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FINITE ELEMENT METHODS: RESEARCH IN INDIA OVER THE LAST DECADE

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This article is a summary of major contributions of Indian mathematicians to the mathematical aspects of the finite element method in the last one decade: 2008-2017. We briefly trace out the historical origins of the topic in India and abroad. A section on the method itself is included so that this review is accessible to anybody with a background in partial differential equations and numerical techniques for solving it.[1]

Key words : Finite element method; conforming; nonconforming; discontinuous Galerkin; *a priori*, *a posteriori*; error estimates.

1. Introduction

The finite element method (FEM) has been a very popular, robust numerical technique for the approximation of solutions of boundary value problems, initial-boundary value problems and functional minimization problems. Unlike the finite difference method which seeks an approximate solution by solving the system of equations obtained after approximating the derivatives in the Partial Differential Equation (PDE) by difference formulae, the FEM approximates the unknown as a linear combination of basis functions constructed specifically so as to achieve computational efficiency. The FEM involves a projection of the variational formulation corresponding to the PDE on to a finite dimensional space spanned by user specified basis functions. As pointed out in [99], the method was first proposed by Courant in 1943 but not pursued even though similar techniques were proposed a decade later in [85, 100]. Also in 1946, Schoenberg [92] proposed that for approximation and interpolation,

[1]The emphasis of this article is on original and novel contributions from our own limited perspective. Obviously, this would mean some omissions for which we offer our apologies.

piecewise polynomials (which were to become the workhorse for FEM) were the most suitable. In spite of all these developments mathematicians did not show any interest to research this idea of approximation any further. It was the engineers who started using this method vigorously as they found other approximation schemes inadequate especially in problems arising from structural mechanics and elasticity. It was in the mid 1950s that structural engineers connected the well established framework with variational methods in continuum mechanics. This led to a discretization method in which a structure was divided into elements with locally defined strains or stresses. The method became highly successful in the American and European aircraft industries. The word FEM appeared first in [30]. A typical reference from the engineering literature is [106].

As FEM became widespread, its advantages and defects began to slowly sink in and further analysis became necessary. As its mathematical foundations were being uncovered, several mathematicians working in finite difference methods moved into FEM [101]. The mathematics of FEM started blooming in the 1970's and became a subject of worthy pursuit for budding mathematicians. In India, P. C. Das (IIT Kanpur) had started work on FEMs after his visit to Dundee for a conference in 1973 and a subsequent visit to Germany. P. K. Bhattacharyya (IIT Delhi) had started his work on FEMs after interactions with pioneers in FEMs like P.G. Ciarlet, O. Pironneau and M. Bernadou during his visits to France and O. Pironneau's visits to India. The courses in mathematical theory of finite elements was intiated (thanks to suggestions of J. L. Lions to K. G. Ramanathan) in the 70's with seminars by O. Pironneau followed by a full-fledged course [29] at IISc Bangalore in the year 1975 under the IISc-TIFR programme. This was followed by the TIFR lectures [74]. The Indo-French instructional conference and symposium held at the IISc-TIFR Centre in 1986 influenced and inspired A. K. Pani, who was then a graduate student of P. C. Das working in the area of FEMs.

In fact, it is A. K. Pani who is mainly responsible for the development of FEMs in such a large scale in India. This was done initially through his research work; then with his graduate students and collaborators, and later on by leading the DST project *National Programme on Differential Equations, theory, computation and applications* from 2012 to 2017. The DST project based out of IIT Bombay involved training of students at undergraduate, postgraduate and graduate students in different parts of the country by national and international experts and played a major role in motivating students to take up Applied Mathematics as a research and career option.

The work on *a posteriori* error estimates and adaptive finite element methods (AFEMs) in India was initiated during the Indo-German Workshop on automatic differentiation, optimal control and adaptivity and applications in IIT Bombay in the year 2006. With the help of Carsten Carstensen (Humboldt Universität Zu Berlin), regular workshops were conducted throughout the country and these workshops motivated graduate students and researchers across the country to actively contribute to the development of adaptive FEM.

In this article, we attempt to outline the major contributions of Indian mathematicians to the mathematical aspects of the finite element method in the last one decade: 2008-2017. Note that the article is not exhaustive and the contributions by all Indian scientists to the area has not been addressed in this work. There are still not many researchers working on the mathematical theory of FEM in the country and there is a need of more outreach programmes which would train students in the area.

This article confines to significant contributions pertaining to linear or nonlinear elliptic PDEs of second and higher order and their applications to some time dependent problems. We would like to mention about the book in FEMs co-authored by an Indian- *Finite Elements: Theory and Algorithms* [41]. The book contains state-of-the art introductory material on FEM followed by PDEs arising in solid and fluid mechanics. It also deals with implementation issues for FEM algorithms and is accessible to post graduate students of Indian universities and also engineering students with a strong mathematical background.

The rest of the article is organised as follows. Section 2 provides a short introduction to some methods in FEM that would provide the necessary background for the reader. Section 3 describes the *medius* analysis [45] which we consider as one of the most fundamental contributions in the last one decade to the mathematical theory of FEM. This is followed by sections on non-linear problems and time-dependent problems.

Throughout this paper, an inequality $A \lesssim B$ abbreviates $A \leq CB$, where $C > 0$ is a mesh-size independent constant that depends only on the domain and the shape of finite elements. Standard notation applies to Lebesgue and Sobolev spaces.

2. Finite Element Methods

This section provides a very short introduction to conforming, nonconforming and discontinuous Galerkin (DG) methods, *a priori* and *a posteriori* error analysis in FEM and adaptive FEM.

2.1 *Conforming FEM and a priori analysis*

Let V be a Hilbert space equipped with the norm $\|\cdot\|_V$ induced by the inner-product $(\cdot,\cdot)_V$ and $u \in V$ be the solution of the variational problem:

$$a(u,v) = f(v) \quad \forall v \in V, \tag{2.1}$$

where $a(\cdot,\cdot)$ is a V-elliptic and V-continuous bilinear form and $f \in V'$, V' being the dual of V. A use of the Lax-Milgram lemma yields the well-posedness of the variational problem (2.1), see [16, 27]. The exact computation of $u \in V$ from (2.1) is practically difficult and hence it is desirable to find an approximate solution u_h from an appropriate finite dimensional space V_h, referred to as the *finite element* space. Note that here and throughout the article, h denotes the discretization parameter associated with the triangulation [27] and the finite dimensional space V_h. The choice of the FEM depends on the admissible solution space V and the bilinear form $a(\cdot,\cdot)$. The complexities in the formulation of (2.1) and a corresponding FEM arise from the order and the nature of the PDE and the boundary conditions. This information is used in the choice of the admissible space V in the variational formulation and the derivation of the bilinear form $a(\cdot,\cdot)$ and their corresponding counterparts in the discrete formulation. Suppose that a finite element method is defined as follows: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h, \tag{2.2}$$

where the bilinear form $a_h(\cdot,\cdot)$ (not necessarily same as $a(\cdot,\cdot)$) and the space V_h (not necessarily a subspace of V) are chosen appropriately. If $a_h(\cdot,\cdot)$ is identical to $a(\cdot,\cdot)$ and $V_h \subset V$, then the method is called a *conforming finite element method*.

In this case the following error estimate, which is called the quasi-best approximation, is well-known due to Céa's Lemma, see [16, 27]:

$$\|u - u_h\|_V \lesssim \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{2.3}$$

We can draw a few observations on the estimate in (2.3). Firstly, the estimate is quasi-optimal in the sense that the approximation error due to FEM is as good as the approximation by the space V_h to the target function u. Secondly, the estimate does not presume any regularity on u beyond what is known from the variational formulation (2.1), that is $u \in V$. This is possible due to mainly two reasons; the consistency of the Finite Element (FE) scheme; that is, u satisfies (2.2) and the fact that the discrete solution u_h belongs to V due to conformity of the space V_h. These two facts lead to the Galerkin orthogonality.

Pertinent questions here are whether the approximate solution u_h converges to u as $h \to 0$ and if so, at what rate does it converge. This is addressed usually in the *a priori* error analysis of finite element theory.

2.2 *A posteriori estimates and adaptive FEM*

In numerical solutions of practical problems the difficulties encountered are the overall deterioration of the numerical solution due to nature of the data in parts of the domain, local singularities arising from re-entrant corners, interior or boundary layers and so on. In these cases, the solution of a boundary value problem is *not* smooth (or regular). For about the last three decades, the question of finding computable error bounds which inherit the local behavior of the solution in the computational domain for a variety of problems have been a very active area of research.

The aim is to increase the accuracy of the FE approximation with an optimal use of additional degrees of freedom, thereby reducing the computaional effort. One way to attain this is to use a *graded mesh* in case the regions where the solution is not smooth is known *a priori*. However, this is not the case in most of the situations and hence it is a better idea to adaptively perform local grid refinement in those parts of the domain where the computed local error is larger than a prescribed threshold value. For this, the FE computations are performed on an initial grid, say $\mathcal{T}_0$, and then error indicators which can *a posteriori* be extracted from the computed numerical solution and the data of the problem are evaluated. The computation of the *a posteriori* indicator should not be expensive and moreover, it should be local. This estimator should also yield reliable upper and lower bounds for the true error in a user-specified norm. A typical *a posteriori* estimator will have the form Verfürth [102]

$$\sum_{T\in\mathcal{T}_h} \eta_T^2 \lesssim \|u-u_h\|_V \lesssim \sum_{T\in\mathcal{T}_h} \eta_T^2,$$

where η_T denotes the residual *a posteriori error indicator* of the triangulation $\mathcal{T}_h$ and depends only on the computed solution u_h and the given data. The residual η_T, which is computed by element-wise integration by parts, consists of two components: *element residuals* that are obtained by plugging in the computed discrete solution element-wise to the strong form of the solution and *edge residuals* which consist of interelement jumps of the trace operator that relates the strong and weak forms of the PDE.

In short, the purpose of the estimator is to identify the parts of the domain that induces large errors and use this information to locally refine and then repeat the finite element computation. Different types of estimators are available in literature but we focus only on the *residual* type estimator in the context of FEMs in this article.

Once we have the *a posteriori* estimator for a problem, the standard adaptive algorithm starts with an initial coarse triangulation $\mathcal{T}_0$ followed by an iterative procedure:

$$\mathbf{SOLVE}, \mathbf{ESTIMATE}, \mathbf{MARK}, \mathbf{REFINE}$$

for different levels, say $l = 0, 1, 2, \cdots$. In the step **SOLVE**, the discrete FE solution is computed corresponding to a triangulation $\mathcal{T}_l$. In the second step **ESTIMATE**, an *a posteriori* error indicator is computed for every element in the triangulation $\mathcal{T}_l$. In the step **MARK**, the elements where the error is more (resp. less) than a prescribed tolerance are marked for refinement (resp. coarsening). The last step **REFINE** involves a refinement (or possible coarsening) of the marked elements.

In practice, it has been observed that the adaptive algorithms do converge. The work on rigorous mathematical justification of this assertion along with the rates of convergence of the adaptive algorithms started more recently. The basic idea is to establish the convergence of AFEM by proving the contraction of the errors between two consecutive adaptive steps. Another question which is of current research interest is to elucidate whether or not the adaptive meshes generated by the algorithm are optimal and have affirmative answers for some problems.

2.3 *Nonconforming FEM*

Recall the results outlined in Subsection 2.1 and in particular (2.2). If the space $V_h \not\subset V$, then the discrete formulation is referred to as *nonconforming FEM*. In this case, the discrete solution u_h does not satisfy (2.1) and this leads to a *consistency error* in addition to the *approximation error* given in (2.3). The error estimates for nonconforming methods are based on the generalizations of Céa's lemma [14] and also permits the evaluation of $a_h(\cdot,\cdot)$ using quadrature formulas. Examples of such nonconforming methods are plenty in the literature of FEM, say constructions of FE spaces using the discontinuous Crouziex-Raviart FE, Morley FE, to name a few prototypes.

There are several reasons for the choice of nonconforming FEMs over conforming FEMs. In the case of fourth order elliptic boundary value problems which has applications in plate bending problems, nonconforming Morley FE based on piecewise quadratic polynomials are much simpler to use. They have fewer degrees of freedom in comparison to conforming C^1 FE subspaces which would for example, use Argyris FE with 21 degrees of freedom in a triangle or Bogner-Fox-Schmit FE that have 16 degrees of freedom in a rectangle.

One of the most popular nonconforming FEMs is the method based on discretizing the velocity with lowest order Crouzeix-Raviart element and the pressure with piecewise constant functions for Stokes and Navier-Stokes equations. The degrees of freedom for these methods are much lower than that of the conforming FEM thus making the implementation simpler without compromising on the convergence rates. Moreover, a use of nonconforming FEM also helps to establish the inf-sup condition sometimes with ease and this condition is crucial to establish the well-posedness of solution of the mixed FE formulation [12].

The challenges encountered in the study of *a priori* and *a posteriori* error analysis and the convergence of adaptive schemes for nonconforming FEM for various problems of practical interest has been an area of active research interest internationally.

2.4 *Discontinuous Galerkin methods*

The DG methods are nonconforming methods and are based on FE spaces that consist of discontinuous piecewise polynomials defined on a partition of the computational domain. They have existed in various forms for more than three decades. Though the method was extensively used to approximate solutions of first order PDEs and time dependent hyperbolic PDEs initially, in the 70's the work picked up for diffusion, see [77, 78]. The DG methods are attractive because they are elementwise conservative, are flexible with respect to local mesh adaptivity, are easy to implement than finite volume schemes and can handle nonuniform degrees of approximations for solutions whose smoothness exhibit variation over the computational domain.

There has been a tremendous development of these methods over the last decade especially for elliptic problems due to the fact that efficient discretizations of second order and fourth order terms have been derived. The derivation of the DG methods to approximate the model problem (3.2) on a given mesh depends on the fact that the jumps of the potential and of the normal component of the diffusive flux vanish across interfaces. A suitable discrete bilinear form that satisfies the consistency requirement and discrete coercivity is derived. The boundary and jump conditions are weakly enforced on the discrete solution using penalty terms. For more details on the method, we refer to [32, 89]. The Indian contributions to the *a priori* and *a posteriori* analysis of DG methods and its variants for non-linear elliptic problems, optimal control problems and variational inequalities have been significant and are outlined later in the paper.

3. Medius Analysis : A Blend of *a Priori* and *a Posteriori* Analysis

In this section, we highlight one of the fundamental contributions to numerical analysis [45] that we think has made an international impact.

For the NCFEM and DGFEM outlined in Subsections 2.3 and 2.4, a natural question to ask is whether the best approximation result (2.3) holds when the space $V_h \not\subset V$ and $a_h(\cdot,\cdot) \neq a(\cdot,\cdot)$. In [45], for the first time, a best approximation result analogous to Céa's lemma and Berger-Scott-Strang lemma [14] is derived without any additional regularity assumptions on the solution u. This is accomplished by combining the analysis of local efficiency estimates in *a posteriori* error analysis [102], introducing appropriate enriching operators [15] and using the norms that are well-defined for the discrete functions and the solution $u \in V$. Precisely, the estimate reads as follows:

$$\|u - u_h\|_h \lesssim \inf_{v_h \in V_h} \|u - v_h\|_h + h.o.t., \tag{3.1}$$

where *h.o.t.* stands for a higher order term for the source term $f \in L^2(\Omega)$ which is a data and $\|\cdot\|_h$ is a norm defined on $V_h + V$. To highlight the ingenious idea of the paper, consider this in the context of a simple prototype example. Let $\Omega \subset \mathbb{R}^2$ be a simply-connected bounded polygonal domain with boundary $\partial\Omega$. For a given right-hand side $f \in L^2(\Omega)$, seek a function u such that

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \;\; u = 0 \text{ on } \partial\Omega. \tag{3.2}$$

The weak formulation corresponding to this problem seeks $u \in V$ such that

$$a(u, v) = (f, v) \quad \forall\, v \in V, \tag{3.3}$$

where $V = H_0^1(\Omega)$ and $a(u,v) = \int\limits_\Omega \nabla u \cdot \nabla v \, dx, \quad (f,v) = \int\limits_\Omega fv \, dx$. If V_h is not a subspace of V then the discrete problem seeks $u_h \in V_h$ such that $a_h(u_h, v_h) = (f, v_h)\ \forall v_h \in V_h$, where

$$a_h(u_h, v_h) = \int\limits_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx,$$

and ∇_h is the piecewise gradient operator. In this case, (2.3) reads

$$\|u - u_h\|_h \le \inf_{w_h \in V_h} \|u - w_h\|_h + \chi,$$

where

$$\chi = \sup_{v_h \in V_h \setminus \{0\}} \frac{a_h(u, v_h) - (f, v_h)}{\|v_h\|_h} \tag{3.4}$$

is the inconsistency error due to the nonconformity of V_h. It is known that the exact solution to the problem satisfies the regularity $u \in H^{1+\alpha}(\Omega)$ and if Ω is convex, then $\alpha = 1$. In this case one can estimate χ in terms of $h\|u\|_{H^2(\Omega)}$. However, the domains Ω arising in applications are not convex and hence it is pertinent to estimate χ when $\alpha \le \frac{1}{2}$. Towards this end, the standard technique has been to write

$$a_h(u, v_h) - (f, v_h) = a_h(u - u_h, v_h - \Pi_h v_h) + a_h(u_h, v_h - \Pi_h v_h) - (f, v_h - \Pi_h v_h),$$

where Π_h is a map from V_h to the conforming space. But to estimate the jump in $\nabla_h(u - u_h)$ in the first term one needs $\alpha > \frac{1}{2}$. To overcome this, in [45], Π_h is replaced by the enriching/companion operator E_h which is also a map from V_h to a conforming FE space of V but is obtained by averaging the finite element solution. It is a one-to-one, well posed operator satisfying

$$||E_h v_h||_V \lesssim ||v_h||_h. \tag{3.5}$$

Next [45] postulates

$$\big|a(u, v) - a_h(u_h, v)\big| \lesssim \|u - u_h\|_h \|v\|_h. \tag{3.6}$$

These estimates are based on the so called bubble function techniques used in discrete local efficiency estimates [102] in *a posteriori* analysis. This technique does not require integration by parts involving u and hence no additional smoothness on u is required. Therefore, α can be less than or equal to half. Next the coercivity assumption in the discrete norm

$$\|v_h\|_h^2 \lesssim a_h(v_h, v_h) \tag{3.7}$$

is made. Using (3.5), (3.6) and (3.7) along with another assumption on the consistency expression (3.6) in terms of $v_h - E_h v_h$, [45] shows that χ can be bounded in terms of

$$Osc(f, \mathcal{T}_h) = \sum_{T \in \mathcal{T}_h} h_T^2 \|f - \tilde{f}\|_{L^2(T)}^2,$$

where $\mathcal{T}_h$ is a conforming triangulation of Ω, h_T is the diameter of T and $\tilde{f}$ is the mean of f over the triangle T. This is a spectacular achievement and is nowadays known as *medius* error analysis as it uses the ideas used in both *a priori* and *a posteriori* analysis for deriving the quasi-best approximation result.

The paper [2] derives the best approximation result for discontinuous FEMs for Stokes problems with much weaker conditions on the source term, for instance $f \in V'$. The comparison results of nonstandard P_1 FEM for Poisson problem [22], P_2 FEM for the biharmonic problem [23] and linear elasticity [20], unified analysis and comparison results for first order finite volume element methods [25], also employ the *medius* analysis. For the analysis of fourth order problems and optimal control problems where the standard analysis fails due to loss of regularity, see [18, 48, 53] and for nonlinear problems of contact, see [51]. The convergence analysis under minimal regularity in lower order norms can be found in [46].

The *medius* analysis is well-accepted internationally and has been applied in many problems; we cite a few important references [1, 13, 19, 39, 42, 54, 76, 98].

4. Nonlinear Problems

This section starts with FEM contributions for non-symmetric and indefinite elliptic problems and discusses second order non-monotone type nonlinear elliptic problems, very thin plate bending problems, Kirchoff type problems and optimal control problems in the subsequent subsections.

4.1 *Nonsymmetric and indefinite elliptic problems*

The general second-order linear elliptic PDE on a simply-connected bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega$ reads as: for a given right-hand side $f \in L^2(\Omega)$, seek a function u such that

$$\mathcal{L}u := -\nabla \cdot (\mathbf{A}\nabla u + u\mathbf{b}) + \gamma\, u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

It is assumed that the coefficients $\mathbf{A},\ \mathbf{b}$ and γ in (4.1) are all essentially bounded functions and that the eigenvalues of the symmetric matrix $\mathbf{A}$ are all positive and uniformly bounded away from zero.

Many problems in engineering demand a good approximation for the unknown solution and its flux, and often this issue is addressed by employing mixed FEMs where a simultaneous approximation of the requisite variables is done. For a mixed formulation corresponding to (4.1), new variables $\mathbf{p} = -(\mathbf{A}\nabla u + u\mathbf{b})$ & $\mathbf{b}^* = \mathbf{A}^{-1}\mathbf{b}$ are introduced. The mixed formulation seeks $(\mathbf{p}, u) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$(\mathbf{A}^{-1}\mathbf{p} + u\mathbf{b}^*, \mathbf{q}) - (\text{div}\ \mathbf{q}, u) = 0 \qquad \text{for all } \mathbf{q} \in H(\text{div}, \Omega) \tag{4.2}$$

$$(\text{div}\ \mathbf{p}, v) + (\gamma\, u, v) = (f, v) \qquad \text{for all } v \in L^2(\Omega), \tag{4.3}$$

where $H(\text{div}, \Omega) := \{\mathbf{q} \in (L^2(\Omega))^2 |\ \text{div}\ \mathbf{q} \in L^2(\Omega)\}$.

The state-of-the-art proof of a global inf-sup condition which is crucial for establishing the well-posedness of the mixed formulation [12] in (4.2)-(4.3) does not allow for an analysis of truly indefinite, second-order linear elliptic PDEs. Therefore, first an analysis of a nonconforming Crouziex-Raviart FE discretization which converges owing to a priori L^2 error estimates even with reduced regularity assumption of the exact solution over non-convex polygonal domains is established. An equivalence result of the nonconforming FEM and mixed FEM leads to the well-posedness of the discrete problem and to *a priori* error estimates for the mixed FEM [24].

The representation formula known as Marini identity [73] that establishes an equivalence between mixed lowest-order Raviart-Thomas and nonconforming Crouzeix-Raviart FE solutions for Poisson problem is well-known in the literature. Marini identity plays a vital role in the *a priori* and *a posteriori analysis* of nonconforming and mixed FEMs and also in multigrid analysis for the Poisson problem.

The main idea in [24] has been to effectively utilize the modified version of the representation formula to establish (a) *a priori* error estimates of general second order indefinite problems under reduced regularity assumptions on the exact solutions and (b) to develop *a posteriori* estimates and adaptive algorithms for the general second order elliptic problems using the mixed FE scheme.

For the convergence analysis of adaptive algorithms, in [35] first of all, an adaptive algorithm and a marking strategy in each step for the local refinement based on the comparison of the edge residual term and the volume residual terms of the *a posteriori estimator* are proposed. The *a posteriori* error estimator, quasi-orthogonality property and quasi-discrete reliability results which are the essential ingredients of the convergence analysis of the adaptive algorithms are derived with the help of the modified version of Marini identity.

4.2 *DG methods for nonlinear elliptic problems of non-monotone type*

The notable contributions to DG methods for non-linear elliptic problems that are of non-monotone type started with [49, 52]. The study is based on linearization and fixed point arguments. The linearized problem is a non-selfadjoint linear elliptic problem with indefinite bilinear form. The analysis is dealt with Gårding's type inequality and requires a sharper estimate in the L^2-norm than that in the energy norm, see [91]. In the case of DG methods, particularly the non-symmetric interior penalty method, the L^2-norm estimate is not sharper than the energy norm. This sub-optimality in L^2-norm leads to major difficulties in dealing with DG methods for indefinite problems.

The analysis in [49, 52] proposes a new method to deal with such difficulties called as *discrete duality argument* in the context of DG methods. The abstract arguments of the Schatz's technique based on Aubin-Nitsche duality argument and its modifications to the DG case are outlined now.

Let the bilinear form $a(\cdot,\cdot)$ in (2.1) corresponding to a second order non-linear elliptic problem be not V-elliptic but satisfies an inequality of the form $C_1\|v\|_V^2 - C_2\|v\|^2_{L^2(\Omega)} \le a(v,v)\ \ \forall v \in V := H^1_0(\Omega)$, where C_1 and C_2 are constants. Arguments as in Céa's lemma lead to

$$\|u-u_h\|_V \lesssim \inf_{v_h\in V_h} \|u-v_h\|_V + \|u-u_h\|_{L^2(\Omega)}. \tag{4.4}$$

To obtain an estimate for the error in $L^2(\Omega)$-norm for (4.4), consider a dual problem that seeks $\phi \in V$ such that

$$a(v,\phi) = (u-u_h, v) \quad \forall v \in V, \tag{4.5}$$

which satisfies the elliptic regularity, say $\|\phi\|_{H^2(\Omega)} \lesssim \|u-u_h\|_{L^2(\Omega)}$. If the bilinear form $a(\cdot,\cdot)$ is adjoint consistent, then using the Galerkin orthogonality, it can be noticed that for $\phi_h \in V_h$ that

$$\|u-u_h\|^2_{L^2(\Omega)} = a(u-u_h,\phi) = a(u-u_h,\phi-\phi_h) \lesssim h\|u-u_h\|_V\|\phi\|_{H^2(\Omega)},$$

and hence $\|u-u_h\|_{L^2(\Omega)} \lesssim h\|u-u_h\|_V$. For sufficiently small mesh size h, from (4.4), it can be deduced that

$$\|u-u_h\|_V \lesssim \inf_{v_h\in V_h} \|u-v_h\|_V. \tag{4.6}$$

However, we note here that if the bilinear form $a(\cdot,\cdot)$ is not adjoint consistent or that the adjoint problem (4.5) does not have $H^{1+\alpha}(\Omega)$ elliptic regularity for any $\alpha > 0$, then the above arguments do not yield a sharper error estimate in L^2-norm and hence the error estimate in energy norm may not be derived as in (4.6). However for some projection $\Pi_h u \in V_h$ of u, consider the discrete dual problem that seeks $\phi_h \in V_h$ such that

$$a(v_h, \phi_h) = (\Pi_h u - u_h, v_h) \quad \forall v_h \in V_h. \tag{4.7}$$

It is not difficult to show that the discrete problem (4.7) has a unique solution by using the regularity of the original problem (2.1) and Aubin-Nitsche duality arguments. Further it can be shown that $\|\phi_h\|_V \lesssim \|\Pi_h u - u_h\|_{L^2(\Omega)}$. Then using the Galerkin orthogonality $a(u - u_h, \phi_h) = 0$, we obtain

$$\begin{aligned}\|\Pi_h u - u_h\|^2_{L^2(\Omega)} &= a(\Pi_h u - u_h, \phi_h) = a(u - u_h, \phi_h) \lesssim \|u - u_h\|_V \|\phi_h\|_V \\ &\lesssim \|u - u_h\|_V \|\Pi_h u - u_h\|_{L^2(\Omega)},\end{aligned}$$

which implies that $\|u - u_h\|_{L^2(\Omega)} \lesssim \|u - u_h\|_V + \|u - \Pi_h u\|_{L^2(\Omega)}$. Hence the energy norm estimate in (4.4) takes the form

$$\|u - u_h\|_V \lesssim \inf_{v_h \in V_h} \|u - v_h\|_V + \|u - \Pi_h u\|_{L^2(\Omega)}.$$

These arguments are useful in the analysis of nonlinear elliptic problems as the corresponding linearized problems involve non-symmetric and non-selfadjoint problems.

Other major contributions for nonsymmetric indefinite problems using DG methods include analysis of superconvergent DG methods for (4.1). With the help of an auxiliary problem which is a discrete version of a linear non-selfadjoint elliptic problem in the divergence form, optimal error estimates for flux and superconvergent results for potential are developed; see [103]. A post-processing technique helps establish superconvergence for the discrete potential. Another work in this direction [104], establishes superconvergence results for quasilinear elliptic problems of the form $-\nabla \cdot (\mathbf{A}(u)\nabla u) = f$ in Ω.

4.3 *The von Kármán plates*

Plate bending problems are ubiquitous in the study of computational mechanics. Several plate models have been studied in literature and prominent ones in applications include linear models like Kirchhoff and Reissner-Mindlin plates for thin and moderately thick plates and von Kármán plates for very thin plates. Indian contributions to the study of FE analysis of general fourth order linear elliptic plate bending problems started in the late 70's; see [8-11, 62, 63]. In this subsection, we present some recent works.

The von Kármán plates proposed in 1910 are modeled by a semi-linear system of fourth-order partial differential equations with transverse displacement and Airy stress as unknown variables. They act as a prototype to *semilinear problems with trilinear nonlinearity*. The displacements in these plates are so large that a non-linear model is essential to consider the membrane action. For a given load function $f \in L^2(\Omega)$, the problem is to seek displacement and Airy stress functions $\psi_1, \psi_2 \in H_0^2(\Omega)$ such that

$$\Delta^2\psi_1 = [\psi_1, \psi_2] + f \text{ and } \Delta^2\psi_2 = -\frac{1}{2}[\psi_1, \psi_1] \quad \text{in } \Omega$$

with the biharmonic operator Δ^2 and the von Kármán bracket $[\bullet, \bullet]$ defined by $\Delta^2\varphi := \varphi_{xxxx} + 2\varphi_{xxyy} + \varphi_{yyyy}$ and $[\eta, \chi] := \eta_{xx}\chi_{yy} + \eta_{yy}\chi_{xx} - 2\eta_{xy}\chi_{xy}$.

Conforming FEM for fourth order problems that are subspaces of $H_0^2(\Omega)$ lead to the choice of C^1 FE spaces and are difficult to implement. Nonconforming and discontinuous Galerkin FEMs circumvent this problem and yield quasi-optimal error estimates. The non-standard methods for this problem is a contribution to the literature on continuum plate theory and simulations and should be of interest to researchers and engineers working in areas of structural mechanics and FE simulations of bodies supporting deflection and rotation such as plates and shells.

Numerical analysis for von Kármán and generalized von Kármán equations are studied in [21, 28, 70, 71]. Existence, uniqueness and optimal order error estimates are obtained for conforming, nonconforming and DG FEMs applied to von Kármán equations under realistic regularity assumptions on the exact solution, that is, $u \in H^{2+\alpha}(\Omega),\ \alpha \in (1/2, 1]$. As in the case of second order problems discussed earlier, the key ingredient in proving well-posedness of the discrete solution relies on rewriting the discrete problem in the framework of a fixed point formulation. Then, an appropriate nonlinear map seeks a unique fixed point which is the solution to the discrete problem. Optimal order error estimates are established for the nonconforming Morley FE [71] and DGFEM [21]. Analysis for a C^0 interior penalty method, which is a DG method based on C^0 Lagrange FE space is also discussed. The Morley or DG FE based on piecewise quadratic polynomials are simpler and have fewer degrees of freedom relative to the conforming counter part and are easier to implement than mixed or hybrid FEMs. However, the difficulties due to non-conformity of the FE space increases the technicalities in the analysis. An important aid in the proof is an enrichment operator that maps the nonconforming FE space to that of the conforming space. A residual-based $a\ posteriori$ error analysis and adaptive FEM for conforming, nonconforming and DG FEMs for the von Kármán equations that allows reliable and efficient error control has also been studied.

Due to their important role in geometry and optimization, there has been a growing interest and a surge of papers in recent years towards developing numerical schemes for the Monge-Ampère equation described by: given data functions f and g, seek u such that

$$[u, u] = f \ \text{ in } \ \Omega, \quad u = g \ \text{ on } \ \partial\Omega,$$

where $[\cdot,\cdot]$ denotes the von Kármán bracket, also referred to as the Monge-Ampère bracket. The numerical methods based on finite difference and vanishing moment techniques were available in literature. While the finite difference schemes restrict the mesh and regularity, the vanishing moment technique lacks in robustness. The work in [17] develops a C^0 penalty method to remedy some of the above difficulties.

4.4 *Kirchoff type problems*

Kirchhoff type nonlocal nonlinear problems that seek u such that $-M(\int_\Omega |\nabla u|^2)\Delta u = f(x,u)$, involves a global integral as a coefficient. This nonlocal coefficient introduces a major challenge in the computation of the numerical solution using Newton-Raphson method because the corresponding Jacobian matrix is dense. The storage of this Jacobian requires a huge memory and involves excessive floating point operations. Using a constrained formulation, this difficulty has been remedied and the solution process can be treated just as in the case of point-wise equations [47]. A substitution $\int_\Omega |\nabla u|^2 = d$ transforms the original problem to $M(d)\Delta u = f(x,u)$ and FE discretization of this problem leads to a nonlinear system of equations whose Jacobian has the structure $J = \begin{bmatrix} A & b \\ c & -1 \end{bmatrix}$, where A is the Jacobian matrix obtained from the discretization of the equation while b and c are vectors. This can be inverted efficiently using the Sherman-Morrison-Woodbuy formula or the block elimination with a single refinement technique. It is shown that the FE solution satisfies the same properties of boundedness and uniqueness that is satisfied by the exact solution. This work is useful for developing FE based methods for many other nonlocal problems of Kirchhoff type; both stationary and evolutionary.

An immediate application of this work is conforming and expanded mixed FEMs for the Kirchhoff equation of elliptic type [33]:

$$-(1 + \|\nabla u^2\|)\Delta u = f \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega. \tag{4.8}$$

The study of the elliptic type problem (4.8) is the first important step towards the understanding of nonlocal time-dependent problems. The conforming FEM is applied to more general cases of the problems discussed in [47]. Here, a mixed formulation in terms of the three variables, that is, displacement u, gradient q and flux $\boldsymbol{\sigma}$ (the coefficient times of the gradient) is proposed, such that

$$\nabla u = \mathbf{q}, \ \ (1 + \|\mathbf{q}^2\|)\mathbf{q} = \boldsymbol{\sigma}, \ \ \text{div } \boldsymbol{\sigma} = f.$$

A priori and *a posteriori* estimates of conforming and expanded mixed FEMs for the Kirchhoff equation of elliptic type are derived.

In [67], the existence of a global strong solution for all finite time for the Kirchhoff's model of parabolic type is derived. Based on an exponential weight function, some new regularity results that reflect the exponential decay property are obtained for the exact solution. The existence of a global attractor is shown, semi-discrete and fully-discrete FE schemes are analysed and optimal error estimates are derived.

4.5 *FEM for control and obstacle problems*

The standard *Dirichlet boundary control problem* is defined by

$$\min_{(y,u)\in L^2(\Omega)\times L^2(\partial\Omega)} J(y,u)$$
$$\text{subject to } (y,-\Delta\varphi)_{L^2(\Omega)} = (f,\varphi)_{L^2(\Omega)} + (u,\partial\varphi/\partial n)_{L^2(\partial\Omega)} \quad \forall\varphi\in H_0^1(\Omega)\cap H^2(\Omega),$$

where $f\in L^2(\Omega)$ is given and the cost functional $J : L^2(\Omega)\times L^2(\partial\Omega)\to\mathbb{R}$ is defined by

$$J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\partial\Omega)}, \quad y\in L^2(\Omega),\ u\in L^2(\partial\Omega),$$

for a given desired state $y_d\in L^2(\Omega)$ and regularizing parameter $\alpha>0$. In this approach, the difficulty is that the state variable y is sought from $L^2(\Omega)$ and hence has low regularity. Furthermore, the well-posedness the problem relies on the regularity of the dual problem and may restrict the smoothness of the domain and the type of boundary conditions. Alternatively, in [26] the Dirichlet boundary control problem is re-formulated as an optimal control problem defined by:

$$\min_{(y,u)\in H^1(\Omega)\times H^1(\Omega)} J(y,u) \quad \text{subject to}$$
$$y = y_0+u \quad y_0\in H_0^1(\Omega) \text{ and } (\nabla y_0,\nabla\varphi) = (f,\varphi) - (\nabla y,\nabla\varphi) \quad \forall\varphi\in H_0^1(\Omega),$$

where $J : H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ is defined by $J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\nabla u\|^2_{L^2(\Omega)},\ y\in H^1(\Omega),\ u\in H^1(\Omega)$. This approach provides sufficiently smooth control and the state on polygonal domains unlike the case where the control variable is sought from $L^2(\partial\Omega)$. It is also observed that if the regularizing parameter small enough, the state converges to the desired y_d.

Another problem of interest is the *distributed optimal control* problem defined by

$$\min_{u\in U_{ad}} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \quad \text{subject to}$$
$$\Delta^2 y = f+u \quad \text{in }\Omega,\ \ y|_{\partial\Omega} = \frac{\partial y}{\partial n}|_{\partial\Omega} = 0,$$

where the load function $f \in H^{-1}(\Omega)$, $U_{ad} \subset L^2(\Omega)$ is a non-empty, convex and bounded admissible set of controls defined by $U_{ad} = \{u \in L^2(\Omega) : u_a \leq u(x) \leq u_b \quad a.e. \text{ in } \Omega\}$, $u_a,\ u_b \in \mathbb{R} \cup \{\pm\infty\},\ u_a < u_b$ are given and y_d is the given observation for y. *A priori* and *a posteriori* error estimates for a C^0 interior penalty method for these problems are analyzed in [53]. In [72], *a priori* error estimates are derived for the mixed FE discretization where the state and co-state are discretized using Raviart-Thomas FE space and the control variable is approximated by piecewise constant functions. More recently in [36, 37], optimal control problems governed by diffusion equation with Dirichlet and Neumann boundary conditions are analyzed in the framework of *gradient discretization* schemes that covers various numerical methods, like conforming FE, nonconforming FE, mimetic finite differences and so on.

The results in [50] establish a reliable residual based *a posteriori* error estimator for elliptic obstacle problems using piecewise quadratic FE where the Lagrange multiplier is approximated from the Crouziex-Raviart nonconforming FE space. In [38], a quadratic FE enriched with element-wise bubble functions is proposed and both *a priori* and *a posteriori* error estimates are analyzed for three dimensional elliptic obstacle problems. The article [34] deals with the analysis of a nonconforming FEM for the discretization of optimal control problems governed by variational inequalities. Here, the state and adjoint variables are discretized using Crouzeix-Raviart nonconforming FE and the control is discretized using a variational discretization approach.

5. Time Dependent Problems: A Brief Survey

In this section, we outline briefly some contributions to parabolic interface problems, partial integro-differential equations, Kelvin-Voigt problems and other applications.

5.1 *Interface problems*

Parabolic interface problems arise in various applications ranging from material science to fluid dynamics when two distinct materials or fluids with different conductivities or densities or diffusion coefficients are involved. These problems usually lead to solutions that are non-smooth across the interface.

Let Ω be a bounded convex polygon in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, Ω_1 be a subdomain of Ω with C^2 boundary. The interface Γ divides the domain Ω into two subdomains Ω_1 and $\Omega_2 := \Omega \setminus \Omega_1$. Consider the linear parabolic interface problem of the form

$$u_t(x,t) - \nabla \cdot (\beta(x)\nabla u) = f(x,t) \text{ in } \Omega \times (0,T]$$

with initial and boundary conditions $u(x,0) = u_0(x)$ in Ω; $u = 0$ on $\partial\Omega \times [0,T]$ and jump conditions

$[u] = 0$, $\left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = 0$ across the interface $\Gamma \times [0,T]$, where the jump $[v]$ is defined by $[v](x) = v_1(x) - v_2(x)$, $x \in \Gamma$ with $v_i(x) = v(x)|_{\Omega_i}$, $i = 1, 2$. The symbol **n** denotes the unit outward normal to the boundary $\partial\Omega_1$. The diffusion coefficient $\beta(x)$ is assumed to be positive and piecewise constant on each subdomain. The initial function $u_0(x)$ and the forcing term $f(x,t)$ are real valued functions. For works related to elliptic and parabolic interface problems, see [31, 82, 94-97].

AFEM for parabolic interface problems is a challenging research area and the work in [93] is an attempt in this direction. New approximation properties of Clément-type interpolation operator are established and have been used to derive elliptic reconstruction error estimate in L^2-norms. This estimate plays an instrumental role in deriving *a posteriori* error bounds in the $L^\infty(L^2)$ norm.

Problems on nonmatching grids arise in many engineering applications and mortar FEM has attracted plenty of attention due to its intriguing features like flexibility to handle different types of nonconformities and complicated geometries. This technique adapts naturally into domain decomposition framework. We refer to [83, 84] for mortar FE related work for parabolic and interface problems.

5.2 *Partial Integro-Differential Equations*

Parabolic Integro-Differential Equations (PIDE) and their variants arise in heat conduction of materials with memory, compression of poro-viscoelasticity media, nuclear reactor dynamics, epidemic phenomena in biology and so on.

In [81], *a priori* error bounds are derived for an hp-local discontinuous Galerkin (LDG) approximation to a PIDE of the following type:

$$\begin{aligned} u_t - \nabla \cdot \left(a(x)\nabla u + \int_0^t b(x,t,s)\nabla u(s)ds \right) \quad &= \quad f(x,t), \quad \text{in } \Omega \times [0,T], \\ u(x,t) = 0 \ \text{ on } \partial\Omega \times [0,T], \qquad & \quad u(x,0) = u_0(x) \ \text{ in } \Omega. \end{aligned}$$

It is shown that the error estimates in L^2-norm of the gradient as well as of the potential are optimal with respect to the discretizing parameter h and suboptimal with respect to the degree of polynomial p. Due to the presence of the integral term, an introduction of an expanded mixed type Ritz-Volterra projection aids in achieving optimal estimates. For developments in mixed FEM, semi-discrete Galerkin methods and domain decomposition procedures for PIDE with non-smooth initial data, we refer to [43, 44, 86].

The paper [57] develops and analyzes a FEM combined with implicit-explicit time semi-discretizations for PIDE arising in pricing American options under Merton's and Kou's jump-diffusion models.

The *a priori* analysis for PIDE has been well-studied and now we describe some work related to *a posteriori* error analysis and space time adaptive algorithms for the numerical solutions to PIDE. The work [75] considers an initial-boundary problem for linear PIDE of the form

$$u_t(x,t) + \mathscr{A}u(x,t) = \int_0^t \mathscr{B}(t,s)u(x,s)ds + f(x,t), \quad (x,t) \in \Omega \times (0,T],$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], u(x,0) = u_0(x), \quad x \in \Omega.$$

Here, $\Omega \subset \mathbb{R}^d, d \geq 1$ is a bounded convex polygonal or polyhedral domain with boundary $\partial\Omega$. Further, $\mathscr{A}$ is a self-adjoint, uniformly positive definite, second-order linear elliptic partial differential operator of the form $\mathscr{A}u = -\nabla \cdot (A\nabla u)$ and the operator $\mathscr{B}(t,s)$ is of the form $\mathscr{B}(t,s)u = -\nabla \cdot (B(t,s)\nabla u)$, where $A = (a_{ij}(x))$ and $B(t,s) = (b_{ij}(x;t,s))$ are two $d \times d$ matrices assumed to be in $L^\infty(\Omega)^{d\times d}$. Moreover, the entries of $B(t,s)$ are assumed to be smooth in both t and s and the initial functions $u_0(x)$ and the source function $f(x,t)$ are assumed to be smooth. A novel operator called Ritz-Volterra reconstruction operator which is a generalization of the elliptic reconstruction operator is constructed for this problem. This operator plays a very crucial role in the derivation of $L^\infty(L^2)$ *a posteriori* error estimates and can be appropriately modified for a class of PIDE. The Ritz-Volterra reconstruction operator is the counterpart of the Ritz-Volterra operator [69] in the *a priori* error analysis and helps to unify the *a posteriori* analysis for parabolic problems and PIDE.

A posteriori error analysis for PIDE in an anisotropic framework is discussed in [88]. Convergence rates similar to those by using isotropic meshes is achieved with reduced number of degrees of freedom and computational effort. Also, a continuous, piecewise quadratic time reconstruction, namely, memory reconstruction and three point reconstructions are introduced to recover optimality for the time-discretization scheme.

5.3 *Kelvin-Voigt Problems*

The Kelvin-Voigt class of viscoelastic fluids constitutes an important group among non-Newtonian fluids. Polymeric fluids, molten plastics, engine oils, paints, ointments, gels, and many biological fluids like egg white and blood are examples in this category. It also appears in the mechanisms of diffuse axonal injury that are unexplained by traumatic brain injury models.

The Kelvin-Voigt fluid model is defined by the following system of equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - k\Delta \mathbf{u}_t - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(x,t),\ x \in \Omega,\ t > 0,$$

along with the incompressibility condition $\nabla \cdot \mathbf{u} = 0,\ x \in \Omega,\ t > 0$, and initial and boundary conditions $\mathbf{u}(x,0) = \mathbf{u}_0$ in $\Omega,\ \mathbf{u} = 0$ on $\partial\Omega,\ t \geq 0$, where $\mathbf{u} = \mathbf{u}(x,t)$ is the velocity vector,

$p = p(x,t)$ is the pressure, $\nu > 0$ is the kinematic coefficient of viscosity and k is the retardation time.

The papers [5, 6] analyses FE Galerkin methods for the equations of motion described by Kelvin-Voigt viscoelastic fluids. The global existence of a unique weak solution has been established first by applying Faedo-Galerkin method and standard compactness arguments. A semidiscrete approximation by discretizing spatial variable with the help of FEM while keeping time variable continuous is done. *A priori* estimates for the exact solution, which are valid uniformly as $t \to \infty$ and $k \to 0$ are derived. Optimal error estimates for the velocity that are uniformly valid in time in $L^1(H^1)$ and $L^2(H^2)$ norms and for the pressure in $L^\infty(L^2)$-norm are established. A complete discretization using a backward Euler method and a second order backward difference scheme to discretize in the temporal direction yields optimal rates of convergence for the error.

A two-grid method based on Newton's type iteration is also studied for the problem in [3, 7]. In [68], the convergence of the solution of the Kelvin-Voigt viscoelastic fluid flow model to its steady state solution with exponential rate is established under certain assumptions. Then, a semidiscrete Galerkin method for spatial direction keeping time variable continuous is considered and asymptotic behavior of the semidiscrete solution is derived. Moreover, optimal error estimates are achieved for large time using steady state error estimates. Based on a linearized backward Euler method, asymptotic behavior for the fully discrete solution is studied and optimal error estimates are derived for large time. Since the Kelvin-Voigt system differs from the Navier-Stokes system only by an additional term $k\frac{\partial}{\partial t}\Delta$, it is more pertinent to explore '*how far the results of the Kelvin-Voigt model carry over to the Navier-Stokes system?*' All the results are even valid as $k \to 0$, that is, when the Kelvin-Voigt model converges to the Navier-Stokes system.

An error analysis of a three steps two level Galerkin FEM for the two dimensional transient Navier-Stokes equations is discussed in [4] to understand this aspect. Optimal error estimates in $L^\infty(\mathbf{L}^2)$-norm, when $h = \mathcal{O}(H^{2-\delta})$ and in $L^\infty(\boldsymbol{H}^1)$-norm, when $h = \mathcal{O}(H^{4-\delta})$ for the velocity and in $L^\infty(L^2)$-norm, when $h = \mathcal{O}(H^{4-\delta})$ for the pressure are established for arbitrarily small $\delta > 0$.

Other applications

A considerable amount of research has been done in the area of FEM for singularly-perturbed problems, for example see [55-60, 64-66, 87, 90, 105], the details of which are not within the scope of this article.

We briefly discuss the contributions related to the arbitrary Eulerian-Lagrangian (ALE) method that permits the computational mesh to be moved in a kinematical fashion that combines the Eulerian and Lagrangian description of fluid flow. This makes it possible to have a versatile description of the fluid domain so as to simulate free surface flows and also tackle greater distortions of the domain. ALE has been applied to fluid flow around a stationary beam attached to a square base, a deforming beam and around a plunging aerofoil with success [79, 80]. Stabilization schemes for PDEs on time dependent domains has been studied, a part of which has been published in [40]. In this work, a stabilized numerical scheme using the Streamlined Upwind Galerkin method (SUPG) for a convection dominated transient equation in a time-dependent domain is proposed. Using a conservative ALE-SUPG FEM, stability estimates using intricately defined mesh-dependent norms for the backward Euler and Crank-Nicolson (CN) schemes are obtained. These stability estimates are shown to be independent of the mesh velocity. For the Euler case there are no conditions for the time step whereas for the CN method the time step depends on the mesh velocity. This is completely counter-intuitive as it is textbook knowledge that CN is unconditionally stable.

6. Conclusions

In this article we have briefly sketched the origins of FEM in India and highlighted some of the work done during the last decade within our own restricted view point. The work in the directions of convection-diffusion problems, fractional PDE, hyperbolic PDE, spectral FEM, parallel computing and engineering applications have not been included in this review. We hope that this article will inspire FEM groups working in these areas to highlight their contributions. The survey indicates that the mathematical aspects of FEM has matured in India since it began in the 1970's. More outreach programmes using GIAN, VAJRA etc could be planned so that it could take off in new directions that has not been explored so far that is relevant to India in the newly created academic ecosystem of IIT's, IISER's, NISER and Central Universities.

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