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CORE INVERTIBILITY OF TRIANGULAR MATRICES OVER A RING

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We obtained several equivalent conditions for the existence of core inverses and dual core inverses of triangular matrices over a ring with involution. As applications, some necessary and sufficient conditions for the (2,2,0) core inverse problem are given.

Key words : Core inverse; dual core inverse; group inverse; ring; triangular matrix.

1. Introduction

The core inverse and the dual core inverse of a complex matrix were introduced by Baksalary and Trenkler in [1]. Let $A \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the ring of all $n \times n$ complex matrices over the complex field $\mathbb{C}$. A matrix $X \in M_n(\mathbb{C})$ is called a core inverse of A, if it is satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where $\mathcal{R}(A)$ denotes the column space of A, P_A is the orthogonal projector onto $\mathcal{R}(A)$. And if such matrix exists, then it is unique and denoted by $A^{\#\!\!\!\!\bigcirc}$. Baksalary and Trenkler [1] give several characterizations of the core inverse by using the Hartwig and Spindelböck's decomposition in [2]. More properties of the core inverse can be found in [6, 7] etc. Let R be a unital ring with involution, that is a ring with unity 1 and an involution $a \mapsto a^*$ satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$ for all $a, b \in R$. We will also use the following notations: $aR = \{ax \mid x \in R\}$ and $Ra = \{xa \mid x \in R\}$. Rakić *et al.* [5] generalized the core inverse of a complex matrix to the case of an element in a ring. Let $a, x \in R$, if

$$axa = a,\ xR = aR,\ Rx = Ra^*,$$

then x is called a core inverse of a and if such element x exists, then it is unique and denoted by $a^{\#\!\!\!\!\bigcirc}$. The set of all core invertible elements in R will be denoted by $R^{\#\!\!\!\!\bigcirc}$. Let $a, x \in R$, if

$$axa = a,\ xR = a^*R,\ Rx = Ra,$$

then x is called a dual core inverse of a and if such element x exists, then it is unique and denoted by $a_{\textcircled{\#}}$. The set of all dual core invertible elements in R will be denoted by $R_{\textcircled{\#}}$. Rakić *et al.* proved [5, Theorem 2.14] that for $a \in R$, the core inverse of a is the unique element $x \in R$ satisfies the following five equations:

$$axa = a, \ \ xax = x, \ \ (ax)^* = ax, \ \ xa^2 = a, \ \ ax^2 = x.$$

In [3, Theorem 1], Hartwig and Shoaf proved that for a matrix $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R_{2\times 2}$, where $a, b, c \in R$ and $R_{2\times 2}$ denote the ring of 2×2 matrices over a ring R, if $a \in R^{\#}$, then we have $M^{\#}$ exists if and only if $c \in R^{\#}$ and $(1 - aa^{\#})b(1 - c^{\#}c) = 0$. In this paper we give some equivalent characterizations such that $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R_{2\times 2}$ is core invertible. We prove that for a matrix $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R_{2\times 2}$, $M^{\textcircled{\#}}$ exists with $M^{\textcircled{\#}} \in UT_{2\times 2}$ if and only if $a, c \in R^{\textcircled{\#}}$ and $(1 - aa^{\textcircled{\#}})b = 0$, where $UT_{2\times 2}$ denotes the ring of upper triangular 2×2 matrices over a ring R. As applications, some necessary and sufficient conditions for the (2,2,0) core inverse problem are given.

An element $a \in R$ (R is not necessary to be a ring with involution) is said to be group invertible if there exists $x \in R$ such that the following equations hold:

$$axa = a, \quad xax = x, \quad ax = xa.$$

The element x satisfies the above equations is called a group inverse of a. If such element x exists, then it is unique and is denoted by $a^{\#}$. The set of all group invertible elements in R will be denoted by $R^{\#}$.

An element $b \in R$ is an inner inverse of $a \in R$ if $aba = a$ holds. The set of all inner inverses of a will be denoted by $a\{1\}$. An element $\tilde{a} \in R$ is called a $\{1, 3\}$-inverse of a if we have $a\tilde{a}a = a$, $(a\tilde{a})^* = a\tilde{a}$. The set of all $\{1, 3\}$-invertible elements will be denoted by $R^{\{1,3\}}$.

2. Core Inverse of Triangular Matrices

In this section, some equivalent conditions of core inverses and dual core inverses of triangular matrices over a ring are obtained.

Lemma 2.1 — [8]. Let $a, x \in R$. Then the following conditions are equivalent:

(1) $a \in R^{\textcircled{\#}}$ with $x = a^{\textcircled{\#}}$;

(2) $a \in R^{\#} \cap R^{\{1,3\}}$ with $x = a^{\#}aa^{(1,3)}$;

(3) $axa = a$, $(ax)^* = ax$, $xa^2 = a$ and $ax^2 = x$;

(4) $(ax)^* = ax$, $xa^2 = a$ and $ax^2 = x$.

Lemma 2.2 — [3, Theorem 1]. Let $a, b, c \in R$ and $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R_{2\times 2}$. If $a \in R^{\#}$, then $M^{\#}$ exists if and only if $c \in R^{\#}$ and $(1 - aa^{\#})b(1 - c^{\#}c) = 0$. In this case, $M^{\#} = \begin{bmatrix} a^{\#} & \alpha \\ 0 & c^{\#} \end{bmatrix}$, where $\alpha = (a^{\#})^2b(1 - c^{\#}c) + (1 - aa^{\#})b(c^{\#})^2 - a^{\#}bc^{\#}$.

Proposition 2.3 — Let $a, b, c \in R$ and $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R_{2\times 2}$. Then $M^{\#}$ exists with $M^{\#} \in UT_{2\times 2}$ if and only if $a, c \in R^{\#}$ and $(1-aa^{\#})b(1-c^{\#}c) = 0$. In this case, $M^{\#} = \begin{bmatrix} a^{\#} & \alpha \\ 0 & c^{\#} \end{bmatrix}$, where $\alpha = (a^{\#})^2b(1 - c^{\#}c) + (1 - aa^{\#})b(c^{\#})^2 - a^{\#}bc^{\#}$.

PROOF : Suppose $M^{\#}$ exists with $M^{\#} \in UT_{2\times 2}$, then $M^{\#}$ has the following form

$$M^{\#} = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}$$

for some $x_1, x_2, x_3 \in R$.

The equation $MM^{\#}M = M$ gives

$$\begin{cases} a = ax_1a \\ b = ax_1b + (ax_2 + bx_3)c \\ c = cx_3c \end{cases} \tag{1}$$

The equation $MM^{\#} = M^{\#}M$ gives

$$\begin{cases} x_1a = ax_1 \\ x_3c = cx_3 \end{cases} \tag{2}$$

The equation $M^{\#}MM^{\#} = M^{\#}$ gives

$$\begin{cases} x_1ax_1 = x_1 \\ x_3cx_3 = x_3 \end{cases} \tag{3}$$

By (1), (2) and (3), we have $a^{\#} = x_1$ and $c^{\#} = x_3$.

Thus, by (1), we have

$$(1 - a^{\#}a)b = (ax_2 + bc^{\#})c$$

which gives

$$(1 - aa^{\#})b(1 - c^{\#}c) = (ax_2 + bc^{\#})c(1 - c^{\#}c) = 0.$$

Conversely, it is clear by Lemma 2.2. $\square$

Lemma 2.4 — Let $a \in R^{\circledast}$ and $b \in R$. Then the following conditions are equivalent:

(1) $(1 - aa^{\circledast})b = 0$;

(2) $(1 - a^{\circledast}a)b = 0$;

(3) $bR \subseteq aR$.

PROOF : $(1) \Rightarrow (2)$. Suppose $(1 - aa^{\circledast})b = 0$, that is $b = aa^{\circledast}b$, thus $a^{\circledast}ab = a^{\circledast}a(aa^{\circledast}b) = aa^{\circledast}b = b$. Hence $(1 - a^{\circledast}a)b = 0$.

$(2) \Rightarrow (3)$. Suppose $(1 - a^{\circledast}a)b = 0$, that is $b = a^{\circledast}ab$, then by $a(a^{\circledast})^2 = a^{\circledast}$ we have $b = a^{\circledast}ab = a(a^{\circledast})^2ab$, hence $bR \subseteq aR$.

$(3) \Rightarrow (1)$. Suppose $bR \subseteq aR$, then $b = ar$, for some $r \in R$. Thus $aa^{\circledast}b = aa^{\circledast}ar = ar = b$, that is $(1 - aa^{\circledast})b = 0$. $\square$

***Theorem* 2.5** — *Let $a, b, c \in R$ and $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R_{2\times 2}$. Then the following conditions are equivalent:*

(1) *$M^{\circledast}$ exists with $M^{\circledast} \in UT_{2\times 2}$;*

(2) *$a, c \in R^{\circledast}$ and $(1 - aa^{\circledast})b = 0$;*

(3) *$a, c \in R^{\circledast}$ and $(1 - a^{\circledast}a)b = 0$;*

(4) *$a, c \in R^{\circledast}$ and $bR \subseteq aR$.*

In this case, $M^{\circledast} = \begin{bmatrix} a^{\circledast} & -a^{\circledast}bc^{\circledast} \\ 0 & c^{\circledast} \end{bmatrix}$.

PROOF : $(1) \Rightarrow (2)$. Suppose $M^{\circledast}$ exists with $M^{\circledast} \in UT_{2\times 2}$, then $M^{\circledast}$ has the following form

$$M^{\circledast} = \begin{bmatrix} x_1 & x_2 \\ 0 & x_3 \end{bmatrix}.$$

for some $x_1, x_2, x_3 \in R$.

By Lemma 2.1, we have

$$MM^{\#}M = M, \quad (MM^{\#})^* = MM^{\#}, \quad M^{\#}M^2 = M, \quad M(M^{\#})^2 = M^{\#}.$$

The equation $MM^{\#}M = M$ gives

$$\begin{cases} a = ax_1a \\ b = ax_1b + (ax_2 + bx_3)c \\ c = cx_3c \end{cases} \tag{4}$$

The equation $(MM^{\#})^* = MM^{\#}$ gives

$$\begin{cases} (ax_1)^* = ax_1 \\ ax_2 + bx_3 = 0 \\ (cx_3)^* = cx_3 \end{cases} \tag{5}$$

The equation $M^{\#}M^2 = M$ gives

$$\begin{cases} x_1a^2 = a \\ x_3c^2 = c \end{cases} \tag{6}$$

The equation $M(M^{\#})^2 = M^{\#}$ gives

$$\begin{cases} a(x_1)^2 = x_1 \\ c(x_3)^2 = x_3 \end{cases} \tag{7}$$

By Lemma 2.1, (5), (6) and (7), we have $a, c \in R^{\#}$ and $a^{\#} = x_1$ and $c^{\#} = x_4$.

By (4) and (5), we have

$$b = ax_1b + (ax_2 + bx_3)c = ax_1b.$$

As $a^{\#} = x_1$, hence $b = aa^{\#}b$, that is $(1 - aa^{\#})b = 0$.

$(2) \Rightarrow (1)$. Suppose $a, c \in R^{\#}$ and $(1 - aa^{\#})b = 0$, then we have

$$b = aa^{\#}b. \tag{8}$$

Let $X=\begin{bmatrix} a^{\#} & -a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix}$, then by (8) we have

$$MX=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a^{\#} & -a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix}=\begin{bmatrix} aa^{\#} & 0 \\ 0 & cc^{\#} \end{bmatrix},$$

$$MXM=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a^{\#} & -a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}=\begin{bmatrix} aa^{\#} & 0 \\ 0 & cc^{\#} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}=M.$$

Thus

$$X\in M\{1,3\}. \tag{9}$$

By [5, Theorem 2.19], we have $a,c\in R^{\#}$ and $a^{\#}=(a^{\#})^2a$ and $c^{\#}=(c^{\#})^2c$.

Since $a^{\#}=a^{\#}aa^{(1,3)}$ by Lemma 2.1, thus $1-a^{\#}a=1-a^{\#}aa^{(1,3)}a=1-a^{\#}a$, similarly $1-c^{\#}c=1-c^{\#}c$.

Since $(1-aa^{\#})b=0$ implies $(1-a^{\#}a)b=0$. (Indeed, $a^{\#}ab=a^{\#}a(aa^{\#}b)=aa^{\#}b$.), thus

$$(1-aa^{\#})b(1-c^{\#}c)=0. \tag{10}$$

By Lemma 2.2 and (10), we have

$$M^{\#}=\begin{bmatrix} a^{\#} & (a^{\#})^2b(1-c^{\#}c)+(1-aa^{\#})b(c^{\#})^2-a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix} \tag{11}$$

Since $M^{\#}=M^{\#}MM^{(1,3)}$ by Lemma 2.1. As we have (9) and (11), thus

$$\begin{aligned}
M^{\#}&=M^{\#}MM^{(1,3)}=M^{\#}MX\\
&=\begin{bmatrix} a^{\#} & (a^{\#})^2b(1-c^{\#}c)+(1-aa^{\#})b(c^{\#})^2-a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a^{\#} & -a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix}\\
&=\begin{bmatrix} a^{\#} & (a^{\#})^2b(1-c^{\#}c)+(1-aa^{\#})b(c^{\#})^2-a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix}\begin{bmatrix} aa^{\#} & 0 \\ 0 & cc^{\#} \end{bmatrix}\\
&=\begin{bmatrix} (a^{\#})^2a & \beta \\ 0 & (c^{\#})^2c \end{bmatrix}\begin{bmatrix} aa^{\#} & 0 \\ 0 & cc^{\#} \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} (a^{\#})^2a^2a^{\#} & \beta cc^{\#} \\ 0 & (c^{\#})^2c^2c^{\#} \end{bmatrix}$$
$$= \begin{bmatrix} a^{\#}aa^{\#} & -(a^{\#})^2ab(c^{\#})^2c^2c^{\#} \\ 0 & c^{\#}cc^{\#} \end{bmatrix}$$
$$= \begin{bmatrix} a^{\#} & -a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix}.$$

where $\beta = ((a^{\#})^2a)^2b(1-(c^{\#})^2cc) + (1-a(a^{\#})^2a)b((c^{\#})^2c)^2 - (a^{\#})^2ab(c^{\#})^2c$.

$(2) \Leftrightarrow (3) \Leftrightarrow (4)$ is clear by Lemma 2.4. □

Lemma 2.6 — Let $a, b \in R$ with $ab = 1$. Then $a \in R^{\#}$ if and only if $a \in R^{-1}$.

PROOF : Suppose $ab = 1$ and $a \in R^{\#}$. Then

$$1 = ab = a^{\#}a^2b = a(a^{\#})^2a^2b = a(a^{\#})^2a(ab) = a(a^{\#})^2a = a^{\#}a.$$

That is $1 = a^{\#}a$ and by the hypothetical condition $ab = 1$, thus $a \in R^{-1}$.

Conversely, if $a \in R^{-1}$, it is clear that $a \in R^{\#}$ by $a^{\#} = a^{-1}$. □

Lemma 2.7 — Let $a, b \in R$ with $ba = 1$. Then $a \in R^{\#}$ if and only if $a \in R^{-1}$.

Example 2.8 : The condition $M^{\#} \in UT_{2\times2}$ in Theorem 2.5 cannot be dropped. Let R be ring such that $ab \neq 1$ and $ba = 1$, for all choices $a, b \in R$. We will present an example using matrices over a ring R with the involution as conjugate transposition. Consider the matrix $M = \begin{bmatrix} a & 1-ab \\ 0 & b \end{bmatrix}$, let $X = \begin{bmatrix} b & 0 \\ 1-ab & a \end{bmatrix}$, then we have $MX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = XM$.That is M is invertible. Thus we have $M^{\#} = X$, but $X \notin UT_{2\times2}$. Since $ab \neq 1$ and $ba = 1$, thus by the Lemma 2.7, we have $a \notin R^{\#}$.

Let us denote the ring of lower triangular 2×2 matrices over a ring R by $LT_{2\times2}$.

***Theorem* 2.9** — *Let $a, b, c \in R$ and $M = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in R_{2\times2}$. Then the following conditions are equivalent:*

(1) *$M^{\#}$ exists with $M^{\#} \in LT_{2\times2}$;*

(2) *$a, c \in R^{\#}$ and $(1-cc^{\#})b = 0$;*

(3) *$a, c \in R^{\#}$ and $(1-c^{\#}c)b = 0$;*

(4) $a, c \in R^{\text{\textcircled{\#}}}$ *and* $bR \subseteq cR$.

In this case, $M^{\text{\textcircled{\#}}} = \begin{bmatrix} a^{\text{\textcircled{\#}}} & 0 \\ -c^{\text{\textcircled{\#}}}ba^{\text{\textcircled{\#}}} & c^{\text{\textcircled{\#}}} \end{bmatrix}$.

Example 2.10 : The condition $M^{\text{\textcircled{\#}}} \in LT_{2\times 2}$ in Theorem 2.9 cannot be dropped. Let R be ring such that $ba \neq 1$ and $ab = 1$, for all choices $a, b \in R$. We will present an example using matrices over a ring R with the involution as conjugate transposition. Consider the matrix $M = \begin{bmatrix} a & 0 \\ 1 & b \end{bmatrix}$, let $X = \begin{bmatrix} b & 1-ba \\ -1 & a \end{bmatrix}$, then we have $MX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = XM$.That is M is invertible. Thus we have $M^{\text{\textcircled{\#}}} = X$, but $X \notin LT_{2\times 2}$. Since $ba \neq 1$ and $ab = 1$, thus by the Lemma 2.6, we have $a \notin R^{\text{\textcircled{\#}}}$.

Remark 2.11 : In Theorem 2.5 and Theorem 2.9, one may obviously replace a, b and c by conformable matrices over R, since all equations used in the proof are remain valid. One also can obtain analogical results for dual core inverse as the Theorem 2.5 and Theorem 2.9.

3. THE (2,2,0) CORE INVERSE PROBLEM

In this section, some equivalent characterizations of (2,2,0) core inverse problem in rings are obtained. Let us start this section with some auxiliary lemmas.

Lemma 3.1 — [4, Corollary 1]. If $a, p \in R$ such that $a^- \in a\{1\}$ and p is right invertible. Then the following are equivalent:

(1) $ap \in R^{\#}$;

(2) $u = apaa^- + 1 - aa^-$ is invertible;

(3) $u' = pa + 1 - a^-a$ is invertible.

In this case, $a^{\#} = u^{-2}ap$.

Lemma 3.2 — If $a, p \in R$ such that $a^{(1,3)} \in a\{1, 3\}$ and $pp^* = 1$. Then ap is $\{1, 3\}$-invertible and $p^*a^{(1,3)} \in (ap)\{1, 3\}$.

PROOF : By $pp^* = 1$, we have
$$app^*a^{(1,3)} = aa^{(1,3)}$$
and
$$app^*a^{(1,3)}ap = aa^{(1,3)}ap = ap,$$

thus ap is $\{1,3\}$-invertible and $p^*a^{(1,3)} \in (ap)\{1,3\}$. $\square$

Proposition 3.3 — Let $a \in R^{\text{#}}$ and $p \in R$. If $pp^* = 1$ and $a^- \in a\{1\}$, then the following are equivalent:

(1) $ap \in R^{\text{#}}$;

(2) $u = apaa^- + 1 - aa^-$ is invertible;

(3) $u' = pa + 1 - a^-a$ is invertible.

In this case, $(ap)^{\text{#}} = u^{-2}apaa^{\text{#}}$.

PROOF : $(1) \Rightarrow (2)$. Suppose $ap \in R^{\text{#}}$, then by [5, Theorem 2.19] we have $ap \in R^{\#}$. Since $pp^* = 1$ gives p is right invertible, thus $u = apaa^- + 1 - aa^-$ is invertible by Lemma 3.1.

$(2) \Rightarrow (1)$. As p is right invertible and $u = apaa^- + 1 - aa^-$ is invertible, then by Lemma 3.2 we have $ap \in R^{\#}$ and

$$(ap)^{\#} = u^{-2}ap \tag{12}$$

By Lemma 2.1, we have $a^{\text{#}} \in a\{1,3\}$. By $pp^* = 1$ and Lemma 3.2, we have $ap \in R^{\{1,3\}}$ and

$$p^*a^{\text{#}} \in ap\{1,3\} \tag{13}$$

By (12), (13) and Lemma 2.1, we have

$$(ap)^{\text{#}} = (ap)^{\#}ap(ap)^{(1,3)} = u^{-2}apapp^*a^{\text{#}} = u^{-2}apaa^{\text{#}}.$$

The proof of $(2) \Leftrightarrow (3)$ is similar to $(1) \Leftrightarrow (2)$. $\square$

Let $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then

$$PP^* = P^*P = I \tag{14}$$

and $MP = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ c & 0 \end{bmatrix}$. Let us denote $A = \begin{bmatrix} b & a \\ c & 0 \end{bmatrix}$, then

$$A = MP. \tag{15}$$

Proposition 3.4 — Let $a, b, c \in R$ and $A = \begin{bmatrix} b & a \\ c & 0 \end{bmatrix}$ with $a, c \in R^{\text{#}}$ and $bR \subseteq aR$. Then the following are equivalent:

(1) A is core invertible;

(2) $U = AMM^{-} + I - MM^{-}$ is invertible;

(3) $U^{'} = PM + I - M^{-}M$ is invertible.

Where $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ and $M^{-} \in M\{1\}$. In this case

$$A^{\#} = U^{-2} \begin{bmatrix} baa^{\#} & acc^{\#} \\ caa^{\#} & 0 \end{bmatrix}$$

PROOF : $(1) \Rightarrow (2)$. Suppose A is core invertible, then by (15) and [5, Theorem 2.19], we have that MP is group invertible, thus U is invertible by [4, Corollary 1].

$(2) \Rightarrow (1)$. Suppose $a, c \in R^{\#}$ and $bR \subseteq aR$, then by Theorem 2.5, we have M is core invertible. Since $PP^{*} = I$, thus MP is core invertible by Proposition 3.3, that is A is core invertible by (15) and

$$\begin{aligned} A^{\#} &= U^{-2}MPMM^{\#} \\ &= U^{-2}AMM^{\#} \\ &= U^{-2} \begin{bmatrix} b & a \\ c & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} a^{\#} & -a^{\#}bc^{\#} \\ 0 & c^{\#} \end{bmatrix} \\ &= U^{-2} \begin{bmatrix} baa^{\#} & -baa^{\#}bc^{\#} + (b^2 + ac)c^{\#} \\ caa^{\#} & -caa^{\#}bc^{\#} + cbc^{\#} \end{bmatrix} \\ &= U^{-2} \begin{bmatrix} baa^{\#} & -b^2c^{\#} + (b^2 + ac)c^{\#} \\ caa^{\#} & -cbc^{\#} + cbc^{\#} \end{bmatrix} \\ &= U^{-2} \begin{bmatrix} baa^{\#} & -b^2c^{\#} + b^2c^{\#} + acc^{\#} \\ caa^{\#} & 0 \end{bmatrix} \\ &= U^{-2} \begin{bmatrix} baa^{\#} & acc^{\#} \\ caa^{\#} & 0 \end{bmatrix}. \end{aligned}$$

The equivalence of (2) and (3) is clear by Lemma 3.1. □

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