable vector bundle E on X_η extends as a semistable vector bundle on X_0. Then E has a theta divisor.

The first part of the theorem is proved by using generalised parabolic bundles on the projective line, the second part follows from the first by semicontinuity of $H^0(\ )$.

9.6 *Interesting Results on related topics*

Biswas, Hogadi and Parameswaran [43] proved that the algebraic fundamental group of a smooth projective variety is same as the algebraic fundamental group of the GIT (Geometric Invariant Theoretic) quotient of the variety by a reductive group. Over the complex numbers, the same result was proved for the topological fundamental group. This result is very useful as constructions of moduli spaces in algebraic geometry are done using GIT.

Biswas, Mahan and Parameswaran [45] proved the following Cheeger-Gromoll type splitting theorem. Any closed manifold M admitting a good complexification has a finite-sheeted regular covering M' such that M' admits a fibre bundle structure with base a torus and fibre that admits a good complexification and also has zero virtual first Betti number. They gave several applications to manifolds of dimension at most 5.

Balaji, Deligne and Parameswaran [12] proved a structure theorem for affine k-subgroup schemes of GL(V) under certain general hypotheses and derived some basic semi-simplicity results as consequence. As an application, they obtained Luna's étale slice theorem in positive characteristic.

10. Hilbert-Kunz function and Hilbert-Kunz Multiplicity

Definition 10.1 — For a commutative Noetherian ring R of prime characteristic $p > 0$ and an ideal $I \subset R$ of finite colength, the *Hilbert-Kunz function* $HK_{R,I} : \mathbb{N} \to \mathbb{N}$, is defined as

$$HK_{R,I}(p^n) = \ell(R/I^{[p^n]}) = e_{HK}(R,I)q^d + O(q^{d-1}),$$

where $I^{[p^n]}$ is the ideal generated by $q = p^n$-th powers of elements of I and $e_{HK}(R,I)$ is a positive real number, called the Hilbert-Kunz multiplicity.

For the invariants Hilbert-Kunz function and multiplicity, there are no general methods (like restricting to general hyperplane sections or going to deformation) for computing them, which makes them difficult to compute them (even in dimension two cases).

It was a big open question: Is e_{HK} a rational number?

In [89-91] Trivedi discussed HK multiplicity of two dimensional graded ring R (of a projective curve) over a field k of char $p > 0$, with respect to a homogeneous ideal I of finite colength. The main point of [89] was to relate $e_{HK}(R,I)$ with a strong Harder-Narasimhan filtration of the syzygy bundle $V_{R,I}$ on Proj R, consequently one got that $e_{HK}(R,I)$ is a rational number. Moreover in [89], Trivedi computed the HK multiplicity for irreducible plane curves in terms of the Frobenius semistablity invariants s and l of the syzygy bundle $V_{\mathcal{O}_X(1)}$. Here the integer $s \geq 1$ is such that $F^{s-1*}V_{\mathcal{O}_X(1)}$ is semistable and $F^{s*}V_{\mathcal{O}_X(1)}$ is not semistable, and the integer l is the measure of how much $F^{s*}V_{\mathcal{O}_X(1)}$ is destabilized.

Combining this, with the Monsky' result on trinomials, she recently showed that the Frobenius semistablity behaviour of the syzygy bundle $V_n =$ Syz (x^n, y^n, z^n), on a trinomial plane curve, given by h, is a function of the congruence classes of p modulo $2\lambda_h$, an integer associated to the the trinomial h [96]. As one of the consequences, it was proved that the reduction mod p of V_1 is strongly semistable, for p in a Zariski dense set of primes. Moreover, for V_1 coming from finitely many trinomials, there is a common Zariski dense set of such primes.

Examples computed so far show that $e_{HK}(R, I)$ can depend on char p of the ring, and a natural question is: How do the HK multiplicities of reductions (mod p) of a given variety (in char. 0) vary as $p \to \infty$?

***Theorem* 10.2** *[91]. The Hilbert-Kunz multiplicities of the reductions to positive characteristics of an irreducible projective curve in characteristic* 0 *have a well-defined limit as the characteristic tends to* ∞.

The problem in higher dimension is still open.

Huneke (in private communication) had posed the following question : *Are there only finitely many Hilbert-Kunz functions for a set of Cohen-Macaulay local rings with a fixed Hilbert-Kunz muliplicity* e_{HK}*?*

Trivedi answered this question negatively in [94]. She gave an example of a family of one dimensional Cohen-Macaulay reduced local (also could be considered standard graded) rings $(R_\alpha, \mathbf{m}_\alpha)$ of characteristic p, parametrized by $\alpha \in \bar{\mathbb{F}}_p$, such that $HK(R_\alpha, \mathbf{m}_\alpha)(p^n) = 4p^n + \Delta_\alpha(n)$ where $\Delta_\alpha(n)$ has period of length equal to the degree of $\bar{\mathbb{F}}_p[\alpha]/\bar{\mathbb{F}}_p$. In particular this family has infinitely many HK functions with the same e_{HK}.

HK multiplicities $e_{HK}(X, \mathcal{L})$ and HK functions $HK(X, \mathcal{L})$, for Hirzebruch surfaces $X = \mathbf{F}_a$, for $a \geq 1$ and every ample line bundle $\mathcal{L}$ on X are computed [95]. Any ample line bundle $\mathcal{L}$ on $\mathbf{F}_a$ is uniquely given by $cA + dB$, for certain natural divisors A and B, where $c, d \in \mathbb{Z}_+$. In [95], $e_{HK}(X, \mathcal{L})$ is written explicitly in terms of a, c and d, and $HK(X, \mathcal{L})(q)$ as periodic functions in q, determined by a, c and d.

One motivation for the computation was to generate more complicated examples of the behaviour of Hilbert-Kunz functions and multiplicities. In spite of extensive study of this subject, concrete examples of general principles are very few. One expects that such examples can be useful in future research, e.g., to make or test conjectures.

Moreover, it answers the question of the following type, in the Hirzebruch surface case:

How do $e_{HK}(X, \mathcal{L})$ *and* $HK(X, \mathcal{L})$ *behave as* $\mathcal{L}$ *varies in the ample cone of line bundles on* X*?*

One knows the answer to this question for $(X, \mathcal{L})$, where X is an elliptic curve and $\mathcal{L}$ an ample line bundie, or for $(X, \mathcal{L}^n)$ where X is a full flag variety and $\mathcal{L} = \mathcal{O}((k-1)\rho)$ by [57] and for nodal plane curves by Monsky.

10.1 *Restriction Theorems, Frobenius pull-backs*

For the result in [91], by making crucial use of the result by Shepherd-Barron and Sun, Trivedi proved the following key lemma :

Let V be a vector bundle on a smooth projective curve in char p. If $p \geq 4(genus(X) - 1)(rank\ V)^3$, then the HN filtration of F^*V is a refinement of the Frobenius pull back of the HN filtration of V.

She extended this to higher dimensional varieties in [90]. Moreover, by applying the numerical characterisation (proved in [89]) of semistability of vector bundles over a curve to examples of Raynaud [85] and Monsky [72], she showed that some lower bound on the characteristic p (in terms of both rank of V and degree of X) is necessary.

Mehta and Trivedi [71] proved the following restriction theorem.

***Theorem* 10.3** — *Let $\mathbb{W}_\tau$ be a homogeneous bundle on $\mathbf{P}^n_k$ associated to an irreducible representation τ of $GL(n)$. Let k be an algebraically closed field of characteristic ≥ 5. Then, for a general hypersurface $H \subset \mathbf{P}^n_k$ of degree $d \geq 2$ the restriction to H of the s^{th} Frobenius pull back $F^{s*}\mathbb{W}_\tau$ is semistable, for any $s \geq 0$.*

The proof uses the following lemma [71]: For any smooth hypersurface X in $\mathbf{P}^n_k$, let $\mathcal{T}_{\mathbf{P}^n_k} \mid_X$ denote the restriction to X of the bundle associated to the standard representation. If $\mathcal{T}_{\mathbf{P}^n_k} \mid_X$ is semistable then, for any irreducible representation τ, the associated bundle $\mathcal{W}_\tau \mid_X$ is semistable.

The syzygy bundles $\mathcal{V}_d :=$ ker $H^0(\mathbf{P}^n_k, \mathcal{O}_{\mathbf{P}^n_k}(d)) \otimes \mathcal{O}_{\mathbf{P}^n_k} \longrightarrow \mathcal{O}_{\mathbf{P}^n_k}(d)$ over a field of characteristic $p > 0$ are considered in [92]. Langer [68] had proved an effective 'strong restriction theorem' for very general hypersurface of degree d provided $d < p$. He remarked that the assumption $d < p$, can be removed if there is a positive answer to one the following questions:

(a)*Is $\mathcal{V}_d$ a semistable bundle, for arbitrary n, d, and $p =$ char k?, or*

(b) is there a good estimate on $\mu_{max}(\mathcal{V}^\star_d)$, in terms of d and n?.

The semistability of the syzygy bundle $\mathcal{V}_d$ is proved in char $k = p > 0$ for certain infinite set of integers $d \geq 0$ (and for every $d \geq 0$, for $\mathbf{P}^2_k$) [92]. Moreover the question (b) is answered affirmatively.

The following sharper version of Sun's conjecture was proved [93].

***Theorem* 10.4** — *Let E be a torsion free sheaf on a smooth projective variety X. For $p \geq$*

$rank(E) + \ dim\ X - (s+2)$, if $\ dim\ X > 1$, and for any p, if $\ dim\ X = 1$:

$$I(F^*E) \leq (l-s)\mu_{max}(\Omega^1_X) + \epsilon \cdot pI(E),$$

where $\epsilon = \min\{1, s\}$.

Here $I(F^*E)$ is the instability degree of E, l is the number of nontrivial subsheaves in the HN filtration of F^*E and s is the number of subsheaves in this HN filtration of F^*E which descend to subsheaves of E.

Sun [88] had proved this result, when $\dim\ X = 1$.

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