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PROPERTIES OF k-FIBONACCI AND k-LUCAS OCTONIONS

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We investigate some binomial and congruence properties for the k-Fibonacci and k-Lucas hyperbolic octonions. In addition, we present several well-known identities such as Catalan's, Cassini's and d'Ocagne's identities for k-Fibonacci and k-Lucas hyperbolic octonions.

Key words : Fibonacci sequence; k-Fibonacci sequence; k-Lucas sequence.

1. INTRODUCTION

The Fibonacci and Lucas sequences are generalised by changing the initial conditions or changing the recurrence relation. The k-Fibonacci sequence is the generalization of the Fibonacci sequence, which is first introduced by Falcon and Plaza [2]. The k-Fibonacci sequence is defined by the numbers which satisfy the second order recurrence relation $F_{k,n} = kF_{k,n-1}+F_{k,n-2}$ with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. Falcon [3] defined the k-Lucas sequence that is companion sequence of k-Fibonacci sequence defined with the k-Lucas numbers which are defined with the recurrence relation $L_{k,n} = kL_{k,n-1} + L_{k,n-2}$ with the initial conditions $L_{k,0} = 2$ and $L_{k,1} = k$. Binet's formulas for the k-Fibonacci and k-Lucas numbers are

$$F_{k,n} = \frac{{r_1}^n - {r_2}^n}{r_1 - r_2}$$

and

$$L_{k,n} = {r_1}^n + {r_2}^n$$

respectively, where $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$ are the roots of the characteristic equation $x^2 - kx - 1 = 0$. The characteristic roots r_1 and r_2 satisfy the properties

$$r_1 - r_2 = \sqrt{k^2+4} = \sqrt{\delta}, \quad r_1 + r_2 = k, \quad r_1 r_2 = -1.$$

The reader can refer to [1, 4-9] for properties and applications of k- Fibonacci and k- Lucas numbers.

The quaternions are generalized numbers. The quaternions first introduced by Irish mathematician William Rowan Hamilton in 1843. Hamilton [10] introduced the set of quaternions form a 4-dimensional real vector space with a multiplicative operation. The quaternions are used in applied sciences such as physics, computer science and Clifford algebras in mathematics. In particular, they are important in mechanics [11], chemistry [12], kinematics [13], quantum mechanics [14], differential geometry and pure algebra. A quaternion a, with real components a_0, a_1, a_2, a_3 and basis $1, i, j, k$, is an element of the form

$$a = a_0 + a_1 i + a_2 j + a_3 k = \langle a_0, a_1, a_2, a_3 \rangle,$$

where

$$\begin{aligned} &i^2 = j^2 = k^2 = ijk = -1, \\ &ij = k = -ji, jk = i = -kj, ki = j = ik. \end{aligned}$$

Horadam [15] defined the n^{th} Fibonacci and n^{th} Lucas quaternions as

$$\bar{F_n} = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k = \langle F_n, F_{n+1}, F_{n+2}, F_{n+3} \rangle$$

and

$$\bar{L_n} = L_n + L_{n+1} i + L_{n+2} j + L_{n+3} k = \langle L_n, L_{n+1}, L_{n+2}, L_{n+3} \rangle$$

respectively.

Ramirez [16] has defined and studied the k-Fibonacci and k-Lucas quaternions as

$$\bar{F_{k,n}} = F_{k,n} + F_{k,n+1} i + F_{k,n+2} j + F_{k,n+3} k = \langle F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3} \rangle$$

and

$$\bar{L_{k,n}} = L_{k,n} + L_{k,n+1} i + L_{k,n+2} j + L_{k,n+3} k = \langle L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3} \rangle$$

respectively, where $F_{k,n}$ is the n^{th} k-Fibonacci sequence and $L_{k,n}$ is the n^{th} k-Lucas sequence.

Different quaternions of sequences have been studied by different researchers. For example, Iyer [17, 18] obtained various relations containing the Fibonacci and Lucas quaternions. Halici [19] studied some combinatorial properties of Fibonacci quaternions. Akyigit *et al*. [20, 21] established and investigated the Fibonacci generalized quaternions and split Fibonacci quaternions. Catarino [22] obtained different properties of the $h(x)$-Fibonacci quaternion polynomials. Polatli and Kesim [23] have introduced quaternions with generalized Fibonacci and Lucas number components.

Hyperbolic k-Fibonacci and k-Lucas Quaternions

A hyperbolic quaternion h is an expression of the form $h = h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 i_4 = \langle h_1, h_2, h_3, h_4 \rangle$, with real components h_1, h_2, h_3, h_4 and i_1, i_2, i_3, i_4 are hyperbolic quaternionic units that satisfy the non-commutative multiplication rules

$$\begin{aligned} &{i_2}^2 = {i_3}^2 = {i_4}^2 = i_2 i_3 i_4 = +1, i_1 = 1, \\ &i_2 i_3 = i_4 = -i_3 i_2, i_3 i_4 = i_2 = -i_4 i_3, i_4 i_2 = i_3 = -i_2 i_4. \end{aligned}$$

The scalar and the vector part of a hyperbolic quaternion h are denoted by $S_h = h_1$ and $\overrightarrow{V}_h = h_2 i_2 + h_3 i_3 + h_4 i_4$, respectively. Thus, a hyperbolic quaternion h is given by $h = S_h + \overrightarrow{V}_h$. For any two hyperbolic quaternions $h^{(1)} = h_1^{(1)} i_1 + h_2^{(1)} i_2 + h_3^{(1)} i_3 + h_4^{(1)} i_4$ and $h^{(2)} = h_1^{(2)} i_1 + h_2^{(2)} i_2 + h_3^{(2)} i_3 + h_4^{(2)} i_4$. Addition and subtraction of the hyperbolic quaternions is defined by

$$\begin{aligned} &h^{(1)} \pm h^{(2)} = \left(h_1^{(1)} i_1 + h_2^{(1)} i_2 + h_3^{(1)} i_3 + h_4^{(1)} i_4\right) \\ &\pm \left(h_1^{(2)} i_1 + h_2^{(2)} i_2 + h_3^{(2)} i_3 + h_4^{(2)} i_4\right) \\ &= \left(h_1^{(1)} \pm h_1^{(2)}\right) i_1 + \left(h_2^{(1)} \pm h_2^{(2)}\right) i_2 + \left(h_3^{(1)} \pm h_3^{(2)}\right) i_3 + \left(h_4^{(1)} \pm h_4^{(2)}\right) i_4\right) \end{aligned}$$

Multiplication of the hyperbolic quaternions is defined by

$$\begin{aligned} h^{(1)} \cdot h^{(2)} &= \left(h_1^{(1)} i_1 + h_2^{(1)} i_2 + h_3^{(1)} i_3 + h_4^{(1)} i_4\right) \\ &\cdot \left(h_1^{(2)} i_1 + h_2^{(2)} i_2 + h_3^{(2)} i_3 + h_4^{(2)} i_4\right) \\ &= \left(h_1^{(1)} h_1^{(2)} + h_2^{(1)} h_2^{(2)} + h_3^{(1)} h_3^{(2)} + h_4^{(1)} h_4^{(2)}\right) \\ &+ \left(h_1^{(1)} h_2^{(2)} + h_2^{(1)} h_1^{(2)} + h_3^{(1)} h_4^{(2)} - h_4^{(1)} h_3^{(2)}\right) i_2 \\ &+ \left(h_1^{(1)} h_3^{(2)} - h_2^{(1)} h_4^{(2)} + h_3^{(1)} h_1^{(2)} + h_4^{(1)} h_2^{(2)}\right) i_3 \\ &+ \left(h_1^{(1)} h_4^{(2)} + h_2^{(1)} h_3^{(2)} - h_3^{(1)} h_2^{(2)} + h_4^{(1)} h_1^{(2)}\right) i_4. \end{aligned}$$

The conjugate of hyperbolic quaternion h is denoted by $\bar{h}$ and it is

$$\bar{h} = h_1 i_1 - h_2 i_2 - h_3 i_3 - h_4 i_4 = \langle h_1, -h_2, -h_3, -h_4 \rangle.$$

The norm of h is defined as

$$N_h = h \cdot \bar{h} = {h_1}^2 - {h_2}^2 - {h_3}^2 - {h_4}^2.$$

In [24], the hyperbolic k-Fibonacci and k-Lucas quaternions $\mathcal{Q^F}_{k,n}$ and $\mathcal{Q^L}_{k,n}$ are defined as

$$\begin{aligned}\mathcal{Q^F}_{k,n} &= F_{k,n}i_1 + F_{k,n+1}i_2 + F_{k,n+2}i_3 + F_{k,n+3}i_4\\ &= \big\langle F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}\big\rangle\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q^L}_{k,n} &= L_{k,n}i_1 + L_{k,n+1}i_2 + L_{k,n+2}i_3 + L_{k,n+3}i_4\\ &= \big\langle L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}\big\rangle,\end{aligned}$$

respectively, where $F_{k,n}$ is n^{th} k-Fibonacci sequence and $L_{k,n}$ is n^{th} k- Lucas sequence. Here, i_1, i_2, i_3, i_4 are hyperbolic quaternionic units which satisfy the multiplication rule

$$\begin{aligned}&{i_2}^2 = {i_3}^2 = {i_4}^2 = i_2i_3i_4 = +1, i_1 = 1,\\ &i_2i_3 = i_4 = -i_3i_2, i_3i_4 = i_2 = -i_4i_3, i_4i_2 = i_3 = -i_2i_4.\end{aligned}$$

The Binet formulas for the hyperbolic k-Fibonacci and k-Lucas quaternions are

$$\mathcal{Q^F}_{k,n} = \frac{\bar{r_1}{r_1}^n - \bar{r}_2{r_2}^n}{r_1 - r_2},$$

and

$$\mathcal{Q^L}_{k,n} = \bar{r_1}{r_1}^n + \bar{r_2}{r_2}^n$$

where,

$$\bar{r_1} = i_1 + r_1i_2 + {r_1}^2i_3 + {r_1}^3i_4 = \big\langle 1, r_1, {r_1}^2, {r_1}^3\big\rangle,$$

$$\bar{r_2} = i_1 + r_2i_2 + {r_2}^2i_3 + {r_2}^3i_4 = \big\langle 1, r_2, {r_2}^2, {r_2}^3\big\rangle,$$

and i_1, i_2, i_3, i_4 are hyperbolic quaternionic units.

Different properties of the hyperbolic k-Fibonacci and k-Lucas quaternions are investigated in [24], some of these are

$$(1)\quad \mathcal{Q^F}_{k,n+2} = k\mathcal{Q^F}_{k,n+1} + \mathcal{Q^F}_{k,n},$$

$$(2)\quad \mathcal{Q^L}_{k,n+2} = k\mathcal{Q^L}_{k,n+1} + \mathcal{Q^L}_{k,n},$$

$$(3)\quad \mathcal{Q^L}_{k,n} = \mathcal{Q^F}_{k,n+1} + \mathcal{Q^L}_{k,n-1},$$

$$(4)\quad \sum_{n=0}^{\infty} \mathcal{Q^F}_{k,tn}x^n = \frac{\mathcal{Q^F}_{k,0} + \big(\mathcal{Q^L}_{k,0}F_{k,t} - \mathcal{Q^F}_{k,t}\big)x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(5)\quad \sum_{n=0}^{\infty} \mathcal{Q^L}_{k,tn}x^n = \frac{\mathcal{Q^L}_{k,0} - \big(\mathcal{Q^L}_{k,0}L_{k,t} - \mathcal{Q^L}_{k,t}\big)x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(6)\quad \sum_{n=0}^{\infty} \mathcal{Q^F}_{k,tn+s}x^n = \frac{\mathcal{Q^F}_{k,s} + (-1)^t x \mathcal{Q^F}_{s,s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(7)\quad \sum_{n=0}^{\infty} \mathcal{Q^L}_{k,tn+s}x^n = \frac{\mathcal{Q^L}_{k,s} + (-1)^t x \mathcal{Q^L}_{s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(8)\quad \sum_{n=0}^{\infty} \frac{\mathcal{Q^F}_{k,tn}}{n!}x^n = \frac{\bar{r_1}e^{{r_1}^t x} - \bar{r_2}e^{{r_2}^t x}}{r_1 - r_2},$$

$$(9)\quad \sum_{n=0}^{\infty} \frac{\mathcal{Q^L}_{k,tn}}{n!}x^n = \bar{r_1}e^{{r_1}^t x} + \bar{r_2}e^{{r_2}^t x},$$

$$(10)\quad \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{Q^F}_{k,i} = \mathcal{Q^F}_{k,2n},$$

$$(11)\quad \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{Q^L}_{k,i} = \mathcal{Q^L}_{k,2n},$$

$$(12)\quad \mathcal{Q^F}_{k,n-t}\mathcal{Q^F}_{k,n+t} - {\mathcal{Q^F}_{k,n}}^2 = (-1)^{n-t}F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, \\ -2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big),$$

$$(13)\quad \mathcal{Q^L}_{k,n-t}\mathcal{Q^L}_{k,n+t} - {\mathcal{Q^L}_{k,n}}^2 = \delta(-1)^{n-t+1}F_{k,t}\big(0, -2F_{k,t+1}, -2F_{k,t+2}, \\ -2F_{k,t+3} + F_{k,t-3} + F_{k,t+1} + F_{k,t-1}\big),$$

$$(14)\quad \mathcal{Q^F}_{k,n-1}\mathcal{Q^F}_{k,n+1} - {\mathcal{Q^F}_{k,n}}^2 = (-1)^n\big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big),$$

$$(15)\quad \mathcal{Q^L}_{k,n-1}\mathcal{Q^L}_{k,n+1} - {\mathcal{Q^L}_{k,n}}^2 = \delta(-1)^{n-1}\big(0, -2F_{k,2}, 2F_{k,3}, F_{k,4}\big),$$

$$(16)\quad \mathcal{Q^F}_{k,t}\mathcal{Q^F}_{k,n+1} - \mathcal{Q^F}_{k,t+1}\mathcal{Q^F}_{k,n} = (-1)^n\big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2}, \\ F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big),$$

$$(17)\quad \mathcal{Q^L}_{k,t}\mathcal{Q^L}_{k,n+1} - \mathcal{Q^L}_{k,t+1}\mathcal{Q^L}_{k,n} = (-1)^{n+1}\delta\big(0, -2F_{k,t-n-1}, 2F_{k,t-n-2}, \\ F_{k,t-n+3} + F_{k,t-n-3} + F_{k,t-n+1} + F_{k,t-n-1}\big),$$

$$(18)\quad {\mathcal{Q^F}_{k,t}}^2 + {\mathcal{Q^L}_{k,t}}^2 = \frac{2(k^2+5)}{k}\mathcal{Q^L}_{k,2t} + \delta(k^2+5)L_{k,2t+3} \\ + 2(-1)^t\frac{(k^2+3)}{\delta}\big(\mathcal{Q^L}_{k,0} - 2\big),$$

$$(19)\quad {\mathcal{Q^F}_{k,t}}^2 - {\mathcal{Q^L}_{k,t}}^2 = \frac{2(k^2+3)}{\delta}\mathcal{Q^L}_{k,2t} + (k^2+3)(k^2+2)L_{k,2t+3} \\ + 2(-1)^{t+1}\frac{(k^2+5)}{\delta}\big(\mathcal{Q^L}_{k,0} - 2\big),$$

$$(20)\quad \mathcal{Q^F}_{k,r+s}\mathcal{Q^L}_{k,r+t} - \mathcal{Q^F}_{k,r+t}\mathcal{Q^L}_{k,r+s} = 2(-1)^{r+t}\big(\mathcal{Q^L}_{k,0} - 2\big)F_{k,s-t},$$

$$(21)\quad \mathcal{Q^F}_{k,s+t} + (-1)^t\mathcal{Q^F}_{k,s-t} = \mathcal{Q^F}_{k,s}L_{k,t},$$

$$(22)\quad \mathcal{Q^L}_{k,s+t} + (-1)^t\mathcal{Q^L}_{k,s-t} = \mathcal{Q^L}_{k,s}L_{k,t},$$

$$(23)\quad \mathcal{Q^F}_{k,s}\mathcal{Q^F}_{k,t} - \mathcal{Q^F}_{k,t}\mathcal{Q^F}_{k,s} = 2(-1)^sF_{k,t-s}\big(0, -1, -k, 1\big),$$

$$(24)\quad \mathcal{Q^L}_{k,s}\mathcal{Q^L}_{k,t} - \mathcal{Q^L}_{k,t}\mathcal{Q^L}_{k,s} = 2(-1)^{s+1}F_{k,t-s}\delta\big(0, -1, -k, 1\big),$$

$$(25)\quad \mathcal{Q}^{\mathcal{F}}{}_{k,t}\mathcal{Q}^{\mathcal{L}}{}_{k,s} - \mathcal{Q}^{\mathcal{F}}{}_{k,s}\mathcal{Q}^{\mathcal{L}}{}_{k,t} = 2(-1)^s F_{k,t-s}\big(\mathcal{Q}^{\mathcal{L}}{}_{k,0} - 2\big),$$

$$(26)\quad \mathcal{Q}^{\mathcal{F}}{}_{k,t}\mathcal{Q}^{\mathcal{L}}{}_{k,s} - \mathcal{Q}^{\mathcal{L}}{}_{k,t}\mathcal{Q}^{\mathcal{F}}{}_{k,s} = 2(-1)^s \bar{r_2}\big[\mathcal{Q}^{\mathcal{F}}{}_{k,t-s} - {r_2}^{t-s}\big(0,1,k,k^2+1\big)\big].$$

Hyperbolic k-Fibonacci and k-LucasOctonions

A hyperbolic octonion $\mathcal{O}$ is an expression of the form

$$\begin{aligned}\mathcal{O} &= h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7\\ &= \big\langle h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7\big\rangle,\end{aligned}$$

with real components h_0, h_1, h_2, h_3, h_4, h_5, h_6, h_7 and i_1, i_2, i_3 are quaternion imaginary units, $e_4({e_4}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions are defined as follows:

$$i_1 e_4 = e_5, i_2 e_4 = e_6, i_3 e_4 = e_7, {e_4}^2 = {e_5}^2 = {e_6}^2 = {e_7}^2 = 1.$$

The bases of hyperbolic octonion $\mathcal{O}$ have multiplication rules as in Table 1.

$\cdot$	i_1	i_2	i_3	e_4	e_5	e_6	e_7
i_1	-1	i_3	$-i_2$	e_5	e_4	$-e_7$	e_6
i_2	$-e_3$	-1	i_1	e_6	e_7	e_4	$-e_5$
i_3	i_2	$-i_1$	-1	e_7	$-e_6$	e_5	e_4
e_4	$-e_5$	$-e_6$	$-e_7$	1	i_1	i_2	i_3
e_5	$-e_4$	$-e_7$	e_6	$-i_1$	1	i_3	$-i_2$
e_6	e_7	$-e_4$	$-e_5$	$-i_2$	$-i_3$	1	i_1
e_7	$-e_6$	e_5	$-e_4$	$-i_3$	i_2	$-i_1$	1

Table 1: Rules for multiplication of hyperbolic octonion bases

Cariow and Cariow [26, 27] state low multiplicative complexity algorithm for multiplying two hyperbolic octonions.

The scalar and the vector part of a hyperbolic octonion $\mathcal{O}$ are denoted by $S_{\mathcal{O}} = h_0$ and $\overrightarrow{V}_{\mathcal{O}} = h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7$, respectively. Thus, a hyperbolic octonion $\mathcal{O}$ is given by $\mathcal{O} = S_{\mathcal{O}} + \overrightarrow{V}_{\mathcal{O}}$. For any two hyperbolic octonions $\mathcal{O}^{(h)} = h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7$ and $\mathcal{O}^{(H)} = H_0 + H_1 i_1 + H_2 i_2 + H_3 i_3 + H_4 e_4 + H_5 e_5 + H_6 e_6 + H_7 e_7$ addition and subtraction of the hyperbolic octonions is defined by

$$\begin{aligned}\mathcal{O}^{(h)} \pm \mathcal{O}^{(H)} &= \big(h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7\big)\\ &\pm \big(H_0 + H_1 i_1 + H_2 i_2 + H_3 i_3 + H_4 e_4 + H_5 e_5 + H_6 e_6 + H_7 e_7\big)\\ &= \big(h_0 \pm H_0\big) + \big(h_1 \pm H_1\big) i_1 + \big(h_2 \pm H_2\big) i_2\big) + \big(h_3 \pm H_3\big) i_3\end{aligned}$$

$$+\big(h_4 \pm H_4\big)e_4 + \big(h_5 \pm H_5\big)e_5 + \big(h_6 \pm H_6\big)e_6 + \big(h_7 \pm H_7\big)e_7.$$

Multiplication of the hyperbolic octonions is defined by

$$\begin{aligned}
\mathcal{O}^{(h)} \cdot \mathcal{O}^{(H)} &= \big(h_0 + h_1 i_1 + h_2 i_2 + h_3 i_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7\big) \\
&\cdot \big(H_0 + H_1 i_1 + H_2 i_2 + H_3 i_3 + H_4 e_4 + H_5 e_5 + H_6 e_6 + H_7 e_7\big) \\
&= \mathcal{O}_0 + \mathcal{O}_1 i_1 + \mathcal{O}_2 i_2 + \mathcal{O}_3 i_3 + \mathcal{O}_4 e_4 + \mathcal{O}_5 e_5 + \mathcal{O}_6 e_6 + \mathcal{O}_7 e_7,
\end{aligned}$$

where,

$$\begin{aligned}
\mathcal{O}_0 &= h_0H_0 - h_1H_1 - h_2H_2 - h_3H_3 + h_4H_4 + h_5H_5 + h_6H_6 + h_7H_7, \\
\mathcal{O}_1 &= h_0H_1 + h_1H_0 + h_2H_3 - h_3H_2 + h_4H_5 - h_5H_4 + h_6H_7 - h_7H_6, \\
\mathcal{O}_2 &= h_0H_2 - h_1H_3 + h_2H_0 + h_3H_1 + h_4H_6 - h_5H_7 - h_6H_4 + h_7H_5, \\
\mathcal{O}_3 &= h_0H_3 + h_1H_2 - h_2H_1 + h_3H_0 + h_4H_7 + h_5H_6 - h_6H_5 - h_7H_4, \\
\mathcal{O}_4 &= h_0H_4 + h_1H_5 + h_2H_6 + h_3H_7 + h_4H_0 - h_5H_1 - h_6H_2 - h_7H_3, \\
\mathcal{O}_5 &= h_0H_5 + h_1H_4 - h_2H_7 + h_3H_6 - h_4H_1 + h_5H_0 - h_6H_3 + h_7H_2, \\
\mathcal{O}_6 &= h_0H_6 + h_1H_7 + h_2H_4 - h_3H_5 - h_4H_2 + h_5H_3 + h_6H_0 - h_7H_1, \\
\mathcal{O}_7 &= h_0H_7 - h_1H_6 + h_2H_5 + h_3H_4 - h_4H_3 - h_5H_2 + h_6H_1 + h_7H_0.
\end{aligned}$$

The conjugate of hyperbolic octonion $\mathcal{O}$ is denoted by $\bar{\mathcal{O}}$ and it is

$$\bar{\mathcal{O}} = h_0 - h_1 i_1 - h_2 i_2 - h_3 i_3 - h_4 e_4 - h_5 e_5 - h_6 e_6 - h_7 e_7.$$

The norm of $\mathcal{O}$ is defined as

$$N_{\mathcal{O}} = \mathcal{O} \cdot \bar{\mathcal{O}} = {h_0}^2 - {h_1}^2 - {h_2}^2 - {h_3}^2 + {h_4}^2 + {h_5}^2 + {h_6}^2 + {h_7}^2.$$

In [25], the hyperbolic k-Fibonacci and k-Lucas octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{O}^{\mathcal{L}}{}_{k,n}$ are defined as

$$\begin{aligned}
\mathcal{O}^{\mathcal{F}}{}_{k,n} &= F_{k,n} + F_{k,n+1} i_1 + F_{k,n+2} i_2 + F_{k,n+3} i_3 + F_{k,n+4} e_4 + F_{k,n+5} e_5 \\
&+ F_{k,n+6} e_6 + F_{k,n+7} e_7 \\
&= \big\langle F_{k,n}, F_{k,n+1}, F_{k,n+2}, F_{k,n+3}, F_{k,n+4}, F_{k,n+5}, F_{k,n+6}, F_{k,n+7} \big\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{O}^{\mathcal{L}}{}_{k,n} &= L_{k,n} + L_{k,n+1} i_1 + L_{k,n+2} i_2 + L_{k,n+3} i_3 + L_{k,n+4} e_4 + L_{k,n+5} e_5 \\
&+ L_{k,n+6} e_6 + L_{k,n+7} e_7 \\
&= \big\langle L_{k,n}, L_{k,n+1}, L_{k,n+2}, L_{k,n+3}, L_{k,n+4}, L_{k,n+5}, L_{k,n+6}, L_{k,n+7} \big\rangle,
\end{aligned}$$

respectively. The conjugate of hyperbolic k-Fibonacci and k-Lucas octonions $\bar{\mathcal{O}^{\mathcal{F}}}_{k,n}$ and $\bar{\mathcal{O}^{\mathcal{L}}}_{k,n}$ are defined as

$$\begin{aligned}\bar{\mathcal{O}^{\mathcal{F}}}_{k,n} &= F_{k,n} - F_{k,n+1}i_1 - F_{k,n+2}i_2 - F_{k,n+3}i_3 - F_{k,n+4}e_4 - F_{k,n+5}e_5\\ &- F_{k,n+6}e_6 - F_{k,n+7}e_7\\ &= \langle F_{k,n}, -F_{k,n+1}, -F_{k,n+2}, -F_{k,n+3}, -F_{k,n+4}, -F_{k,n+5}, -F_{k,n+6}, -F_{k,n+7}\rangle,\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{O}^{\mathcal{L}}}_{k,n} &= L_{k,n} - L_{k,n+1}i_1 - L_{k,n+2}i_2 - L_{k,n+3}i_3 - L_{k,n+4}e_4 - L_{k,n+5}e_5\\ &- L_{k,n+6}e_6 - L_{k,n+7}e_7\\ &= \langle L_{k,n}, -L_{k,n+1}, -L_{k,n+2}, -L_{k,n+3}, -L_{k,n+4}, -L_{k,n+5}, -L_{k,n+6}, -L_{k,n+7}\rangle,\end{aligned}$$

respectively, where $F_{k,n}$ is n^{th} k-Fibonacci sequence and $L_{k,n}$ is n^{th} k-Lucas sequence. Here, i_1, i_2, i_3 are quaternion imaginary units, $e_4(e_4{}^2 = 1)$ is a counter imaginary unit, and the bases of hyperbolic octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{O}^{\mathcal{L}}{}_{k,n}$ are defined as $i_1e_4 = e_5, i_2e_4 = e_6, i_3e_4 = e_7, e_4{}^2 = e_5{}^2 = e_6{}^2 = e_7{}^2 = 1$. The bases of hyperbolic octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{O}^{\mathcal{L}}{}_{k,n}$ have multiplication rules as in Table 1.

The Binet Formulas for the hyperbolic k-Fibonacci and k-Lucas octonions $\mathcal{O}^{\mathcal{F}}{}_{k,n}$ and $\mathcal{O}^{\mathcal{L}}{}_{k,n}$ are

$$(i)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n} = \frac{\bar{r_1}r_1{}^n - \bar{r_2}r_2{}^n}{r_1 - r_2}$$

$$(ii)\quad \mathcal{O}^{\mathcal{L}}{}_{k,n} = \bar{r_1}r_1{}^n + \bar{r_2}r_2{}^n,$$

$$(iii)\quad \bar{\mathcal{O}^{\mathcal{F}}}_{k,n} = \frac{\bar{r_3}r_1{}^n - \bar{r_4}r_2{}^n}{r_1 - r_2}$$

$$(iv)\quad \bar{\mathcal{O}^{\mathcal{L}}}_{k,n} = \bar{r_3}r_1{}^n + \bar{r_4}r_2{}^n,$$

where,

$$\begin{aligned}\bar{r_1} &= 1 + r_1i_1 + r_1{}^2i_2 + r_1{}^3i_3 + r_1{}^4e_4 + r_1{}^5e_5 + r_1{}^6e_6 + r_1{}^7e_7\\ &= \langle 1, r_1, r_1{}^2, r_1{}^3, r_1{}^4, r_1{}^5, r_1{}^6, r_1{}^7\rangle,\\ \bar{r_2} &= 1 + r_2i_1 + r_2{}^2i_2 + r_2{}^3i_3 + r_2{}^4e_4 + r_2{}^5e_5 + r_2{}^6e_6 + r_2{}^7e_7\\ &= \langle 1, r_2, r_2{}^2, r_2{}^3, r_2{}^4, r_2{}^5, r_2{}^6, r_2{}^7\rangle,\\ \bar{r_3} &= 1 - r_1i_1 - r_1{}^2i_2 - r_1{}^3i_3 - r_1{}^4e_4 - r_1{}^5e_5 - r_1{}^6e_6 - r_1{}^7e_7\\ &= \langle 1, -r_1, -r_1{}^2, -r_1{}^3, -r_1{}^4, -r_1{}^5, -r_1{}^6, -r_1{}^7\rangle,\\ \bar{r_4} &= 1 + r_2i_1 - r_2{}^2i_2 - r_2{}^3i_3 - r_2{}^4e_4 - r_2{}^5e_5 - r_2{}^6e_6 - r_2{}^7e_7\\ &= \langle 1, -r_2, -r_2{}^2, -r_2{}^3, -r_2{}^4, -r_2{}^5, -r_2{}^6, -r_2{}^7\rangle.\end{aligned}$$

The properties presented here are the extension of Cassini, Catalan and d'Ocagne's identity to hyperbolic k-Fibonacci and k-Lucas octonions.

***Theorem* 1.1** *(Catalan's Identity). For any integer t and s, we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,n-t}{\mathcal{O}^{\mathcal{F}}}_{k,n+t} - {{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2 = (-1)^{n-t} F_{k,t} \bar{\mathcal{U}}_{k,t},$$
$$(ii) \quad {\mathcal{O}^{\mathcal{L}}}_{k,n-t}{\mathcal{O}^{\mathcal{L}}}_{k,n+t} - {{\mathcal{O}^{\mathcal{L}}}_{k,n}}^2 = \delta(-1)^{n-t+1} F_{k,t} \bar{\mathcal{V}}_{k,m-n}.$$

In Catalan's identity, taking $t = 1$, it reduces to Cassini's Identity.

***Theorem* 1.2** *(Cassini's Identity). For all $n \geq 1$, we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,n-1}{\mathcal{O}^{\mathcal{F}}}_{k,n+1} - {{\mathcal{O}^{\mathcal{F}}}_{k,n}}^2 = (-1)^{n-1} F_{k,t} \bar{\mathcal{U}}_{k,1},$$
$$(ii) \quad {\mathcal{O}^{\mathcal{L}}}_{k,n-1}{\mathcal{O}^{\mathcal{L}}}_{k,n+1} - {{\mathcal{O}^{\mathcal{L}}}_{k,n}}^2 = \delta(-1)^{n} F_{k,t} \bar{\mathcal{V}}_{k,1}.$$

***Theorem* 1.3** *(d'Ocagene's Identity). Let n be any non-negative integer and t a natural number. If $t \geq n+1$, then we have*

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,t}{\mathcal{O}^{\mathcal{F}}}_{k,n+1} - {\mathcal{O}^{\mathcal{F}}}_{k,t+1}{\mathcal{O}^{\mathcal{F}}}_{k,n} = (-1)^{n} \bar{\mathcal{V}}_{k,t-n},$$
$$(ii) \quad {\mathcal{O}^{\mathcal{L}}}_{k,t}{\mathcal{O}^{\mathcal{L}}}_{k,n+1} - {\mathcal{O}^{\mathcal{L}}}_{k,t+1}{\mathcal{O}^{\mathcal{L}}}_{k,n} = (-1)^{n+1} \delta \bar{\mathcal{V}}_{k,t-n}.$$

Different properties of the hyperbolic k-Fibonacci and k-Lucas octonions are investigated in [25]. These properties are the generalisations of Cassini, Catalan and d'Ocagne's identity to hyperbolic k-Fibonacci and k-Lucas octonions.

$$(1) \quad {\mathcal{O}^{\mathcal{F}}}_{k,n+2} = k{\mathcal{O}^{\mathcal{F}}}_{k,n+1} + {\mathcal{O}^{\mathcal{F}}}_{k,n},$$
$$(2) \quad {\mathcal{O}^{\mathcal{L}}}_{k,n+2} = k{\mathcal{O}^{\mathcal{L}}}_{k,n+1} + {\mathcal{O}^{\mathcal{L}}}_{k,n},$$
$$(3) \quad {\mathcal{O}^{\mathcal{L}}}_{k,n} = {\mathcal{O}^{\mathcal{F}}}_{k,n+1} + {\mathcal{O}^{\mathcal{F}}}_{k,n-1},$$
$$(4) \quad {\bar{\mathcal{O}}^{\mathcal{F}}}_{k,n+2} = k{\bar{\mathcal{O}}^{\mathcal{F}}}_{k,n+1} + {\bar{\mathcal{O}}^{\mathcal{F}}}_{k,n},$$
$$(5) \quad {\bar{\mathcal{O}}^{\mathcal{L}}}_{k,n+2} = k{\bar{\mathcal{O}}^{\mathcal{L}}}_{k,n+1} + {\bar{\mathcal{O}}^{\mathcal{L}}}_{k,n},$$
$$(6) \quad {\bar{\mathcal{O}}^{\mathcal{L}}}_{k,n} = {\bar{\mathcal{O}}^{\mathcal{F}}}_{k,n+1} + {\bar{\mathcal{O}}^{\mathcal{F}}}_{k,n-1},$$
$$(7) \quad \sum_{n=0}^{\infty} {\mathcal{O}^{\mathcal{F}}}_{k,tn} x^n = \frac{{\mathcal{O}^{\mathcal{F}}}_{k,0} + \left({\mathcal{O}^{\mathcal{L}}}_{k,0} F_{k,t} - {\mathcal{O}^{\mathcal{F}}}_{k,t}\right)x}{1 - xL_{k,t} + x^2(-1)^t},$$
$$(8) \quad \sum_{n=0}^{\infty} {\mathcal{O}^{\mathcal{L}}}_{k,tn} x^n = \frac{{\mathcal{O}^{\mathcal{L}}}_{k,0} - \left({\mathcal{O}^{\mathcal{L}}}_{k,0} L_{k,t} - {\mathcal{O}^{\mathcal{L}}}_{k,t}\right)x}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(9)\quad \sum_{n=0}^{\infty} \mathcal{O}^{\mathcal{F}}{}_{k,tn+s}x^n = \frac{\mathcal{O}^{\mathcal{F}}{}_{k,s} + (-1)^t x \mathcal{O}^{\mathcal{F}}{}_{s,s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(10)\quad \sum_{n=0}^{\infty} \mathcal{O}^{\mathcal{L}}{}_{k,tn+s}x^n = \frac{\mathcal{O}^{\mathcal{L}}{}_{k,s} + (-1)^t x \mathcal{O}^{\mathcal{L}}{}_{s-t}}{1 - xL_{k,t} + x^2(-1)^t},$$

$$(11)\quad \sum_{n=0}^{\infty} \frac{\mathcal{O}^{\mathcal{F}}{}_{k,tn}}{n!}x^n = \frac{\bar{r_1}e^{r_1{}^t x} - \bar{r_2}e^{r_2{}^t x}}{r_1 - r_2},$$

$$(12)\quad \sum_{n=0}^{\infty} \frac{\mathcal{O}^{\mathcal{L}}{}_{k,tn}}{n!}x^n = \bar{r_1}e^{r_1{}^t x} + \bar{r_2}e^{r_2{}^t x},$$

$$(13)\quad \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{O}^{\mathcal{F}}{}_{k,i} = \mathcal{O}^{\mathcal{F}}{}_{k,2n},$$

$$(14)\quad \sum_{i=0}^{n} \binom{n}{i} k^i \mathcal{O}^{\mathcal{L}}{}_{k,i} = \mathcal{O}^{\mathcal{L}}{}_{k,2n},$$

$$(15)\quad \mathcal{O}^{\mathcal{F}}{}_{k,t}{}^2 + \mathcal{O}^{\mathcal{L}}{}_{k,t}{}^2 = \frac{1}{\delta}\big[(1+\delta)\bar{\mathcal{W}}_{k,2t} + (\delta-1)(-1)^t \mathcal{O}^{\mathcal{L}}{}_{k,0}\big],$$

$$(16)\quad \mathcal{O}^{\mathcal{F}}{}_{k,t}{}^2 - \mathcal{O}^{\mathcal{L}}{}_{k,t}{}^2 = \frac{1}{\delta}\big[(1-\delta)\bar{\mathcal{W}}_{k,2t} - (1+\delta)(-1)^t \mathcal{O}^{\mathcal{L}}{}_{k,0}\big],$$

$$(17)\quad \mathcal{O}^{\mathcal{F}}{}_{k,r+s}\mathcal{O}^{\mathcal{L}}{}_{k,r+t} - \mathcal{O}^{\mathcal{F}}{}_{k,r+t}\mathcal{O}^{\mathcal{L}}{}_{k,r+s} = 2(-1)^{r+t}\big(\mathcal{O}^{\mathcal{L}}{}_{k,0} - 2\big)F_{k,s-t},$$

$$(18)\quad \mathcal{O}^{\mathcal{F}}{}_{k,s+t} + (-1)^t \mathcal{O}^{\mathcal{F}}{}_{k,s-t} = \mathcal{O}^{\mathcal{F}}{}_{k,s}L_{k,t},$$

$$(19)\quad \mathcal{O}^{\mathcal{L}}{}_{k,s+t} + (-1)^t \mathcal{O}^{\mathcal{L}}{}_{k,s-t} = \mathcal{O}^{\mathcal{L}}{}_{k,s}L_{k,t},$$

$$(20)\quad \mathcal{O}^{\mathcal{F}}{}_{k,s}\mathcal{O}^{\mathcal{F}}{}_{k,t} - \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{F}}{}_{k,s} = (-1)^s \delta^{-\frac{1}{2}}\ \bar{u_5}F_{k,t-s},$$

$$(21)\quad \mathcal{O}^{\mathcal{L}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t} - \mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s} = (-1)^t \delta^{\frac{1}{2}}\ \bar{u_5}F_{k,s-t},$$

$$(22)\quad \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s} - \mathcal{O}^{\mathcal{F}}{}_{k,s}\mathcal{O}^{\mathcal{L}}{}_{k,t} = 2(-1)^s F_{k,t-s}\mathcal{O}^{\mathcal{L}}{}_{k,0},$$

$$(23)\quad \mathcal{O}^{\mathcal{F}}{}_{k,t}\mathcal{O}^{\mathcal{L}}{}_{k,s} - \mathcal{O}^{\mathcal{L}}{}_{k,t}\mathcal{O}^{\mathcal{F}}{}_{k,s} = 2(-1)^s \bar{\mathcal{V}}_{k,t-s},$$

$$(24)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} - \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n} = (-1)^n \delta^{-\frac{1}{2}}(2\bar{u_{10}} - \bar{u_{14}}),$$

$$(25)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{L}}{}_{k,n} + \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,n} = 2\delta^{-\frac{1}{2}}(L_{k,t+1} + L_{k,t+3} - L_{k,t+9} - L_{k,t+13})$$
$$+ (-1)^n \delta^{-\frac{1}{2}}(2\bar{u_{15}} - \bar{u_{16}}),$$

$$(26)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n}\mathcal{O}^{\mathcal{L}}{}_{k,n} - \bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n}\bar{\mathcal{O}}^{\mathcal{F}}{}_{k,n} = \bar{\mathcal{X}}_{k,2n} + \delta^{-\frac{1}{2}}\big((-1)^n\bar{u_5} - \bar{u_1} + \bar{u_2}\big) - \bar{\mathcal{Y}}_{k,2n},$$

where, ($\bar{u_1}$- $\bar{u_{18}}$) are defined in Lemma 2.5 and $\bar{\mathcal{U}}_{k,t}$, $\bar{\mathcal{V}}_{k,t}$, $\bar{\mathcal{W}}_{k,t}$, $\bar{\mathcal{X}}_{k,t}$ and $\bar{\mathcal{Y}}_{k,t}$ are defined in Lemma 2.8 of [25]. In current paper, our main goal is to investigate some binomial and congruence properties for the k-Fibonacci and k-Lucas hyperbolic octonions.

2. PROPERTIES OF HYPERBOLIC k-FIBONACCI AND k-LUCAS OCTONIONS

In this section, we explore some binomial and congruence properties of the hyperbolic k-Fibonacci and k-Lucas octonions.

Lemma 2.1 — For $n \geq 0$, we have

(i) $\quad {r_1}^n = r_1 F_{k,n} + F_{k,n-1},$

(ii) $\quad {r_2}^n = r_2 F_{k,n} + F_{k,n-1},$

(iii) $\quad {r_1}^{2n} = {r_1}^n L_{k,n} - (-1)^n,$

(iv) $\quad {r_2}^{2n} = {r_2}^n L_{k,n} - (-1)^n,$

(v) $\quad {r_1}^{tn} = {r_1}^n \dfrac{F_{k,tn}}{F_{k,n}} - (-1)^n - \dfrac{F_{k,(t-1)n}}{F_{k,n}},$

(vi) $\quad {r_2}^{tn} = {r_2}^n \dfrac{F_{k,tn}}{F_{k,n}} - (-1)^n - \dfrac{F_{k,(t-1)n}}{F_{k,n}},$

(vii) $\quad {r_1}^{sn} F_{k,rn} - {r_1}^{rn} F_{k,sn} = (-1)^{sn} F_{k,(r-s)n},$

$(viii)$ $\quad {r_2}^{sn} F_{k,rn} - {r_2}^{rn} F_{k,sn} = (-1)^{sn} F_{k,(r-s)n},$

(ix) $\quad 1 + kr_1 + {r_1}^{2(2^{n+1}+1)} = L_{k,2^{n+1}} {r_1}^{2(2^n+1)},$

(x) $\quad 1 + kr_2 + {r_2}^{2(2^{n+1}+1)} = L_{k,2^{n+1}} {r_1}^{2(2^n+1)},$

(xi) $\quad$ For every $n, t \geq 1$ and $l_n = \displaystyle\sum_{i=1}^{n} L_{k,2^i}$, we have

$$(a) \quad 1 + {r_1}^{2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}} {r_1}^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}} {r_1}^{2^{n-t}} - l_{n-1} \displaystyle\sum_{i=2}^{t} \dfrac{1}{l_{n-i}}, & \text{for } t = 2, 3, 4, \ldots, n-2\ ; \\ l_{n-1} {r_1}^2 - l_{n-1} \displaystyle\sum_{i=2}^{n-1} \dfrac{1}{l_{n-i}} \end{cases},$$

$$(b) \quad 1 + {r_2}^{2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}} {r_2}^{2^{n-1}}; \\ \dfrac{l_{n-1}}{l_{n-t-1}} {r_2}^{2^{n-t}} - l_{n-1} \displaystyle\sum_{i=2}^{t} \dfrac{1}{l_{n-i}}, & \text{for } t = 2, 3, 4, \ldots, n-2\ ; \\ l_{n-1} {r_2}^2 - l_{n-1} \displaystyle\sum_{i=2}^{n-1} \dfrac{1}{l_{n-i}} \end{cases},$$

(xii) $\quad r_1^{2t} = \dfrac{F_{k,2t}}{k} r_1 \sqrt{\delta} - \dfrac{L_{k,2t-1}}{k},$

$(xiii)$ $\quad r_2^{2t} = -\dfrac{F_{k,2t}}{k} r_2 \sqrt{\delta} - \dfrac{L_{k,2t-1}}{k}.$

$$(xiv) \quad r_1^{2t+1} = \frac{L_{k,2t+1}}{k} r_1 - \frac{F_{k,2t}}{k} \sqrt{\delta},$$
$$(xv) \quad r_2^{2t+1} = \frac{L_{k,2t+1}}{k} r_2 + \frac{F_{k,2t}}{k} \sqrt{\delta}.$$

PROOF : (i). We use induction principle on n, for $n = 1$, we have

$$r_1{}^1 = 1 \cdot r_1 + 0 = r_1 F_{k,1} + F_{k,0}.$$

For $n = 2$, since r_1 is root of $r^2 - kr - 1 = 0$ therefore we have

$$r_1{}^2 = kr_1 + 1 = r_1 F_{k,2} + F_{k,1}.$$

Now, consider

$$r_1{}^{n+1} = r_1{}^n \cdot r_1 = (r_1 F_{k,n} + F_{k,n-1}) r_1 = r_1{}^2 F_{k,n} + r_1 F_{k,n-1}.$$

Using $r_1{}^2 = kr_1 + 1$, we have

$$= (kr_1 + 1) F_{k,n} + r_1 F_{k,n-1} = r_1 (k F_{k,n} + F_{k,n-1}) + F_{k,n} = r_1 F_{k,n+1} + F_{k,n}.$$

This complete the proof of (i).

(iii). Using (i), we have

$$\begin{aligned} r_1{}^{2n} &= F_{k,n} r_1{}^{n+1} + r_1{}^n F_{k,n-1} \\ &= F_{k,n}(r_1 F_{k,n+1} + F_{k,n}) + r_1{}^n F_{k,n-1} \\ &= r_1 F_{k,n} F_{k,n+1} + F_{k,n-1} r_1{}^n + F_{k,n}^2 \\ &= (r_1{}^n - F_{k,n-1}) F_{k,n+1} + F_{k,n-1} r_1{}^n + F_{k,n}^2 \\ &= r_1{}^n (F_{k,n+1} + F_{k,n-1}) + F_{k,n}^2 - F_{k,n} F_{k,n-1}. \end{aligned}$$

Using $F_{k,n-1} F_{k,n+1} - F_{k,n}^2 = (-1)^n$ and $F_{k,n+1} + F_{k,n-1} = L_{k,n}$, we obtain

$$r_1{}^{2n} = L_{k,n} r_1{}^n - (-1)^n.$$

This complete the proof of (iii).

The proofs of (ii), (iv), (v), (vii), (viii), (ix), (x), (xi),(xii), (xiii), (xiv) and (xv) are similar to (i) and (iii). □

***Theorem* 2.2** — *For all* $n, r, s, t \geq 1$*, we* have

$$(i) \quad {\mathcal{O}^{\mathcal{F}}}_{k,n+t} = F_{k,n}{\mathcal{O}^{\mathcal{F}}}_{k,t+1} + F_{k,n-1}{\mathcal{O}^{\mathcal{F}}}_{k,t},$$

$$(ii) \quad {\mathcal{O}^{\mathcal{F}}}_{k,2n+t} = L_{k,n}{\mathcal{O}^{\mathcal{F}}}_{k,n+t} - (-1)^n{\mathcal{O}^{\mathcal{F}}}_{k,t},$$

$$(iii) \quad {\mathcal{O}^{\mathcal{F}}}_{k,sn+t} = \frac{F_{k,sn}}{F_{k,n}}{\mathcal{O}^{\mathcal{F}}}_{k,n+t} - (-1)^n\frac{F_{k,(s-1)n}}{F_{k,n}}{\mathcal{O}^{\mathcal{F}}}_{k,t},$$

$$(iv) \quad {\mathcal{O}^{\mathcal{F}}}_{k,sn+t}F_{k,rn} - {\mathcal{O}^{\mathcal{F}}}_{k,rn+t}F_{k,sn} = (-1)^{sn}{\mathcal{O}^{\mathcal{F}}}_{k,t}F_{k,(r-s)n},$$

$$(v) \quad {\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n+1}+2} = \frac{{\mathcal{O}^{\mathcal{F}}}_{k,t} + k{\mathcal{O}^{\mathcal{F}}}_{k,t+1} + {\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}},$$

$$(vi) \quad {\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n+1}+2} = \frac{{\mathcal{O}^{\mathcal{L}}}_{k,t} + k{\mathcal{O}^{\mathcal{L}}}_{k,t+1} + {\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n+2}+2}}{L_{k,2^{n+1}}},$$

(vii) *For every* $n, t \geq 1$ *and* $l_n = \displaystyle\sum_{i=1}^{n} L_{k,2^i}$*, we have*

$$(a) \quad {\mathcal{O}^{\mathcal{F}}}_{k,t+2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}}{\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n-1}} - {\mathcal{O}^{\mathcal{F}}}_{k,t}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}{\mathcal{O}^{\mathcal{F}}}_{k,t+2^{n-s}} - l_{n-1}\sum\limits_{i=2}^{s}(1+\dfrac{1}{l_{n-i}}){\mathcal{O}^{\mathcal{F}}}_{k,t}, \\ \qquad \text{If } s = 2,3,4,\ldots,n-2\,; \\ l_{n-1}{\mathcal{O}^{\mathcal{F}}}_{k,t+2} - l_{n-1}\sum\limits_{i=2}^{n-1}(\dfrac{1}{l_{n-i}}+1){\mathcal{O}^{\mathcal{F}}}_{k,t}. \end{cases},$$

$$(b) \quad {\mathcal{O}^{\mathcal{L}}}_{k,t+2^n} = \begin{cases} \dfrac{l_{n-1}}{l_{n-2}}{\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n-1}} - {\mathcal{O}^{\mathcal{L}}}_{k,t}; \\ \dfrac{l_{n-1}}{l_{n-t-1}}{\mathcal{O}^{\mathcal{L}}}_{k,t+2^{n-s}} - l_{n-1}\sum\limits_{i=2}^{s}(1+\dfrac{1}{l_{n-i}}){\mathcal{O}^{\mathcal{L}}}_{k,t}, \\ \qquad \text{If } s = 2,3,4,\ldots,n-2\,; \\ l_{n-1}{\mathcal{O}^{\mathcal{L}}}_{k,t+2} - l_{n-1}\sum\limits_{i=2}^{n-1}(\dfrac{1}{l_{n-i}}+1){\mathcal{O}^{\mathcal{L}}}_{k,t}. \end{cases},$$

$$(c) \quad {\mathcal{O}^{\mathcal{F}}}_{k,2^r n+t} = \begin{cases} \sum\limits_{i+j=n}\binom{n}{i}(\dfrac{l_{r-1}}{l_{r-2}})^i(-1)^j{\mathcal{O}^{\mathcal{F}}}_{k,2^{r-1}i+t}; \\ \sum\limits_{i+j=n}\binom{n}{i}(\dfrac{l_{r-1}}{l_{r-s-1}})^i(-1)^j(\sum_{h=2}^{s}(1+\dfrac{l_{r-1}}{l_{r-h}})^j{\mathcal{O}^{\mathcal{F}}}_{k,2^{n-s}i+t}, \\ \qquad \text{If } s = 2,3,4,\ldots,n-2\,; \\ \sum\limits_{i+j=n}\binom{n}{i}(l_{r-1})^i(-1)^j(\sum_{h=2}^{s}(1+\dfrac{l_{r-1}}{l_{r-h}})^j{\mathcal{O}^{\mathcal{F}}}_{k,2i+t}. \end{cases},$$

$$(d)\quad \mathcal{O^L}_{k,2^rn+t} = \begin{cases} \sum\limits_{i+j=n} \binom{n}{i}(\frac{l_{r-1}}{l_{r-2}})^i(-1)^j\mathcal{O^L}_{k,2^{r-1}i+t}; \\ \sum\limits_{i+j=n} \binom{n}{i}(\frac{l_{r-1}}{l_{r-s-1}})^i(-1)^j(\sum_{h=2}^{s}(1+\frac{l_{r-1}}{l_{r-h}})^j\mathcal{O^L}_{k,2^{n-s}i+t}, \\ \qquad\qquad \textit{If } s=2,3,4,\ldots,n-2\ ; \\ \sum\limits_{i+j=n} \binom{n}{i}(l_{r-1})^i(-1)^j(\sum_{h=2}^{s}(1+\frac{l_{r-1}}{l_{r-h}})^j\mathcal{O^L}_{k,2i+t}. \end{cases},$$

$$(viii)\quad \mathcal{O^F}_{k,s+2t} = \frac{F_{k,2t}}{k}\mathcal{O^L}_{k,s+1} - \frac{L_{k,2t-1}}{k}\mathcal{O^F}_{k,s},$$

$$(ix)\quad \mathcal{O^L}_{k,s+2t} = \frac{F_{k,2t}}{k}\delta\mathcal{O^F}_{k,s+1} - \frac{L_{k,2t-1}}{k}\mathcal{O^L}_{k,s},$$

$$(x)\quad \mathcal{O^F}_{k,s+2t} - \frac{F_{k,2t}}{k}\mathcal{O^F}_{k,s+2} + \frac{F_{k,2t-2}}{k}\mathcal{O^F}_{k,s} = 0,$$

$$(xi)\quad \mathcal{O^L}_{k,s+2t} - \frac{F_{k,2t}}{k}\mathcal{O^L}_{k,s+2} + \frac{F_{k,2t-2}}{k}\mathcal{O^L}_{k,s} = 0,$$

$$(xii)\quad \mathcal{O^F}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k}\mathcal{O^F}_{k,s+1} - \frac{F_{k,2t}}{k}\mathcal{O^L}_{k,s},$$

$$(xiii)\quad \mathcal{O^L}_{k,s+2t+1} = \frac{L_{k,2t+1}}{k}\mathcal{O^L}_{k,s+1} - \delta\frac{F_{k,2t}}{k}\mathcal{O^F}_{k,s},$$

$$(xiv)\quad \mathcal{O^F}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{O^F}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\mathcal{O^L}_{k,s} = 0,$$

$$(xv)\quad \mathcal{O^L}_{k,s+2t+1} - \frac{L_{k,2t+1}}{k(k^2+3)}\mathcal{O^L}_{k,s+3} + \frac{F_{k,2t-2}}{k(k^2+3)}\delta\mathcal{O^F}_{k,s} = 0.$$

PROOF : (i). From (i) and (ii) of Lemma 2.1, we have

$$r_1{}^n = r_1F_{k,n} + F_{k,n-1}, \tag{2.1}$$

$$r_2{}^n = r_2F_{k,n} + F_{k,n-1}. \tag{2.2}$$

Multiplying (2.1) by $\frac{\bar{r_1}r_1{}^t}{r_1-r_2}$ and (2.2) by $\frac{\bar{r_2}r_2{}^t}{r_1-r_2}$, we get

$$\frac{\bar{r_1}r_1{}^{n+t}}{r_1-r_2} = \frac{\bar{r_1}r_1{}^{t+1}}{r_1-r_2}F_{k,n} + \frac{\bar{r_1}r_1{}^t}{r_1-r_2}F_{k,n-1}, \tag{2.3}$$

$$\frac{\bar{r_2}r_2{}^{n+t}}{r_1-r_2} = \frac{\bar{r_2}r_2{}^{t+1}}{r_1-r_2}F_{k,n} + \frac{\bar{r_2}r_2{}^t}{r_1-r_2}F_{k,n-1}. \tag{2.4}$$

Subtracting (2.3) and (2.4), we obtain

$$\frac{\bar{r_1}r_1{}^{n+t} - \bar{r_2}r_2{}^{n+t}}{r_1-r_2} = \frac{\bar{r_1}r_1{}^{t+1} - \bar{r_2}r_2{}^{t+1}}{r_1-r_2}F_{k,n} + \frac{\bar{r_1}r_1{}^t - \bar{r_2}r_2{}^t}{r_1-r_2}F_{k,n-1}.$$

Using Binet formula of the hyperbolic k-Fibonacci octonion $\mathcal{O}^{\mathcal{F}}{}_{k,n}$, we get

$$\mathcal{O}^{\mathcal{F}}{}_{k,n+t} = F_{k,n}\mathcal{O}^{\mathcal{F}}{}_{k,t+1} + F_{k,n-1}\mathcal{O}^{\mathcal{F}}{}_{k,t}.$$

The proofs of (ii)-(xv) are similar to (i) using Lemma 2.1. □

***Theorem* 2.3** — *For all* $n, r, s, t \geq 1$*, we have*

$$(i)\quad \mathcal{O}^{\mathcal{F}}{}_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}\mathcal{O}^{\mathcal{F}}{}_{k,i+t},$$

$$(ii)\quad \mathcal{O}^{\mathcal{L}}{}_{k,rn+t} = \sum_{i=0}^{n}\binom{n}{i}F_{k,r}^{i}F_{k,r-1}^{n-i}\mathcal{O}^{\mathcal{L}}{}_{k,i+t},$$

$$(iii)\quad \mathcal{O}^{\mathcal{F}}{}_{k,2rn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}L_{k,r}^{i}\mathcal{O}^{\mathcal{F}}{}_{k,ri+t},$$

$$(iv)\quad \mathcal{O}^{\mathcal{L}}{}_{k,2rn+t} = \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}L_{k,r}^{i}\mathcal{O}^{\mathcal{L}}{}_{k,ri+t},$$

$$(v)\quad \mathcal{O}^{\mathcal{F}}{}_{k,trn+l} = \frac{1}{F_{k,r}^{n}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}F_{k,(t-1)r}^{n-i}F_{k,tr}^{i}\mathcal{O}^{\mathcal{F}}{}_{k,ri+l},$$

$$(vi)\quad \mathcal{O}^{\mathcal{L}}{}_{k,trn+l} = \frac{1}{F_{k,r}^{n}}\sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)(r+1)}F_{k,(t-1)r}^{n-i}F_{k,tr}^{i}\mathcal{O}^{\mathcal{L}}{}_{k,ri+l},$$

$$(vii)\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\mathcal{O}^{\mathcal{F}}{}_{k,r(n-i)+i+t}F_{k,r}^{i} = \mathcal{O}^{\mathcal{F}}{}_{k,t}F_{k,r-1}^{n},$$

$$(viii)\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\mathcal{O}^{\mathcal{L}}{}_{k,r(n-i)+i+t}F_{k,r}^{i} = \mathcal{O}^{\mathcal{L}}{}_{k,t}F_{k,r-1}^{n},$$

$$(ix)\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\mathcal{O}^{\mathcal{F}}{}_{k,ri+t}F_{k,r-1}^{(n-i)} = \mathcal{O}^{\mathcal{F}}{}_{k,n+t}F_{k,r}^{n},$$

$$(x)\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}\mathcal{O}^{\mathcal{L}}{}_{k,ri+t}F_{k,r-1}^{(n-i)} = \mathcal{O}^{\mathcal{L}}{}_{k,n+t}F_{k,r}^{n},$$

$$(xi)\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}F_{k,sm}^{(n-i)}F_{k,rm}^{(i)}\mathcal{O}^{\mathcal{F}}{}_{k,m[rn+i(s-r)]+t} = (-1)^{smn}\mathcal{O}^{\mathcal{F}}{}_{k,t}F_{k,(r-s)m}^{n},$$

$$(xii)\quad \sum_{i=0}^{n}\binom{n}{i}(-1)^{(n-i)}F_{k,sm}^{(n-i)}F_{k,rm}^{(i)}\mathcal{O}^{\mathcal{L}}{}_{k,m[rn+i(s-r)]+t} = (-1)^{smn}\mathcal{O}^{\mathcal{L}}{}_{k,t}F_{k,(r-s)m}^{n},$$

$$(xiii)\quad \mathcal{O}^{\mathcal{F}}{}_{k,n+t} = \sum_{i+j+s=n}\binom{n}{i,j}k^{-n}(-1)^{j+s}L_{k,2^{r+1}}{}^{i}\mathcal{O}^{\mathcal{F}}{}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$(xiv)\quad \mathcal{O^L}_{k,n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2^{r+1}}}^i \mathcal{O^L}_{k,2^{r+1}(i+2j)+2(i+j)+t},$$

$$(xv)\quad \mathcal{O^F}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,2^{r+1}}}^i \mathcal{O^F}_{k,(2^{r+1}+2)i+j+t},$$

$$(xvi)\quad \mathcal{O^L}_{k,(2^{r+2}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j}(-1)^{j+s} {L_{k,2^{r+1}}}^i \mathcal{O^L}_{k,(2^{r+1}+2)i+j+t},$$

$$(xvii)\quad \mathcal{O^F}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j} k^{j} {L_{k,2^{r+1}}}^{-n} \mathcal{O^F}_{k,(2^{r+1}+2)i+j+t},$$

$$(xviii)\quad \mathcal{O^F}_{k,(2^{r+1}+2)n+t} = \sum_{i+j+s=n} \binom{n}{i,j}) k^{j} {L_{k,2^{r+1}}}^{-n} \mathcal{O^F}_{k,(2^{r+1}+2)i+j+t},$$

$$(xix)\quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} (L_{k,2t-1})^{(n-i)} \mathcal{O^F}_{k,2ti+s}$$
$$= \begin{cases} k^{-n}(\mathcal{F}_{k,2t})^n \delta^{\frac{n}{2}} \mathcal{O^F}_{k,n+s}, & \textit{if } n \textit{ is even};\\ k^{-n}(\mathcal{F}_{k,2t})^n \delta^{\frac{n-1}{2}} \mathcal{O^L}_{k,n+s}, & \textit{if } n \textit{ is odd}, \end{cases},$$

$$(xx)\quad \sum_{i=0}^{n} \binom{n}{i} k^{(i-n)} (L_{k,2t-1})^{(n-i)} \mathcal{O^L}_{k,2ti+s}$$
$$= \begin{cases} k^{-n}(F_{k,2t})^n \delta^{\frac{n}{2}} \mathcal{O^L}_{k,n+s}, & \textit{if } n \textit{ is even};\\ k^{-n}(F_{k,2t})^n \delta^{\frac{n+1}{2}} \mathcal{O^F}_{k,n+s}, & \textit{if } n \textit{ is odd}. \end{cases},$$

$$(xxi)\quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} k^{-i} (L_{k,2t+1})^{i} \mathcal{O^F}_{k,2t(n-i)+n}$$
$$= \begin{cases} \delta^{\frac{n}{2}} \mathcal{O^F}_{k,0}, & \textit{if } n \textit{ is even};\\ \delta^{\frac{n-1}{2}} \mathcal{O^L}_{k,0}, & \textit{if } n \textit{ is odd}, \end{cases},$$

$$(xxii)\quad \sum_{i=0}^{n} \binom{n}{i} (-1)^{(n-i)} k^{-i} (L_{k,2t+1})^{i} \mathcal{O^L}_{k,2t(n-i)+n}$$
$$= \begin{cases} \delta^{\frac{n}{2}} \mathcal{O^L}_{k,0}, & \textit{if } n \textit{ is even};\\ \delta^{\frac{n+1}{2}} \mathcal{O^F}_{k,0}, & \textit{if } n \textit{ is odd}. \end{cases}$$

PROOF : (i). From (i) and (ii) of Lemma 2.1, we have

$$r_1^r = F_{k,r} r_1 + F_{k,r-1},$$
$$r_2^r = F_{k,r} r_2 + F_{k,r-1}.$$

Now, by the binomial theorem, we have

$$r_1^{rn} = (F_{k,r}r_1 + F_{k,r-1})^n = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} r_1^i, \tag{2.5}$$

$$r_2^{rn} = (F_{k,r}r_2 + F_{k,r-1})^n = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} r_2^i. \tag{2.6}$$

Multiplying (2.5) by $\dfrac{\bar{r_1}}{r_1 - r_2}$ and (2.6) by $\dfrac{\bar{r_2}}{r_1 - r_2}$ and subtracting, we obtain

$$\frac{\bar{r_1} r_1^{rn+t} - \bar{r_2} r_2^{rn+t}}{r_1 - r_2} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} (\frac{\bar{r_1} r_1^{i+t} - \bar{r_2} r_2^{i+t}}{r_1 - r_2}).$$

Using Binet formula of ${\mathcal{O}^{\mathcal{F}}}_{k,rn+t}$ and ${\mathcal{O}^{\mathcal{F}}}_{k,i+t}$, we get

$${\mathcal{O}^{\mathcal{F}}}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} {\mathcal{O}^{\mathcal{F}}}_{k,i+t}.$$

(ii). Multiplying (2.5) by $\bar{r_1}$ and (2.6) by $\bar{r_2}$ and adding, we obtain

$$\bar{r_1} r_1^{rn+t} + \bar{r_2} r_2^{rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} (\bar{r_1} r_1^{i+t} + \bar{r_2} r_2^{i+t}).$$

Using Binet formula of ${\mathcal{O}^{\mathcal{L}}}_{k,rn+t}$ and ${\mathcal{O}^{\mathcal{L}}}_{k,i+t}$, we get

$${\mathcal{O}^{\mathcal{L}}}_{k,rn+t} = \sum_{i=0}^{n} \binom{n}{i} F_{k,r}^i F_{k,r-1}^{n-i} {\mathcal{O}^{\mathcal{L}}}_{k,i+t}.$$

The proofs of (iii)-(xxii) are similar to (i) and (ii) using Lemma 2.1. □

Next theorem deals with congruence properties of the hyperbolic k-Fibonacci and k-Lucas octonions.

***Theorem* 2.4** — *For $n, t \geq 1$ and $\mathcal{G}_{k,n} = {\mathcal{O}^{\mathcal{F}}}_{k,n}$ or ${\mathcal{O}^{\mathcal{L}}}_{k,n}$, we have*

$$(i) \qquad {\mathcal{O}^{\mathcal{F}}}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n} (-1)^n {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2^{r+1}}},$$

$$(ii) \qquad {\mathcal{O}^{\mathcal{L}}}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n} (-1)^n {\mathcal{O}^{\mathcal{L}}}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2^{r+1}}},$$

$$(iii) \qquad {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j} (-1)^n {\mathcal{O}^{\mathcal{F}}}_{k,j+t} \equiv 0 \pmod{L_{k,2^{r+1}}},$$

$$(iv) \qquad {\mathcal{O}^{\mathcal{L}}}_{k,(2^{r+2}+2)n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{j} (-1)^n {\mathcal{O}^{\mathcal{L}}}_{k,j+t} \equiv 0 \pmod{L_{k,2^{r+1}}}.$$

PROOF : (i). From (xiii) of Theorem 2.3, for all $n, t \geq 1$, we have

$$\begin{aligned}
{\mathcal{O}^{\mathcal{F}}}_{k,n+t} &= \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2^{r+1}}}^{i} {\mathcal{O}^{\mathcal{F}}}_{k,2^{r+1}(i+2j)+2(i+j)+t} \\
&+ \sum_{i+j+s=n;_{i=0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2^{r+1}}}^{i} {\mathcal{O}^{\mathcal{F}}}_{k,2^{r+1}(i+2j)+2(i+j)+t}, \\
&= \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2^{r+1}}}^{i} {\mathcal{O}^{\mathcal{F}}}_{k,2^{r+1}(i+2j)+2(i+j)+t} \\
&+ \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)j+t}. \\
{\mathcal{O}^{\mathcal{F}}}_{k,n+t} &- \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)j+t} \\
&= \sum_{i+j+s=n;_{i\neq 0}} \binom{n}{i,j} k^{-n}(-1)^{j+s} {L_{k,2^{r+1}}}^{i} {\mathcal{O}^{\mathcal{F}}}_{k,2^{r+1}(i+2j)+2(i+j)+t}, \\
&\therefore L_{k,2} \quad \text{divides} \quad \left({\mathcal{O}^{\mathcal{F}}}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)j+t}\right), \\
&\therefore {\mathcal{O}^{\mathcal{F}}}_{k,n+t} - \sum_{j=0}^{n} \binom{n}{j} k^{-n}(-1)^{n} {\mathcal{O}^{\mathcal{F}}}_{k,(2^{r+2}+2)j+t} \equiv 0 \pmod{L_{k,2}}.
\end{aligned}$$

The proofs of (ii), (iii) and (iv) are similar to (i), using Theorem 2.3. □

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