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OPERATOR ALGEBRAS IN INDIA IN THE PAST DECADE

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Operator algebras come in many flavours. For the purpose of this article, however, the term is only used for one of two kinds of self-adjoint algebras of operators on Hilbert space, *viz.*, C^*-algebras (which are norm-closed) or von Neumann algebras (which are closed in the topology of pointwise strong convergence or equivalently, in the weak-* topology it inherits as a result of being a Banach dual space). To be fair, there are a number of people in India (eg., Gadadhar Misra, Tirthankar Bhattacharyya, Jaydeb Sarkar, Santanu Dey, etc.,) who work on non-selfadjoint algebras, mostly from the point of view of connections with complex function theory; but in the interest of restricting the size of this paper, I confine myself here to selfadjoint algebras. I apologise for ways in which my own personal taste and limitations colour this depiction of operator algebras. Another instance of this arbitrary personal taste is a decision to concentrate on the work of younger people. Thus, the work of the more senior people who have worked in operator algebras is only seen via their collaborations with younger people: e.g., KRP via Srinivasan and Rajarama Bhat, Kalyan Sinha via Debashish, Partha, Arup, Raja, etc,. and me via Vijay, Srinivasan and Panchugopal.

Not long ago, interest in operator algebras in India was restricted to the three centres of the Indian Statistical Institute. Now, I am happy to note that it has spread to IMSc, some IITs, IISERs, NISER, JNU, ... My role in this article has been merely that of compiling inputs from many active Indian operator algebraists that came to my mind. I wrote soliciting a response from a certain number of them, then put together the responses received. (I apologise to those people who were omitted in this process). My colleague, Partha, with the help of his collaborator Arup, agreed to take care of the C^*-related inputs, while I take care of the von Neumann-related ones with the help of my collaborator Vijay.

What follows are some areas of ongoing research done in von Neumann algebras in India and some names of people doing such work: (a) subfactors and planar algebras, (Vijay Kodiyalam of IMSc, Chennai); (b) quantum dynamical systems and complete positivity (Rajarama Bhat of ISI, Bengaluru); (c) E_0 semigroups (R. Srinivasan of CMI, Chennai, and Panchugopal Bikram of NISER, Bhubhaneswar), and (d) Masas in II_1 von Neumann algebras and free Araki-Woods factors (Kunal Mukherjee of IIT, Chennai, and Panchugopal Bikram of NISER, Bhubhaneswar).

Key words : von Neumann algebras; subfactors; planar algebras; free probability; TQFTs, Hopf (and Kac) algebras; Skein theories; quantum dynamical systems; complete positivity; (sub- and super-) product systems; E_0 semigroups on $B(H)$, resp., on general factors; types I-III of E_0 semigroups; extendability of E_0 semigroups; Masas; q-deformed Araki-Woods von Neumann algebras; Ergodic theory; C^*-algebras; semigroup C^*-algebras; operator-algebraic quantum groups; Elliott's classification program; noncommutative geometry (NCG); spectral triples for quantum groups; dimensional invariants; quantum isometry groups; local index formula in NCG; metric properties; Dirac differential graded algebra; Yang Mills functionals; compact quantum metric spaces; rapid decay property.

1. SUBFACTORS

A subfactor is a unital inclusion $N \subseteq M$ of factors - von Neumann algebras with trivial centre. The focus will be on II_1-subfactors where both N and M are factors of type II_1 in the Murray-von Neumann classification. The origins of modern subfactor theory lie in the paper [86] which defined a numerical invariant called the index of such a subfactor and showed strikingly that it was quantised.

Over the next two decades, there was much work done on subfactors. A major discovery was that of the Jones polynomial for knots [88] which opened up a new chapter in knot theory. Other work focused on an object known as the standard invariant of a subfactor which had several apparently different axiomatisations such as the paragroups of Ocneanu and the λ-lattices of Popa. The standard invariant was given a topological-combinatorial interpretation as planar algebras in [87].

Very briefly, a planar algebra P is a collection of vector spaces $P_{(k,\pm)}$ equipped with an action of the operad of planar tangles that is subject to some conditions. Those planar algebras that arise from finite-index subfactors satisfy further 'niceness' properties - including the important finite-dimensionality (of all the vector spaces $P_{(k,\pm)}$) - and are called subfactor planar algebras.

Much of the work done at Chennai during the last decade in the area of subfactors and planar algebras can be subsumed under one of the following areas described in the next 4 sections.

1.1 *Planar algebras and free probability theory*

One of the exciting developments in planar algebras has been the realisation that they are connected with Voiculescu's free probability theory - the appearance of planar objects in both theories hinting at this connection.

The paper [94] considers a planar algebra associated to non-crossing partitions and relates it to the 2-cabling of the Temperley-Lieb planar algebra. In the course of establishing the main result a combinatorial identity relating various features of a planar configuration consisting of a straight line and a system of closed curves each of whose components intersects the line is obtained, which has since found many applications.

In [95] a purely planar algebraic proof of a result of Popa is given that constructs a subfactor with a given planar algebra. This uses the ideas of a proof of Guionnet-Jones-Shlyakhtenko in [77].

When the initial planar algebra arises from a Kac algebra the GJS-construction produces interpolated free group factors. This is the main result of [96] which is then extended to finite depth planar algebras in general in [98]. The construction of a factor from a graph is studied further in [12] without assumptions of being bipartite and it is again shown that the factors that arise are interpolated free group factors.

1.2 *Subfactors and TQFTs*

Subfactors, via their representation categories, yield $2+1$-dimensional topological quantum field theories and invariants of 3-manifolds. The TQFT defined in [92] is of a rather different kind - being a $1+1$-dimensional theory defined on a certain cobordism category. It is shown that unitary TQFTs on this cobordism category are essentially the same as subfactor planar algebras.

In [97], a new construction is given of Kuperberg's 3-manifold invariant associated to a finite-dimensional involutive Hopf algebra. This uses - and indeed was motivated by - Heegaard diagram presentations of the manifold, which closely resemble the planar tangles used in Jones' theory.

1.3 *Planar algebras associated to Hopf and Kac algebras*

It was realised right from the outset that subfactors arising from finite groups, or more generally, finite-dimensional Kac algebras, yield particularly nice planar algebras as their invariants. These planar algebras are irreducible ($dim\ P_{(1,\pm)} = 1$), and of depth 2 ($P_{(3,\pm)}$ are matrix algebras).

Demonstrating the power of planar algebra techniques in a simple example, it was shown in [59] how the Ocneanu-Szymanski theorem, which relates irreducible depth 2 subfactors with finite-dimensional Kac algebras, has a natural and diagrammatic planar algebraic proof.

Associated to a subfactor is its asymptotic inclusion subfactor which is regarded as the subfactor analogue of the Drinfeld double construction for Hopf algebras. In [84] the planar algebra of the asymptotic inclusion subfactor is described when the original subfactor arises from a Kac algebra.

In [61], to a finite-dimensional Hopf algebra H, a certain natural inclusion of infinite-dimensional algebras $A \subseteq B$ is associated, such that B is the smash product of A and the Drinfeld double $D(H)$

of H. Moreover, it is shown that $D(H)$ is the only finite-dimensional Hopf algebra with this property. While there is no appeal to planar algebras overtly, and indeed the result applies to Hopf algebras for which there is no natural associated planar algebra, all the proofs were obtained by planar algebraic techniques and then translated to the algebraic context.

The algebraic result above is exploited in [60] to produce an explicit embedding of the planar algebra of the Drinfeld double of a finite-dimensional, semisimple and cosemisimple Hopf algebra H into the two-cabling of the planar algebra of the dual Hopf algebra H^* and to characterise its image.

1.4 *Skein theories for planar algebras*

Analogous to free groups, there is a notion of a universal planar algebra associated with a set and it is of interest to express naturally occurring planar algebras as quotients of such by explicit sets of relations. Borrowing terminology from knot theory, such presentations of planar algebras are called skein theories. Building on earlier work on the planar algebra of a Kac algebra, an explicit skein theory for a planar algebra associated to a semisimple and cosemisimple Hopf algebra - even defined over a field of arbitrary characteristic - is presented in [93].

A class of planar algebras "without analysis" is that of finite depth planar algebras, and interesting and very non-trivial skein theories for some of these were established in several papers - [23, 111, 126]. It is shown in [99] that every finite depth planar algebra has a finite skein theory and is further singly generated with the generator subject to finitely many relations. This is refined further in [100] to get information about the degree of the generator.

2. Quantum Dynamical Systems and Complete Positivity

One parameter semigroups of contractive completely positive maps on C^*-algebras are known as quantum dynamical semigroups and they have been studied extensively. The main reason being that they are used to describe quantum open systems. Dilating these semigroups one obtains semigroups of $*$-endomorphisms, known as E-semigroups. Unital E-semigroups are known as E_0-semigroups.

One of the main invariants for classification of E_0-semigroups are tensor product systems of Hilbert spaces. A family of Hilbert spaces $\{\mathcal{H}_t : t > 0\}$, with an associative family of unitaries $\{U_{s,t} : s, t > 0\}$,

$$U_{s,t} : \mathcal{H}_s \otimes \mathcal{H}_t \to \mathcal{H}_{s+t}$$

is said to be a product system of Hilbert spaces when they satisfy some technical measurability conditions.

The product systems are broadly classified into three types. Type I product systems are well-understood, whereas type II and III are exotic. Whole new classes of E_0-semigroups with type III product systems were obtained by Izumi and Srinivasan in [82, 83]. Complete classification of product systems appearing here and computations of their automorphism groups seems to be a very challenging problem.

The notion of inclusion systems was introduced by Bhat and Mukherjee [14] where the linking unitaries of product systems get replaced by co-isometries (independently such systems were called sub-product systems by Shalit and Solel [139]). In the last decade the notion has found many uses. While studying quantum dynamical semigroups, it is inclusion systems one obtains first, and product systems appear later only through inductive limits. The concept was used in Bhat and Mukherjee [14] to compute the index of some special classes of amalgamated products of tensor product systems. Further applications can be found in Mukherjee [116, 117].

Given a state ϕ on a unital C^*-algebra $\mathcal{A}$, consider the unital quantum dynamical semigroups $\{\tau_t : t \geq 0\}$ on $\mathcal{A}$ such that $\tau_{t_0}(\cdot) = \phi(\cdot)I$ for some $t_0 > 0$. It is true that for the von Neumann algebra $\mathcal{B}(\mathcal{H})$, such quantum dynamical semigroups dilate to E_0-semigroups in standard form in the sense of Powers and conversely all E_0-semigroups in standard form arise this way [132].

What happens if one considers non-unital completely positive maps and semigroups of such maps? The theory here has many surprises. It is possible to have even nilpotent completely positive maps. For the first time, some majorization type inequalities for such maps have been obtained in Bhat and Mallick [13]. Generators of quantum dynamical semigroups which decay to zero have been studied in Bhat and Srivastava [17].

Another important development has been the growing importance of the theory of Hilbert C^*-modules in studying completely positive maps. It is known that product systems of Hilbert C^*-modules are needed for studying CP semigroups on general C^*-algebras. (Bhat and Skeide [16]). A Stinespring type theorem for a class of maps on Hilbert C^*-modules was proved in Bhat, Ramesh and Sumesh [15].

The Bures distance was originally defined by Bures as a metric for states on von Neumann algebras. The definition has a natural extension to completely positive maps. Bhat and Sumesh [18] initiate a study of this idea from the point of view of Hilbert C^*-modules and provide several examples and counter examples. It is seen that the concept has applications to quantum dynamical semigroups.

Very little is known about E_0-semigroups on general von Neumann algebras and factors. In this context, a beginning has been done by Srinivasan and his co-authors in [25, 108]. Here, the theory of

Hilbert von Neumann modules is a basic tool and treated in Bikram *et al*. [27].

In view of the Choi-Kraus representation, the study of normal contractive completely positive maps on $\mathcal{B}(\mathcal{H})$ amounts to a study of row contractions. Starting with [133] and [63] there have been a large number of papers in this field. The dilation theory takes inspiration from Sz. Nagy dilation, and one talks about models and characteristic functions for row contractions. See for instance papers of Santanu Dey, Tirthankar Bhattacharyya, Jaydeb Sarkar and their collaborators.

3. E_0-SEMIGROUPS

3.1 *Introduction*

E_0-semigroups are semigoups of unital $*-$ endomorphisms on a von Neumann algebra, which are further assumed to be $\sigma-$weakly continuous. E_0-semigroups are classified up to an identification called cocycle conjugacy. The study of E_0-semigroups was initiated by R.T. Powers in the eighties (see [131]), and nursed from its infancy to a rich theory from Arveson. The monograph [4] provides an extensive treatment of the theory of E_0-semigroups on type I factors. E_0-semigroups arise naturally in the study of open quantum systems, the theory of interactions, conformal field theory, and in quantum stochastic calculus. The study of E_0-semigroups led to the study of interesting objects like product systems, super product systems and C^*-semiflows, which arise as its associated invariants. The subject has established deep connections with other areas like probability theory. Invariably in all our examples, to construct E_0-semigroups, we will be using the semigroup $\{S_t\}$ of right shifts on $L^2((0,\infty),k)$, where k is a complex Hilbert space.

3.2 *Type III E_0-semigroups on $B(H)$*

An analogous statement, for an E_0-semigroup on $B(H)$, to the celebrated Wigner's theorem on strongly continuous semi-groups of automorphisms of $B(H)$, would be the statement that an E_0-semigroup in $B(H)$ is completely determined, up to cocycle conjugacy, by the set of all strongly continuous intertwining semigroups of isometries, called *units*. A unit for an E_0-semigroup $\{\alpha_t : t \geq 0\}$ is a strongly continuous semigroups of isometries $\{U_t\}$, satisfying $\alpha_t(X)U_t = U_tX$ for all $X \in B(H)$. A subclass of E_0-semigroups, where the above mentioned analogy is indeed true, are called type I E_0-semigroups. An E_0-semigroup is said to be type III if there exists no such intertwining semigroup.

For a complex Hilbert space k, we denote by $\Gamma(k)$ the symmetric Fock space associated with k. The Weyl operators $\{W(x) : x \in k\} \subseteq B(\Gamma(k))$ satisfy the canonical commutation relations, $W(x)W(y) = e^{-iIm\langle y,x\rangle}W(x+y)$. There exits a unique E_0-semigroup α acting on

$B(\Gamma(L^2((0,\infty),k)))$ satisfying $\alpha_t(W(f)) = W(S_t f)$. This E_0-semigroup is called the CCR flow of index $\dim k$. Arveson showed CCR flows are classified up to cocycle conjugacy by their index ($= \dim(k)$) and they exhaust all isomorphism classes of type I examples.

In 1987, Powers constructed (see [130]) the first example of a type III E_0-semigroup. In 2000, Tsirelson produced a family consisting of uncountably many type III mutually non-cocycle-conjugate E_0-semigroups (see [149]). Tsirelson's construction involves complicated techniques from probability theory. In [134], Bhat and Srinivasan systematically studied these examples and provided a purely operator theoretic construction. In Tsirelson's work, the type III property and non-isomorphism were proved by arriving at a contradiction, after messy computations. Bhat and Srinivasan clarified both by providing sufficient conditions in this operator algebraic frame work. They also proved the important dichotomy result that only type I and III are possible in this construction. The E_0-semigroup can also explicitly be described as generalized CCR flows, as explained below.

3.3 *Generalized CCR flows*

For a real Hilbert space G, let $G^{\mathbb{C}}$ denote its complexification and $H = \Gamma(G^{\mathbb{C}})$. Let $\{S_t\}$ and $\{T_t\}$ be strongly continuous semigroups of linear operators on G. We say that $\{T_t\}$ is a perturbation of $\{S_t\}$ if ${T_t}^* S_t = 1$. and $S_t - T_t$ is a Hilbert Schmidt operator, for all $t \in (0,\infty)$. Given a perturbation pair $(\{T_t\},\{S_t\})$, Izumi and Srinivasan proved that there exists an unique E_0-semigroup $\{\alpha_t\}$ on $B(\Gamma(G^{\mathbb{C}}))$ satisfying $\alpha_t(W(x+iy)) = W(S_t x + iT_t y)$ for all $x, y \in G$ and it is called the generalized CCR flow associated with the pair $\{S_t\}$ and $\{T_t\}$.

In [82], Izumi and Srinivasan proved several results about generalized CCR flows, clarifying its relation to the so called sum systems, and a necessary and sufficient criterion for them to be type III, which is more powerful than the earlier criterion given in [134]. They systematically studied the generalized CCR flows associated with perturbations of the right shift on $L^2(0,\infty)$ and produced several interesting results, in particular a new family of generalized CCR flows, which can not be distinguished from type I examples, by the invariants introduced by Tsirelson. By associating type III factors, arising as local algebras, as invariants to these type III E_0-semigroups, they showed that there exists uncountably many pairwise non-isomorphic examples in this family.

3.4 *Toeplitz CAR flows*

Though Tsirelson's path breaking result, initiated a flurry of activity on type III E_0-semigroups, there was not much work done following the original construction of Powers. Powers' construction [131] could produce large families of E_0-semigroups by varying the associated quasi-free states. But the problem is to find invariants to distinguish them up to cocycle conjugacy. Izumi and Srinivasan generalized Powers construction and called them 'Toeplitz CAR flows'.

Let $K = L^2((0,\infty),\mathbb{C}^N)$ and $\mathcal{A}(K)$ be the associated CAR algebra. A quasi-free state on $\mathcal{A}(K)$ is uniquely determined by a positive contraction $A \in B(K)$, through the values on the $(n,m)-$point functions. Let (H_A,π_A,Ω_A) be the associated GNS triple and $M_A := \pi_A(\mathcal{A}(K))''$. If A satisfies $Tr(A-A^2) < \infty$, then M_A is a type I factor. Further if A is also a Toeplitz operator (i.e it satisfies $S_t^* A S_t = A$), then there exists a unique E_0-semigroup on M_A satisfying $\alpha_t(\pi_A(a(x))) = \pi_A((a(S_t x)))$, for all $t \geq 0$, $x \in K$, which is called the Toeplitz CAR flow.

We regard K as a closed subspace of $\tilde{K} = L^2(\mathbb{R},\mathbb{C}^N)$, denote by P_+ the projection from $\tilde{K}$ onto K and identify $B(K)$ with $P_+ B(\tilde{K}) P_+$. For $\Phi \in L^\infty(\mathbb{R}, M_N(\mathbb{C}))$, define $C_\Phi \in B(\tilde{K})$ by $(\hat{C_\Phi f})(p) = \Phi(p)\hat{f}(p)$, where $\hat{}$ denotes Fourier transform. Then the Toeplitz operator $A = A_\Phi \in B(K)$ with the symbol Φ is defined by $A_\Phi f = P_+ C_\Phi f$ for all $f \in K$. Arveson determined the most general form of a Toeplitz operator A_Φ further satisfying $Tr(A_\Phi - A_\Phi^2) < \infty$, which clarifies the mysterious choice used by Powers. Such a symbol is called admissible and the associated E_0-semigroup is denoted by α^Φ. Izumi and Srinivasan showed the following in [83]. The original example of Powers corresponds to the case $\nu = \frac{1}{5}$.

***Theorem* 3.1** — *For $\nu > 0$, let $\theta_\nu(p) = (1+p^2)^{-\nu}$, and let*

$$\Phi_\nu(p) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta_\nu(p)} \\ e^{-i\theta_\nu(p)} & 1 \end{pmatrix}.$$

Then Φ_ν is admissible. Let $\alpha^\nu := \alpha^{\Phi_\nu}$ be the corresponding Toeplitz CAR flow.

(i) *If $\nu > 1/4$, then α^ν is of type I$_2$.*

(ii) *If $0 < \nu \leq 1/4$, then α^ν is of type III.*

(iii) *If $0 < \nu_1 < \nu_2 \leq 1/4$, then α^{ν_1} and α^{ν_2} are not cocycle conjugate.*

To distinguish the E_0-semigroups α^ν in the type III region $0 < \nu \leq \frac{1}{4}$, they used the type I factorizations of Araki-Woods, arising from local von Neumann algebras for the product systems, as invariants. They further showed that Toeplitz CAR flows are either of type I or of type III, a dichotomy result similar to the case of generalized CCR flows.

3.5 *E_0-semigroups on II$_1$ factors*

Till now, the focus was on E_0-semigroups on type I factors, but they can be studied on a general von Neumann algebra. Powers [131] had initiated the study of E_0-semigroups on type II$_1$ factors. In his initial paper [131] in 1988, Powers introduced two countable families of E_0-semigroups on II_1 factors called Clifford flows and even Clifford flows. Since then it was open whether these families contain mutually non-cocycle-conjugate E_0-semigroups. In 2004, Alevras defined an index through the associated boundary representations, introduced by Powers in [130], which is invariant under conjugacy (see [1]). But it is not clear whether the boundary representation is a cocycle conjugacy invariant.

In [108], Margetts and Srinivasan introduced two new numerical invariants, the coupling index and the gauge index, for E_0-semigroups on II_1 factors, and computed them for these examples. But both the numerical invariants turned out to be trivial for the above mentioned two families. In this context they also introduced another invariant called super-product systems, by generalizing the concept of product systems of Arveson, using modular theory. But it was not clear (then!) how to distinguish those super-product systems. But their non-cocycle-conjugacy was proved rather indirectly, by combining Alevras' result on boundary representations and a new invariant called C^*-semiflows.

3.6 *The extendability problem*

For any factor $M \subseteq L^2(M)$ and for any given E_0-semigroup on a factor M, one may define the complementary E_0-semigroup on its commutant (in $B(L^2(M))$), using the canonical modular conjugation operator J, given by the Tomita-Takesaki theory. The extendability question is whether these two endomorphisms extend to a common endomorphism $B(L^2(M))$. This question was systematically studied by Amosov, Bulinskii, Shirokov in [2], but there was a mistake in their proof implying the Clifford flows are extendable. Indeed Bikram, Izumi, Srinivasan and Sunder showed that Clifford flows on II_1 factors are actually not extendable in [25]. It is still open whether there exists an extendable E_0-semigroup on II_1 factors. On the other hand type III factors admit both extendable and non-extendable E_0-semigroups. It was shown by Margetts and Srinivasan [109] that the E_0-semigroups given by quasi-free representations of CCR relation are extendable E_0-semigroups, and Bikram [24] proved that the E_0-semigroups given by quasi-free representations of CAR relations are not extendable. This consequently shows that on type III factors the CCR flows and CAR flows are not cocycle conjugate, in contrast to the case of type I factors.

4. MASAS, FREE ARAKI-WOODS, ETC.

4.1 *Masas*

Constructing functions with prescribed properties has been the tradition of analysts over centuries. In a similar spirit, construction of maximal abelian subalgebras (masa) with prescribed properties has been a focal point in the structure theory of von Neumann algebras since the birth of the subject.

This originated in what is known today as the group measure space construction of Murray and von Neumann but was investigated more closely by Dixmier in [64]. The examples of masas cited in [64] opened a new area of research which was extensively followed by Connes, Krieger, Feldman-Moore, Jones and finally explored by Popa to the fullest extent from 1980s to date [48, 49, 69, 70, 102, 127-129]. The names of contributors and the references are by no means complete and just; we have chosen them only to keep the writing within reasonable size. For a comprehensive account on masas in II_1 factors check [140] and the references therein.

The most successful invariant to distinguish masas in II_1 factors is the Pukanzsky invariant. A refined invariant was considered in [67], which was subsequently used to distinguish some masas in the free group factors. This invariant (known as the measure-multiplicity invariant) was systematically explored in [113] and shown to 'detect' the notions of regularity (Cartan), semiregularity (Cartan inside a subalgebra) and singularity of masas in finite von Neumann algebras. The results concerning 'weak asymptotic homomorphism' of conditional expectations in the paper are a two-variable version of a well known theorem of Wiener regarding Fourier coefficients. The normalizing algebras of masas were shown to behave well under tensoring, i.e., $N_{M_1}(A_1)''\overline{\otimes}N_{M_2}(A_2)'' = N_{M_1\overline{\otimes}M_2}(A_1\overline{\otimes}A_2)''$, where A_i is a masa in a finite von Neumann algebras M_i, $i = 1, 2$, and $N_M(\mathcal{A})$ denotes the group of unitary operators in M which normalise A. This formula is crucial in providing examples of singular masas in the hyperfinite II_1 factor, as the latter is stable under tensoring with itself.

The theme of how the measure-multiplicity invariant recognises properties of masas, like existence of non trivial central sequences, strong mixing, etc., was pursued in [114], which also established that for any (possibly even empty) subset $E \subseteq \mathbb{N}$, there exist uncountably many pairwise non conjugate singular masas (i.e., not in the same orbit under the natural action of the automorphism group of the ambient von Neumann algebra) in $L(\mathbb{F}_k)$, $k = 2, 3, \cdots, \infty$, for each of which the Pukanzsky invariant is E [114].

The study of singular masas is related to the study of weakly mixing dynamical systems. With this in mind, the study of masas arising out of mixing dynamical systems was initiated by Jolissaint and Stalder in [85]. This paper motivated the study of various algebraic and analytical properties of subalgebras with mixing properties in [29]. Some basic results about mixing inclusions of von Neumann algebras were proved using ultra filters and connections were established between mixing properties and normalisers of von Neumann subalgebras. The special case of mixing subalgebras arising from inclusions of countable discrete groups finds applications to ergodic theory, in particular, a new generalisation of a classical theorem of Halmos on the automorphisms of a compact abelian group. For a finite von Neumann algebra M and von Neumann subalgebras A, B of M, the notion of weak mixing of B relative to A is introduced and shown to be equivalent to the requirement that if $x \in M$ and if there exist a finite number of elements $x_1, x_2, \cdots, x_n \in M$ such that $Ax \in \sum_i x_i B$, then $x \in B$. This essentially involves studying an appropriate basic construction of Jones. Examples of mixing subalgebras were shown to arise from the amalgamated free product and crossed product constructions.

A follow up paper [28] studied strongly mixing masas in detail. Some rigidity results were proved in the sense that strongly mixing masas arising out of group inclusions in group von Neumann algebras produce masas whose bimodule is coarse and Pukanzsky invariant is a singleton set when the smaller group is torsion free. Investigating examples from classical Ergodic theory, notably staircase transformations, led to the result that there exist uncountably many pairwise non conjugate strongly mixing masas in $L(\mathbb{F}_k)$, $k = 2, 3, \cdots, \infty$, for each of which the Pukanzsky invariant is $\{1, \infty\}$. Needless to say these masas are singular, while the masas described in [114] were not strong mixing.

To any strongly continuous orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathfrak{h}_{\mathbb{R}}$, Hiai in [79] constructed q-deformed Araki-Woods von Neumann algebras for $1 < q < 1$, which are W^*-algebras arising from non tracial representations of the q-commutation relations, the latter yielding an interpolation between the Bosonic and Fermionic statistics. These are von Neumann algebras that play a role in quantum field theory to provide examples of field theories that allow small violations of Pauli's exclusion principle [76]. The structure of these algebras is a current topic of study in the subject. While many properties of these algebras have been investigated in the last decade, it is still unknown if these von Neumann algebras have trivial center. It was shown in [26] that if the orthogonal representation is not ergodic then these von Neumann algebras are factors whenever $dim(\mathfrak{h}_{\mathbb{R}}) \geq 2$ and $q \in (-1, 1)$. In such case, the centralizer of the q-quasi free state has trivial relative commutant, thus one can completely determine its S-invariant. Moreover, in case the factor is type III_1 it satisfies the bicentralizer conjecture of Connes. In the process, bimodules of 'generator masas' in these factors are studied and it is established that they are strongly mixing. The possibility of removing the 'non-ergodicity' hypothesis imposed and concluding the factoriality of these algebras, whenever $dim(\mathfrak{h}_{\mathbb{R}}) \geq 3$, is the subject of future study.

4.2 *Ergodic Theory*

The classical theory of joinings of measurable dynamical systems is extended to the noncommutative setting from several interconnected points of view in [11]. Among these is a particularly fruitful identification of joinings with equivariant quantum channels between W^*-dynamical systems that provides noncommutative generalisations of many fundamental results of classical joining theory.

Fully general analogues of the main classical disjointness characterisations of ergodicity, primeness and mixing phenomena, are obtained. This approach to the characterisation of weak mixing shows that any finite-dimensional invariant subspace of the induced unitary representation of a φ-preserving action of a group on the standard Hilbert space $L^2(M,\varphi)$ lies inside the image $M^\varphi\Omega_\varphi$ of the centraliser, implying that an ergodic dynamics is chaotic on large regions of the phase space.

Moving beyond states, it is shown that the canonical unitary representation of a locally compact separable group arising from an ergodic action of the group on a von Neumann algebra with separable predual preserving a $f.n.s.$ (infinite) weight is weak mixing. On the other hand, there exists a non ergodic automorphism of a von Neumann algebra, preserving a $f.n.s.$ trace, such that the induced unitary representation has countable Lebesgue spectrum [101].

4.3 *Free Probability*

Quantum exchangeable random variables (namely, random variables whose distributions are invariant for the natural co-actions of Wang's quantum permtuation groups were characterized by Koestler and Speicher to be those sequences of identically distributed random variables that are free with respect to the conditional expectation onto their tail algebra (that is, free with amalgamation over the tail algebra). In [65], Dykema, Koestler and Williams considered, for any unital C^*-algebra A, the analogous notion of quantum symmetric states on the universal unital free product C^*-algebra $\mathfrak{A} = *_1^\infty A$.

[56] show that the space of tracial quantum symmetric states of an arbitrary unital C^*-algebra is a Choquet simplex and is a face of the tracial state space of the universal unital C^*-algebra free product of A with itself infinitely many times. Using free Brownian motion, it is also shown that the extreme points of this simplex are dense, making it the Poulsen simplex when A is separable and nontrivial.

Replacing traciality by KMS condition with respect to a fixed one parameter automorphism group, quantum symmetries which are also KMS states for the infinite free product automorphism group are characterised. Such states are shown to form a Choquet simplex as well whenever it is non-empty and its extreme points are characterised in [66].

Contributions in the field of C^*-algebras and NCG

A C^*-algebra is a norm closed involutive subalgebra of the algebra of bounded operators in a Hilbert space. Commutative C^*-algebras can be identified as the algebra of continuous functions vanishing at infinity on some locally compact Hausdorff space. Thus one would like to treat C^*-algebras as space of functions on some 'Ghost space' which does not make sense within the framework of usual topology. They are looked at from various angles. One may begin with a classical context and attach a C^*-algebra to it and then analyse its properties. Or one may look at C^*-algebras as algebras of continuous functions on some generalised spaces and try to lift concepts from topology. Very successful developments along these lines include extending ideas of topological K-theory and the study of topological groups. Finally one may try to use these topological ideas to the classification problem. In the coming sections we will see a study of C^*-algebras constructed out of topological semigroups, lifting of various ideas from group theory to the setting of compact quantum groups and finally some contributions in the classification problem.

5. Semigroup C^*-algebras

In the nineties, Murphy studied C^*-algebras associated to discrete semigroups and their crossed products in a series of papers [118-120]. Semigroup C^*-algebras again came into vogue with Cuntz' work [54] on the C^*-algebra associated to the $ax+b$-semigroup. Subsequently, a systematic study of discrete semigroup C^*-algebras was undertaken by Li [104] in collaboration with Cuntz and others. For more on the discrete semigroup C^*-algebras, we refer the reader to [104] and the references therein.

5.1 *Topological semigroup C^*-algebras*

The semigroup C^*-algebra can also be studied in the topological context. Thus let G be a locally compact second countable Hausdorff topological group and $P \subset G$ be a closed subsemigroup containing the identity element e. For technical reasons, one assumes that $Int(P)$ is dense in P. To keep the formulae simple, we will assume that G is unimodular. Consider $L^2(P)$ as a closed subspace of $L^2(G)$.

For $g \in G$, let U_g be the right translation operator on $L^2(G)$ and let W_g be the cut-down of U_g onto $L^2(P)$. For $a \in P$, let $V_a := W_a$. Then $\{V_a : a \in P\}$ is a strictly continuous semigroup of isometries. The reduced C^*-algebra of P, denoted by $C^*_{red}(P)$, is defined as the C^*-algebra generated by $\{\int f(a)V_a da : f \in L^1(P)\}$. The related C^*-algebra, called the Wiener-Hopf C^*-algebra $\mathcal{W}(P,G)$, is defined as the C^*-algebra generated by $\{\int f(g)W_g dg : f \in L^1(G)\}$. Clearly $C^*_{red}(P) \subset \mathcal{W}(P,G)$. It is not known in general if $\mathcal{W}(P,G)$ coincides with $C^*_{red}(P)$. However, if $PP^{-1} = G$ then $\mathcal{W}(P,G)$ coincides with $C^*_{red}(P)$ (see [135]).

For the rest of this exposition, we assume that $PP^{-1} = G$. Let X be a compact metric space. By a right action of P on X, we mean a map $X \times P \ni (x,a) \to xa \in X$ such that $xe = x$ and $x(ab) = (xa)b$ for $a,b \in P$. We say that the action is injective if for every $a \in P$, the map $X \ni x \to xa \in X$ is injective. Let X be a compact metric space on which P acts on the right injectively. Let $X \rtimes P$ be the set $\{(x,g,y) \in X \times G \times X : \exists\, a,b \in P$ such that $g = ab^{-1},\ xa = yb\}$ with its obvious groupoid structure. This is often called the Deaconu-Renault groupoid.

Let us now explain the groupoid considered in [112]. Endow $L^\infty(G)$ with the weak *-topology and let G act on $L^\infty(G)$ by right translations. Denote the weak*-closure of $\{1_{P^{-1}a} : a \in P\}$ in $L^\infty(G)$ by Ω_P. Observe that Ω_P is invariant under right translation by elements of P. The compact space Ω_P is called the Wiener-Hopf or the order compactification of P. In the article [135], Renault and Sundar showed that the Wiener-Hopf C^*-algebra $\mathcal{W}(P,G)$ is isomorphic to the reduced C^*-algebra of the Deaconu-Renaut groupoid $\Omega_P \rtimes P$ where Ω_P is the Wiener-Hopf compactification of P. It should be noted that this was first proved in [112] for normal semigroups and later generalised to Lie semigroups by Hilgert & Neeb [80].

Another main result of the paper [135] is the abstract realisation of the Wiener-Hopf compactification. Let X be a compact metric space on which P acts on the right injectively. For $x \in X$, let $Q_x := \{g \in G : \exists\, y \in X$ such that $(x,g,y) \in X \rtimes P\}$. Then the P-space X is homeomorphic to the Wiener-Hopf compactification Ω_P if and only if (i) there exists $x_0 \in X$ such that $\{x_0 a : a \in P\}$ is dense in X and $Q_{x_0} = P$, (ii) the map $x \mapsto Q_x$ is injective.

This abstract realisation of the Wiener-Hopf compactification is exploited in [143] to show that the K-theory of the Wiener-Hopf algebra associated to a symmetric cone is zero. A prototypical example of a symmetric cone is the cone of positive self-adjoint matrices of size n.

Let us conclude by summarising the related works. The question of defining the universal semigroup C^*-algebra is considered in [141]. It is shown that there exists a universal groupoid C^*-algebra, $C^*(\mathcal{G})$ generated by symbols $\{f : f \in C_c(G)\}$ such that if $P \ni a \to V_a \in B(\mathcal{H})$ is a strongly continuous semigroup of isometries and if the final projections $E_a := V_a V_a^*$ commute then there exists a unique $*$-homomorphism $\pi : C^*(\mathcal{G}) \to B(\mathcal{H})$ such that $\pi(f) = \int f(a) V_a da$. A version of Coburn's theorem for the quarter plane $[0,\infty) \times [0,\infty)$ is obtained as a consequence.

Semigroup crossed products defined the same way as group crossed products are considered in [144]. In particular, groupoid realisations of semigroup crossed products are achieved in [144] and in [142]. Also K-theoretic results for semigroup crossed products are derived.

6. Quantum Groups, Operator Algebraic Aspects

In this section we discuss the operator algebraic description of quantum groups. The theory of operator algebraic quantum groups has its roots in the early work of Kac [89, 90] in his attempt to generalize Pontrajin duality. However, the subject really came into its own following the seminal works of Vaksman and Soibelman [150, 151] and Woronowicz [158-160] in the 1980's. Woronowicz formulated a compact quantum group as a tuple $G = (A, \Delta)$, where A is a unital C^*-algebra and $\Delta : A \to A \otimes A$ is a C^*-morphism, which is coassociative i.e. $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ and for which the left and right cancellation laws hold, i.e. the sets $(A \otimes 1)\Delta(A)$ and $(1 \otimes A)\Delta(A)$ are total in $A \otimes A$. Then there exists a unique state, called the Haar state, $h_G : A \to \mathbb{C}$, that satisfies $(h_G \otimes id)\Delta(.) = h_G(.)1 = (id \otimes h_G)\Delta$. Further, one defines a (finite-dimensional) representation of G to be an unitary element $u = ((u_{ij})) \in M_n(A)$ such that for any $1 \leq i, j \leq n$, we have that $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. Two representations u and v are said to be equivalent if there exists $T \in GL_n(\mathbb{C})$ such that $(T \otimes 1)u = v(T \otimes 1)$ and a representation u is said to be irreducible if for any $T \in M_n(\mathbb{C})$ such that $(T \otimes 1)u = u(T \otimes 1)$ implies that $T = \lambda \cdot 1$ for some $\lambda \in \mathbb{C}$. The set of equivalence classes of irreducible representations of G is denoted by Irr(G). Suppose now that G is a compact group. Consider the commutative C^*-algebra $C(G)$, which is the algebra of complex valued continuous functions on G. In this case, a comultiplication can be defined in the following way: $\Delta : C(G) \to C(G) \otimes C(G) \cong C(G \times G)$ with $\Delta(f)(s,t) = f(st)$, where $s, t \in G$. Then it is easy to see that the tuple $(C(G), \Delta)$ is a compact quantum group. In this case, the Haar state is the state on $C(G)$ corresponding to the Haar measure on G. Conversely given a compact quantum group $G = (A, \Delta)$ where A is commutative is necessarily $A = C(G)$ for some compact group G, with Δ defined as above in the compact group case. This is a version of Gelfand duality.

The linear span of matrix coefficients of irreducible representations of G is a Hopf $*$-algebra, which is dense in the C^*-algebra. This algebra is denoted by Pol(G), and its enveloping C^*-algebra is denoted as $C_m(G)$, called the maximal C^*-algebra associated to G. On the other hand, the image of A under the GNS representation with respect to the Haar state h_G is also a C^*-algebra, denoted as $C(G)$ and is called the reduced C^*-algebra associated to G. The weak closure of $C(G)$ is denoted as $L^\infty(G)$, and is the von-Neumann algebra associated to G. An instructive example in this context is the dual of discrete group example. Let Γ be a discrete group, and $C_r^*(\Gamma)$ denote the reduced group C^*-algebra of Γ. Then the mapping $\lambda_g \mapsto \lambda_g \otimes \lambda_g$, where λ_g denotes the left translation operator in $C_r^*(\Gamma)$ associated to a group element $g \in \Gamma$, extends to a morphism $\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes C_r^*(\Gamma)$ and the tuple $G = (C_r^*(\Gamma), \Delta)$ is a compact quantum group. In this case, it can be shown that Pol$(G) = \mathbb{C}[\Gamma]$, the complex group algebra associated to Γ, Irr$(G) = \Gamma$ as a set, with the group unitaries λ_g being all the one-dimensional and irreducible representations of G. Further, the Haar state in this case can be shown to be the canonical trace on $C_r^*(\Gamma)$. It then follows that the reduced C^*-algebra, the von-Neumman algebra and the maximal C^* algebra associated to G in this case are, respectively, $C_r^*(\Gamma)$, $L(\Gamma)$ and $C^*(\Gamma)$, the full group C^*-algebra associated to Γ.

Let us next describe the directions of research being pursued currently in this field.

Since operator algebraic quantum groups in general and compact quantum groups in particular are generalizations of the idea of groups, these objects are often studied from a group theoretic perspective. In [125], generalizing the center of a compact group (see also [45]) a notion of center for a compact quantum group has been introduced. In this paper, normal subgroups, introduced by Wang [157], are studied and a certain notion of "inner" automorphisms of compact quantum groups was also introduced. It was then shown that any normal subgroup of a compact quantum group is stable under the action of any inner automorphism, as in the classical case, but that the converse is not true. In other words, there are examples of subgroups of compact quantum groups which are stable under the action of any normal subgroup, but are not normal. The underlying motivation for this was to somehow define a quantum inner automorphism group of a compact quantum group G, which should be isomorphic to the compact quantum group $G/Z(G)$, where $Z(G)$ denotes the center of G, and further, to show that stability of a subgroup of G under the action of this quantum inner automorphism group forces the subgroup to be normal. Results of this type were very recently obtained by Kasprzak, Skalski and Soltan [91].

Another important direction of research in the area has been to define and understand various "approximation properties" like Haagerup property for quantum groups and their associated operator algebras. To give examples of quantum groups having Haagerup property, it is useful to consider constructions and permanence of the Haagerup property under various constructions. Keeping this in mind, the paper of Fima, Mukherjee and Patri [72], considers the bicrossed product construction and the crossed product constructions and proves permanence results for these constructions, which leads to examples of quantum groups having the Haagerup property.

Closely related is the celebrated Property (T), which is a strong negation of the Haagerup property in a certain sense. Property (T) was first defined and studied for quantum groups by Fima [71] and later by Kyed [103]. However, no non-trivial family of examples were known. Finally, in the paper by Fima, Mukherjee and Patri [72], it was shown that the bicrossed product of a matched pair consisting of a discrete group with Property (T) and a finite group yields a compact quantum group whose dual has the Property (T). This was then used to construct an infinite family of mutually non-isomorphic non-trivial quantum groups having Property (T).

Just like groups, quantum groups also act on operator algebras and hence, study of these actions of quantum groups forms a major area of research in the theory of quantum groups (see for example [47, 62, 81, 110, 146]. However, in the paper of Mukherjee and Patri [115], a different route is taken.

In this paper, the study of discrete group actions on compact quantum groups by quantum group automorphisms (i.e. automorphisms which "commute" with the comultiplication of the quantum group) is initiated. Any such quantum group automorphism automatically preserves the Haar state of the compact quantum group, so one has an interesting non-commutative dynamical system. This study was motivated by the study of group actions on compact groups by group automorphisms, which was initiated by a paper of Halmos [78] and still is a thriving field of study (see the monograph [138]). Suppose a discrete group Γ is acting on a compact quantum group G by quantum group automorphisms. We then call the corresponding dynamical system as a CQG dynamical system. In the paper [115], an in-depth study is carried out of spectral properties of CQG dynamical properties, viz. ergodicity, weak mixing, mixing, compactness, etc. It is shown for example that ergodic CQG dynamical systems are automatically weak mixing, echoing the classical case. Combinatorial conditions are obtained for these spectral properties, in terms of the induced Γ action on the set Irr(G), and this is used to study several examples. Further, in the same paper, it is shown that the spectral "measure" of any ergodic CQG dynamical system, with $\Gamma = \mathbb{Z}$, is the Haar state and the spectral multiplicity of such an action is singleton. But unlike the classical case, it is not clear whether this is infinite and this leads to very interesting connections to some deep open problems in combinatorial group theory. Then, the authors go on to study the structure theory of CQG dynamical systems and under some conditions, show the existence and uniqueness of the maximal ergodic normal subgroup of such CQG dynamical systems. Finally, the authors use the results in the paper to explore maximal abelian subalgebras (MASAs) of von-Neumann algebra $L^\infty(G)$ associated to a compact quantum group G. In particular, the authors consider an abelian group Γ, with the abelian algebra $L(\Gamma)$ being a subalgebra of $L^\infty(G)$. They find equivalent conditions for $L(\Gamma)$ to be a MASA in $L^\infty(G)$ and prove a striking result that the normalizer of any such MASA in $L^\infty(G)$ is always of the form $L(\Lambda)$, for some discrete group Λ. This allows the author to prove a rigidity result that if $L(\Gamma)$ is a Cartan MASA in $L^\infty(G)$, then necessarily all the quantum group is co-commutative, and hence there exists a discrete group Λ with $L^\infty(G) = L(\Lambda)$.

7. The Classification Program

The main goal of the Elliott programme is to classify simple, separable, nuclear C^*-algebras by their K-theory and related invariants. With the remarkable work of Kirchberg, Phillips, Elliott, Gong, Li and others, the programme achieved great success in the 1990s and early 2000s.

However, in the early 2000s, examples by Villadsen and Rørdam showed that some regularity conditions were needed to ensure that an algebra is classifiable in the sense described above. An important counterexample to the Elliott conjecture was due to Toms [147], who described two non-isomorphic, unital, simple, nuclear C^*-algebras, which agree on any continuous homotopy-invariant functor. This brought to light the importance of the Cuntz semigroup (the invariant that distinguished Toms' examples), and also led to the Toms-Winter conjecture [148] that seeks to identify the regularity conditions mentioned above. This conjecture has been verified for a large class of algebras, but has yet to be completely proved.

Assuming one such regularity condition (finite nuclear dimension), Elliott, Gong, Lin and Niu [68] have proved a classification theorem for simple, nuclear C^*-algebras which satisfy the Universal Coefficient Theorem (UCT), under the additional assumption that all traces are quasidiagonal. While it is unknown whether all separable, nuclear C^*-algebras satisfy the UCT, Tikuisis, White and Winter [145] have proved that, for a simple, nuclear C^*-algebra satisfying the UCT, every faithful trace is quasidiagonal. In particular, this completes the classification (begun by Kirchberg and Phillips) of unital, simple, C^*-algebras which satisfy the UCT and have finite nuclear dimension.

The classification of non-simple C^*-algebras seeks to extend results from the simple realm by representing a non-simple C^*-algebra as a continuous fields over its primitive ideal space X. Initial results focussed on the case where X was finite, but the introduction of powerful approximation theorems due to Dadarlat [57] made the infinite case more tractable. These theorems were used by Dadarlat and Vaidyanathan [58] to describe the equivariant KK-theory group for certain continuous fields over $[0,1]$. These results constitute the first such explicit description for continuous fields over an infinite space, and are used to prove that algebras in this class are classified by their filtrated K-theory.

8. Noncommutative Geometry

There are several flavors of non commutative geometry in mathematics. The one we are concerned with in the following should really be called noncommutative differential geometry, which was introduced by Alain Connes during the 1980's. Following standard practice, however, we will continue to refer to it by just noncommutative geometry or NCG. This is an extension of noncommutative topology and was initially developed in order to handle certain spaces like the leaf space of foliations or duals of groups which are difficult to study using machinery available in classical geometry or topology. The basic idea is the same as in most other 'noncommutative mathematics', namely, study a set equipped with some structures in terms of an appropriate algebra of functions on the set. Here one starts with a separable unital C^*-algebra A which is the noncommutative version of a compact Hausdorff space. Associated to this, one already has certain invariants like the K-groups and the

K-homology groups. In geometry, one next equips the space with a smooth structure. In the noncommutative situation, the parallel is an appropriate dense subalgebra of A that plays the role of smooth functions on the space. Given this dense subalgebra, one can then compute various cohomology groups associated with it, namely the Hochschild cohomology, cyclic cohomology and the periodic cyclic cohomology, which are noncommutative and far-reaching generalizations of ordinary de Rham homology and cohomology. In ordinary differential geometry, in order that one can talk about shapes and sizes of spaces, one needs to bring in extra structure. One example is the Riemannian structure, which gives rise to a Riemannian connection, which in turn enables one to talk about curvature and so on. Other examples of such extra structures are $Spin$ and $Spin^c$ structures. In the presence of these extra structures, one has an operator theoretic data that completely encodes the structure. In noncommutative geometry, one takes this operator theoretic data as the starting point. This operator theoretic data is what is called a **spectral triple**. A spectral triple $(\mathcal{H}, \pi, D)$ for an associative unital *-algebra $\mathcal{A}$ consists of a (complex separable) Hilbert space $\mathcal{H}$, a *-representation $\pi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$ (usually assumed faithful) and a self-adjoint operator D with compact resolvent such that $[D, \pi(a)] \in \mathcal{L}(\mathcal{H})$ for all $a \in \mathcal{A}$. Often one also have a $\mathbb{Z}_2$-grading present, satisfying certain conditions. In the $Spin$ manifold situation one considers the Hilbert space of square integrable spinors and Dirac operator plays the role of D. This is called the canonical spectral triple associated with a $Spin$ manifold. It is for this reason that even in the general context of a spectral triple, the operator D is often referred to as the Dirac operator. Relevant concepts of Differential Geometry or Index Theory are transferred to the noncommutative setting by the usual two step procedure of first describing it for a $Spin$ manifold in terms of the canonical spectral triple and then one takes this description as the defining condition in the general context. This has been amply demonstrated in [51]. Of course the canonical spectral triple has many more properties and while transferring these concepts to the noncommutative side, one often demands more properties of the spectral triple like finite dimensionality, regularity, discreteness of dimension spectrum etc.

9. Quantum Groups and NCG

9.1 *Spectral triples for quantum groups*

Even though Connes had developed NCG in order to investigate spaces that are intractable using classical machinery, soon mathematicians realized that the scope of this new formalsim is possibly much broader. One should be able to study spaces that are much more noncommutative in nature, i.e. further away from classical spaces which are usually point sets with some extra structure. For example, the noncommutative two-torus was studied extensively by noncommutative geometers. At the same time, for a considerable time, there were not too many other examples of noncommutative geometric spaces that are somewhat far away from classical spaces. The q-deformations of the classical Lie groups were particularly intriguing. On one hand, one would expect them to be nice noncommutative geometric spaces, just as their classical counterparts have nice geometries, but on the other hand, the initial studies seemed to suggest that they do not fall perfectly in Connes' set up. This situation changed with the papers [35, 36] by Chakraborty and Pal where quantum SU(2) was shown to admit finitely summable spectral triples. The Dirac operator constructed in [35] was analyzed by Connes in [52]. He analyzed the spectral triple from the local index formula perspective. These two papers generated a flurry of activities in the area in subsequent years. Meanwhile, an alternative construction of a spectral triple for quantum $SU(2)$ was provided by Dabrowski *et al*. [55], which was soon proved by Chakraborty and Pal [37] to be essentially equivalent to their earlier construction in [35]. Inspired by Connes' work, Chakraborty and Pal extended the results of [35] to the case of odd dimensional quantum spheres [39], which are nothing but the quotient spaces $SU_q(n+1)/SU_q(n)$. They characterized all $\mathrm{SU}_q(n+1)$ equivariant spectral triples for these spaces. They also identified explicitly the C^*-extensions given by these in [38]. A very natural question in this context is whether these spectral triples can be endowed with further properties like Poincaré duality. It turned out that they do not satisfy Poincaré duality [40]. But in the same paper, they also produced explicit finitely summable spectral triples for odd dimensional spheres that satisfy Poincaré duality [40]. Continuing the study of quantum homogeneous spaces, Chakraborty and Sundar [43] computed and obtained explicit generators for the K-groups of the quantum Steiffel manifolds $SU_q(n)/SU_q(n-2)$. One must point out here two important papers by Neshveyev and Tuset [121, 122]. In [121], they proved the existence of a Dirac operator for all q-deformations of simply connected simple compact Lie groups by starting with the classical Dirac and twisting it in a certain way, and in [122], they proved the KK-equivalence of a family of quantum homogeneous spaces with their classical counterparts. However, even though they resolve the existence issues for a wider class of quantum groups, the Dirac operator is not as tractable, and many questions remain unanswered. So there is a need for lot more work in this direction.

9.2 *Dimensional invariants*

In the examples mentioned in the previous subsection, i.e., for the quantum $SU(2)$ and the quantum spheres, instead of using KK-theoretic machinery, Chakraborty and Pal tries to characterize all equivariant spectral triples. In the process, they observed that even without the condition of nontriviality of the corresponding K-cycle, in many cases there are canonical spectral triples encoding essential dimensional information about the space. This led Chakraborty and Pal [34] to produce dimensional invariants for ergodic C^*-dynamical systems utilising the notion of spectral triples. One should mention here that study of cycles not necessarily nontrivial is not new. For example, Voiculescu's work [153, 154] on norm ideal perturbations or Rieffel's work [137] on extending the notion of metric spaces. Computation and understanding of these invariants promises to be an elaborate program.

9.3 *Quantum isometry groups*

We have looked at quantum groups as examples of noncommutative geometric spaces till now. Quantum groups have another equally, if not more, important role to play in noncommutative geometry by virtue of being objects governing symmetries of noncommutative spaces. Investigation of quantum groups from this point of view can be traced back to the papers of Manin [106, 107], Wang [155, 156], Banica [5, 6, 8] and others. In the context of noncommutative geometry, study of this aspect of compact quantum groups were initiated and pursued extensively by Goswami and his collaborators. In [73], Goswami formulated and studied the quantum analogue of the group of Riemannian isometries called the quantum isometry group. In Riemannian geometry, an isometry is characterized by the fact that its action commutes with the Laplacian. Following this lead, Goswami considered a category of compact quantum groups that act on a noncommutative manifold given by a spectral triple such that the action commutes with the Laplacian coming from the spectral triple. He goes on to prove [73] that a universal object in the category of such quantum groups exists if one makes some mild assumptions on the spectral triple all of which are valid for a compact Riemannian spin manifold. He calls this universal object the quantum isometry group. Later on, in a joint work with Bhowmick [19], the notion of a quantum group analogue of the group of orientation preserving isometries was given and its existence as the universal object in a suitable category was proved.

This was followed by a number of computations for quantum isometry groups by Goswami, Bhowmick and several others including Banica, Skalski, Soltan etc (see for example [9, 10, 19-21, 105]. Some of the main tools for making explcit computations were provided by the results about the effect of deformation and taking suitable inductive limit on quantum isometry groups. In particular, many interesting noncommutative manifolds were obtained by deforming classical Riemannian manifolds in a suitable sense and it was proved that the quantum isometry group of such a deformed (noncommutative) manifold is nothing but a similar deformation or twist of the quantum isometry group of the original, undeformed classical manifold. This led to the problem of computing quantum isometry groups of classical Riemannian manifolds.

It was quite remarkable that none of the initial attempts of computing quantum isometry groups of connected classical manifolds including the spheres and the tori (with the usual Riemannian metrics) could produce any genuine quantum group, i.e. the quantum isometry groups for all these manifolds turned out to be the same as the classical isometry groups. On the other hand, it follows from the work of Banica *et al*. [7] that most of the known compact quantum groups, including the quantum permutation groups of Wang, can never act faithfully and isometrically on a connected compact Riemannian manifold. All these led to a conjecture due to Goswami that it is not possible to have smooth faithful actions of genuine compact quantum groups on $C(M)$ where M is a compact connected smooth manifold. After a series of initial attempts, the above conjecture has been finally proved by Goswami and Joardar [75] for two important cases: (i) when the action is isometric for some Riemannian metric on the manifold, and (ii) when the quantum group is finite dimensional. In particular, the quantum isometry group of an arbitrary compact, connected Riemannian manifold M is classical, i.e. same as $C(ISO(M))$. This also implies that the quantum isometry group of a noncommutative manifold obtained by cocycle twisting of a classical connected compact manifold is a similar cocycle twisted version of the isometry group of the classical manifold, thus allowing one to explicitly compute quantum isometry groups of a large class of noncommutative manifolds. More recently, Goswami has formulated and proved the existence of a quantum isometry group for a compact metric space without any geometric structure [74].

10. Local Index Formula in NCG

The Connes-Moscovici local index formula [50] constructs a local Chern character as a sum of several multilinear functionals. For Dirac operators associated with a closed Riemannian spin manifold, most of these functionals vanish (cf. remark II.1, page 63, [50]). Therefore most of the terms in the local Chern character are visible in truly noncommutative cases and hence should be interpreted as a signature of noncommutativity. To have a better understanding of the contribution of these terms, it is desirable to have examples where these terms survive. Even though the Connes-Moscovici paper ended with the hope that in the foliations example these contributions will be visible, it turned out to be not very tractable in the codimension one foliations case. So the task of illustrating the local index formula in simpler examples remained open. The first simple illustration was given by Connes in [52]. He first showed that the spectral triple given by Chakraborty and Pal is regular, with discrete dimension spectrum, so that the Connes-Moscovici local index formula is applicable, and then goes on to compute the functionals. Connes' idea was used by Pal and Sundar [124] to establish regularity and discreteness of the dimension spectrum of the equivariant spectral triples for odd dimensional quantum spheres constructed earlier by Chakraborty and Pal [39]. Extending this technique further and drawing inspiration from heat kernel expansion, Chakraborty and Sundar [44] then gave a functorial construction of a spectral triple for the double suspension of a C^*-algebra A

starting with a spectral triple for A. Using heat kernel techniques, they were able to show that if the original spectral triple is regular with discrete dimension spectrum, then the new one also shares the same properties. As a consequence, if Connes-Moscovici formula applies to the original triple, it would apply to the new one as well. To completely extend Connes' work [52], the final step would be to compute the cocycles. The paper by Chakraborty and Saurabh [42] is the next step towards that goal.

11. Metric Properties and Differential Calculus

If spectral triples encode essential geometric data, the natural question is what next? Can one recover, say the differential graded algebra given by the De Rham complex? Can this machinery be utilised to extend gauge theoretic notions like Yang-Mills? What about the metric? In his book [51], Connes addressed these issues. Now we will have a glimpse of these developments pursued by the Indian school.

11.1 *Dirac Differential Graded Algebra*

Starting with a spectral triple Connes' gave a functorial construction of a differential graded algebra that we will refer to as the Dirac differential graded algebra. This is a unitary invariant of the spectral triple. For the canonical spectral triple associated with a Riemannian spin manifold, he identified this as the de Rham complex. However, so far this has been computed for very few special cases. Chakraborty and Sinha [41] computed this for the quantum Heisenberg manifolds, while Chakraborty and Pal [36] did the same for the quantum $SU(2)$. All these were isolated computations. In [32], Chakraborty and Guin have identified suitable hypotheses on a spectral triple that helps one to compute the associated Connes' calculus for its quantum double suspension. This allows one to compute the Dirac differential graded algebra for spectral triples obtained by iterated quantum double suspension of the spectral triple associated with a first order differential operator on a compact smooth manifold. This gives the first systematic computation of Connes calculus for a large family of spectral triples.

11.2 *Yang Mills functionals*

One of the motivations behind the initiation of the subject of NCG was to formulate the so called action principles of Physics in this extended framework. For that one had to make sense of action functionals first. Yang-Mills is one such and indeed Connes and Rieffel [53] introduced this concept in the framework of NCG. Later Connes gave a spectral formulation of Yang-Mills. For Noncommutative two torus, both the notions agree (see [51]). In [31, 33], Chakraborty and Guin have shown that these two notions of Yang-Mills agree both for higher noncommutative torus and the quantum Heisenberg manifolds of Rieffel.

11.3 *Compact quantum metric spaces*

The requirements of Riemannian Geometry is much more elaborate compared to that of metric spaces. Therefore it is natural to expect that it would be simpler to extend the notions of compact metric spaces to the noncommutative framework. Moreover if we keep in mind various deformation schemes studied in the particle physics literature it may not always be possible to interpret them as noncommutative geometric spaces in the sense of Connes but interpreting them as compact quantum metric spaces could be a much more modest program. With this in mind Rieffel [136] introduced the concept of compact quantum metric spaces as a unital C^*-algebra equipped with a seminorm L, called the Lip norm, such that $L(a) = 0$ for any a implies that a is a scalar, L is lower semicontinuous and the metric $\rho_L(\mu, \nu) := \sup_{f:L(f)\leq 1} |\mu(f) - \nu(f)|$ induces the weak*-topology on $S(A)$. Examples of compact quantum metric spaces often arise from spectral triples. One of the earliest examples of spectral triples on noncommutative spaces were those by Connes for group algebras. For a finitely generated discrete group Γ and a proper length function l on Γ, the spectral triple is given by $(\mathbb{C}\Gamma, \ell^2(\Gamma), D_l)$ where $\mathbb{C}\Gamma$ acts on $\ell^2(\Gamma)$ via the left regular representation while D_l is defined by the self adjoint extension of the operator defined by $D(\delta_\gamma) = l(\gamma)\delta_\gamma$. In [136], Rieffel showed that the spectral triple $(\mathbb{C}\Gamma, \ell^2(\Gamma), D_l)$ is a compact quantum metric space if one takes the length function to be the canonical word length function on the group $\Gamma = \mathbb{Z}^n$. Very recently, Christ and Rieffel [46] extended this to the case of all groups of polynomial growth. The case of word hyperbolic groups was taken care by Ozawa and Rieffel in [123]. On the other hand, Antonescu and Christensen [3] obtained compact quantum metric space structures on the reduced group C^* algebras of groups with the property of Rapid Decay.

The result of Antonescu and Christensen [3] was extended to the case of discrete quantum groups with Rapid Decay by Bhowmick et all in [22]. A discrete quantum group $C_0(\widehat{G})$ dual to the compact quantum group $C(G)$ is a suitable completion of a direct sum of matrix algebras, the size of the matrix blocks being equal to the dimension of an equivalence class of irreducible corepresentation of $C(G)$. It is a fact that every discrete quantum group is the dual of a compact quantum group and vice versa. Vergnioux [152] studied the notion of a length function of a discrete quantum group and defined a notion of Rapid Decay. Unfortunately, his definition rules out duals of compact quantum groups whose Haar states are not tracial. This defect was corrected in [22] where a twisted version of the Sobolev s-norms were defined in order to have a different definition of the property of Rapid Decay. For quantum groups whose Haar states are tracial, this definition agrees with Vergnioux's definition. Now with a modification of the Lip norm used by Antonescu and Christensen, Bhowmick *et al*. [22] proved that the reduced quantum group C^*-algebras of finitely generated discrete quantum groups having Rapid Decay property are compact quantum metric spaces.

There is another way to look at certain examples coming from quantum groups or their homogeneous spaces, for example, those associated to the quotients of quantum $SU(n)$. They often fit into short exact sequences. To exploit this, Chakraborty obtains a general principle to construct compact quantum metric spaces out of certain C^*-algebra extensions [30]. In particular this shows that quantum $SU(n)$ and quantum Steiffel manifolds do admit compact quantum metric space structures.

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