Indian J. Pure Appl. Math., **50**(3): 795-799, September 2019

DOI: 10.1007/s13226-019-0355-2

DIFFERENTIAL GEOMETRY IN INDIA

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This is a brief survey of work done in India in differential geometry during the period 2000 to 2015. The survey is not exhaustive and highlights aspects that the author is familiar with.

Key words : Riemannian manifolds; nonpositive curvature; Ricci flow; harmonic manifolds.

1. Nonpositive Curvature and Differential Topology

Investigation of the topology of Riemannian manifolds with restrictions on curvature is a long-standing theme in Riemannian geometry. In particular the topology of Riemannian manifolds with nonpositive sectional curvature (NPC) has been extensively studied. Considerable work has been done in this area by Aravinda and his collaborators. To describe their results, we begin by recalling the basic result of Cartan and Hadamard which states that the universal cover of a compact Riemannian n-manifold M with NPC is diffeomorphic to $\mathbb{R}^n$. This implies that M is a $K(\pi, 1)$ space, i.e., the higher homotopy groups $\pi_k(M) = 0$ for $k \geq 2$. By a result of Whitehead, it follows that the homotopy type of M is determined by $\pi_1(M)$. When we restrict ourselves to the class of locally symmetric spaces with NPC there is a far more dramatic result, due to Mostow the *isometry* type is determined by the fundamental group. This raises the following natural question: if we drop the locally symmetric condition and work within the class of all NPC manifolds, does the fundamental group determine the homeomorphism or even the diffeomorphism type ? In fact, Borel conjectured that just the property of being aspherical should force π_1 to determine the homeomorphism type. Borel's conjecture is known to hold for NPC manifolds. The question of diffeomorphism type is more mysterious and it is not clear what topological or curvature conditions force the rigidity of smooth structures.

These questions have been explored over the last four decades by Farrell-Jones and their collaborators. The results are largely in the negative, i.e., examples have been constructed displaying the lack of smooth rigidity in various cases. These constructions involve a beautiful blend of techniques from higher-dimensional topology and Riemannian geometry. Two significant results in this direction are due to Aravinda and Farrell. The background to their results is earlier work [8] of Farrell and Jones asserting that there are compact NPC manifolds homeomorphic to, but not diffeomorphic to, real or complex hyperbolic space-forms. A natural question is whether such examples exist in the setting of other symmetric spaces of non-compact type. A remarkable theorem of Gromov [4] asserts that a compact NPC manifold homotopy equivalent to a compact locally symmetric space of rank ≥ 2 is, in fact, isometric to it. That leaves the case of other rank-1 symmetric spaces, namely quaternionic-hyperbolic space and the Cayley hyperbolic plane. It is reasonable to expect that examples do not exist in these cases, keeping in mind that they tend to be "more" rigid, in light of Margulis's Superrigidity theorem and the Geometric Superigidity theorem of Mok-Siu-Yeung (see, for example, [2, 3]) and share many features with the higher-rank spaces. However, it is a surprising fact that examples can be constructed in the remaining rank-1 cases: in [3] and [2], Aravinda and Farrell show that there are compact NPC manifolds homeomorphic but not diffeomorphic to quotients of the Cayley hyperbolic plane and quaternionic hyperbolic space.

An interesting corollary of these constructions, as pointed out in a subsequent paper [1] of Aravinda and Farrell, is that there are compact smoooth manifolds which admit Riemannian metrics with negative sectional curvature but do not admit any Riemannian metrics with negative curvature operator. While the latter condition is much stronger than the former, there are very few global techniques to distinguish the two and the Aravinda-Farrell examples are the only known ones.

2. Ricci Flow and Isotropic Curvature

The notion of *isotropic curvature* was introduced by Micallef and Moore [15] in 1988 in their formulation of the second variation of energy of harmonic maps of Riemann surfaces into Riemannian manifolds. It turns out that positivity of isotropic curvature, which is needed for applying the second variation formula, generalizes several well-studied conditions such as positivity of curvature operator and strictly quarter-pinched sectional curvature. Micallef-Moore proved that a compact simply-connected Riemannian manifold with positive isotropic curvature (PIC) has to be homeomorphic to a sphere. Subsequently Gromov conjectured that the fundamental group of any compact PIC manifold should be virtually free. In [10] Gadgil and Seshadri point out that if Gromov's conjecture is true then we have a stronger conclusion: if M is a compact PIC n-manifold then M is homeomorphic to a connected sum of copies of $S^{n-1} \times S^1$. This may be an optimal result, assuming the validity of Gromov's conjecture, as it is unknown if exotic spheres can support PIC metrics.

Apart from the second variation formula and the Bochner formula for harmonic 2-forms, there is another important technique involving isotropic curvature, namely the Ricci flow (RF). It is a remarkable fact (see [6]), especially in light of the fact that RF does not preserve positivity of sectional or Ricci curvatures, that the PIC condition is preserved by RF. In other words if $g(t)$, $0 \leq t \leq T$, is a RF and $g(0)$ is PIC then $g(t)$ is PIC for all $t \leq T$. In [19], Seshadri explores the weaker condition of nonnegative isotropic curvature (NNIC). For certain curvature conditions, sectional curvature being an important instance, the differences between nonnegativity and positivity are very subtle and not fully understood. In [19] the RF invariance of PIC is exploited to prove that unless the NNIC manifold (M, g) is locally symmetric or is Kähler and biholomorphic to $\mathbb{C}P^n$, the RF $g(t)$ starting at g is actually PIC for any $t > 0$. The crucial ingredient here is a strong maximum principle due to Bony-Brendle-Schoen [5]. A corollary of the main result in [19] is an affirmative answer to a conjecture of Micallef-Wang [14]: a compact NNIC Kähler manifold with second betti number ≥ 2 is necessarily a Hermitian symmetric space of rank ≥ 2.

Further results due to Seshadri and his collaborators in this area deal with the general question of which positive curvature conditions are preserved by RF. The following framework is necessary to describe these results: let $C(\mathbb{R}^n)$ denote the vector space consisting of algebraic operators on $\mathbb{R}^n$. One is interested in cones in $C(\mathbb{R}^n)$ which are preserved by a certain ODE associated to the RF. It is known, by early work of Hamilton, that to every such cone one gets a notion of positivity preserved by the actual RF on a manifold. In [21] Wilking gave an elegant sufficiency criterion for constructing such cones. This criterion recovers all known positivity curvature conditions preserved by RF. Gururaja-Maity-Seshadri investigated Wilking's criterion further in [11] and proved that all the cones constructed by Wilking are contained in the cone of PIC curvature operators. It is also proved that the class of RF invariant cones can be partitioned into two subsets: the cones for which one has a "Sphere Theorem" and those which are stable under connected sums. The latter topic was pursued by Hoelzel [13] and a more general theory was developed.

3. Harmonic Manifolds

A Riemannian manifold (M, g) is *harmonic* if the mean curvature of small geodesic spheres depends only on the radius of the sphere. This condition arises naturally when one attempts to solve Laplace's equation on geodesic balls in Riemannian manifolds. A fundamental conjecture in this area is due to Lichnerowicz : A complete simply connected harmonic manifold is necessarily a rank-1 locally symmetric space. This was proved in the compact case by Szabo in 1990 [12] and counterexamples constructed in the noncompact case by Damek and Ricci in 1992 [7]. The problem of classifying noncompact harmonic manifolds is still open. Several important results in this area are due to by

Ranjan and Shah and their collaborators.

Complete simply connected noncompact harmonic manifolds have certain common properties with simply connected Riemannian manifolds with nonpositive sectional curvature. For instance, the exponential map at any point of the manifold is a diffeomorphism. In [18], another "nonpositive curvature-like" property is established: if the volume growth is polynomial then the manifold is flat. The proof involves a careful study of Busemann functions and horospheres. In a subsequent paper [17] Ranjan and Shah prove the remarkable fact that Buseman functions on complete simply connected noncompact harmonic manifolds are, in fact, real-analytic.

Subsequent work of Shah concerns *asymptotically* harmonic manifolds (ASH). Here one demands the weaker condition that horospheres have mean constant depending only on the radius. In [9] Heber-Knieper-Shah proved that ASH 3-manifolds with uniformly bounded negative sectional curvature are of constant curvature. Schroeder and Shah improved this result by replacing the negative curvature condition with the nonexistence of conjugate points in [20].

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