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NONLOCAL CONFORMABLE FRACTIONAL CAUCHY PROBLEM WITH SECTORIAL OPERATOR

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After giving the Duhamel formula in terms of the standard semigroup, we prove the existence, uniqueness, stability and regularity of mild solution for a nonlocal fractional Cauchy problem with the conformable fractional derivative. Also, we give an example to illustrate the applicability of our results.

Key words : Fractional differential equations; semigroups of linear operators; nonlocal condition; conformable fractional derivative; sectorial operator.

1. Introduction

The Cauchy problem with nonlocal condition $x(0) = x_0 + g(x)$ is introduced in [2]. For physical interpretations of this condition we refer to [3, 7]. The nonlocal Cauchy problem has attracted several authors including the works [4, 6, 9]. Here, we are going to study same Cauchy problem with a new fractional derivative named "conformable fractional derivative" [5]. This novel derivative satisfied the properties of classical one and can be used for non-regular functions. Precisely, we consider the conformable fractional Cauchy problem of the form:

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)), \quad x(0) = x_0 + g(x), \tag{1}$$

where $0 \leq t \leq \tau$ and $\frac{d^\alpha x(t)}{dt^\alpha}$ is the conformable fractional derivative of order $\alpha \in]0,1]$. $(A, D(A))$ is a sectorial operator which generates a strongly analytic semigroup $(T(t))_{t\geq 0}$ on Banach space $(X, \| \ . \ \|)$ and $x_0 \in \overline{D(A)}$. We denote by $\mathcal{C}$ the Banach space of continuous functions from $[0,\tau]$ into X with the norm $\mid x \mid = \sup_{t\in[0,\tau]} \| x(t) \|$. For our study, the functions $f : [0,\tau] \times X \longrightarrow X$ and $g : \mathcal{C} \longrightarrow \overline{D(A)}$ satisfy some conditions.

The content of this paper is organized as follows. In Section 2, we recall some needed definitions and results. In Section 3, after giving the Duhamel formula, we prove the existence, uniqueness, stability and regularity of mild solution for the Cauchy problem (1). Section 4 is devoted to a concrete application of the main results.

2. PRELIMINARIES

In this section, we recall some concepts on conformable fractional calculus.

Definition 1 — [5]. The conformable fractional derivative of x of order α at $t > 0$ is defined as

$$\frac{d^\alpha x(t)}{dt^\alpha} = \lim_{\varepsilon \longrightarrow 0} \frac{x(t + \varepsilon t^{1-\alpha}) - x(t)}{\varepsilon}.$$

When the limit exists, we said that x is (α)-differentiable at t.

If x is (α)-differentiable and $\lim_{t \longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha}$ exists, then define

$$\frac{d^\alpha x(0)}{dt^\alpha} = \lim_{t \longrightarrow 0^+} \frac{d^\alpha x(t)}{dt^\alpha}.$$

The (α)-fractional integral of a function x is defined by:

$$I^\alpha(x)(t) = \int_0^t s^{\alpha-1} x(s) ds.$$

***Theorem* 1** — *If x is a continuous function in the domain of I^α. Then, we have*

$$\frac{d^\alpha(I^\alpha(x)(t))}{dt^\alpha} = x(t).$$

The standard Laplace transform is not compatible with the conformable fractional derivative. In consequence, the adapted Laplace transform is given as follows:

Definition 2 — [1]. The fractional Laplace transform of order α of a function x is defined by

$$\mathcal{L}_\alpha(x(t))(\lambda) := \int_0^{+\infty} t^{\alpha-1} e^{-\lambda \frac{t^\alpha}{\alpha}} x(t) dt.$$

The following proposition gives us the actions of the fractional integral and the fraction Laplace transform on the conformable fractional derivative, respectively.

Proposition 1 — If $x(t)$ is differentiable. Then, we have

$$I^\alpha \left(\frac{d^\alpha x}{dt^\alpha} \right)(t) = x(t) - x(0),$$

$$\mathcal{L}_\alpha\left(\frac{d^\alpha x(t)}{dt^\alpha}\right)(\lambda) = \lambda\mathcal{L}_\alpha(x(t))(\lambda) - x(0).$$

Now, we recall some definitions and results on the sectorial operator theory [8].

Definition 3 — $A : D(A) \subset X \longrightarrow X$ is said to be sectorial operator of type (M, ω, θ) if there exist $M > 0$, $\omega \in \mathbb{R}$ and $0 < \theta < \frac{\pi}{2}$ such that:

1. A be a closed and linear operator,
2. $\forall\lambda \notin \omega + S_\theta$, the resolvent $(\lambda I - A)^{-1}$ of A exists,
3. $\forall\lambda \notin \omega + S_\theta$, $\mid (\lambda I - A)^{-1} \mid \leq \frac{M}{|\lambda-\omega|}$,

where

$$\omega + S_\theta := \{\omega + \lambda \mid \quad \lambda \in \mathbb{C} \ \text{ with } \ \mid Arg(-\lambda) \mid < \theta\}.$$

***Theorem* 2** — *A densely sectorial operator generates a strongly analytic semigroup* $(T(t))_{t\geq 0}$. *Moreover, we have*

$$T(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}(\lambda I - A)^{-1} d\lambda,$$

with γ *being a suitable path* $\lambda \notin \omega + S_\theta$.

Now we give the main contribution results.

3. Main Results

Before presenting our main results, we introduce the following assumptions:

(H_1) The function $f(t, .) : X \longrightarrow X$ is continuous and for all $r > 0$ there exists a function $\mu_r \in L^\infty([0, \tau], \mathbb{R}^+)$ such that $\sup\limits_{\|x\|\leq r} \parallel f(t, x) \parallel \leq \mu_r(t)$, for all $t \in [0, \tau]$.

(H_2) The function $f(., x) : [0, \tau] \longrightarrow X$ is continuous, for all $x \in X$.

(H_3) There exists a constant $l_1 > 0$ such that $\parallel g(y) - g(x) \parallel \leq l_1 \mid y - x \mid$, for all $x, y \in \mathcal{C}$.

3.1 *Existence and uniqueness of the mild solution*

Applying the fractional Laplace transform in equation (1), we obtain

$$\mathcal{L}_\alpha(x(t))(\lambda) = (\lambda - A)^{-1}[x_0 + g(x)] + (\lambda - A)^{-1}\mathcal{L}_\alpha(f(t, x(t)))(\lambda).$$

According to the inverse fractional Laplace transform, we find the following Duhamel formula:

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^t s^{\alpha-1}T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

We remark that for $\alpha = 1$, we recover the classical Duhamel formula. Thus, we can introduce the following definition.

Definition 4 — We say that $x \in \mathcal{C}$ is a mild solution of equation (1), if the following assertion is true:

$$x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds, \ t \in [0, \tau].$$

***Theorem* 3** — *If $(T(t))_{t>0}$ is compact and $(H_1) - (H_3)$ are satisfied, then the Cauchy problem (1) has at least one mild solution, provided that*

$$l_1 \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| < 1.$$

PROOF : Choosing

$$r \geq \frac{\sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| [\| x_0 \| + \| g(0) \| + \frac{\tau^\alpha}{\alpha} \mid \mu_r \mid_{L^\infty([0,\tau],\mathbb{R}^+)}]}{1 - l_1 \sup\limits_{t\in[0,\tau]} |T(\frac{t^\alpha}{\alpha})|}.$$

Let $B_r = \{x \in \mathcal{C} \mid \mid x \mid \leq r\}$. For $x \in B_r$ and $t \in [0, \tau]$ define the operators Γ_1 and Γ_2 by:

$$\Gamma_1(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)],$$

$$\Gamma_2(x)(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

By $(H_1) - (H_3)$, we can show that Γ_1 is a contraction operator on B_r and $\Gamma_1(x) + \Gamma_2(y) \in B_r$ whenever x, $y \in B_r$. Now, we will prove that Γ_2 is continuous and compact.

Continuity of Γ_2 :

Let $(x_n) \subset B_r$ such that $x_n \longrightarrow x$ in B_r. We have

$$\Gamma_2(x_n)(t) - \Gamma_2(x)(t) = \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s, x_n(s)) - f(s, x(s))]ds, \ t \in [0, \tau].$$

Then, we obtain

$$\mid \Gamma_2(x_n) - \Gamma_2(x) \mid \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \int_0^\tau s^{\alpha-1} \| f(s, x_n(s)) - f(s, x(s)) \| ds.$$

By using assumption (H_1), we have $\| \ s^{\alpha-1}[f(s, x_n(s)) - f(s, x(s))] \ \| \leq 2\mu_r(s)s^{\alpha-1}$ and $f(s, x_n(s)) \longrightarrow f(s, x(s))$ as $n \longrightarrow +\infty$. According to the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\longrightarrow+\infty} \mid \Gamma_2(x_n) - \Gamma_2(x) \mid = 0.$$

Compactness of Γ_2 :

Claim 1 : We prove that $\{\Gamma_2(x)(t) \mid \ x \in B_r\}$ is relatively compact in X.

For some fixed $t \in]0, \tau]$ let $\varepsilon \in]0, t[$, $x \in B_r$ and define the operator Γ_2^{ε} by:

$$\Gamma_2^{\varepsilon}(x)(t) = \int_0^{(t^\alpha - \varepsilon^\alpha)^{\frac{1}{\alpha}}} s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds.$$

We can write Γ_2^{ε} as follows:

$$\Gamma_2^{\varepsilon}(x)(t) = T\left(\frac{\varepsilon^\alpha}{\alpha}\right) \int_0^{(t^\alpha - \varepsilon^\alpha)^{\frac{1}{\alpha}}} s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha - \varepsilon^\alpha}{\alpha}\right) f(s, x(s)) ds.$$

According to compactness of $(T(t))_{t>0}$, we conclude that the set $\{\Gamma_2^{\varepsilon}(x)(t) \mid \ x \in B_r\}$ is relatively compact in X. Using (H_1), we have

$$\| \Gamma_2^{\varepsilon}(x)(t) - \Gamma_2(x)(t) \| \leq | \mu_r |_{L^\infty([0,\tau],\mathbb{R}^+)} \sup_{t \in [0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \frac{\varepsilon^\alpha}{\alpha}.$$

This proves that the set $\{\Gamma_2(x)(t) \mid \ x \in B_r\}$ is relatively compact in X. It is clear that $\{\Gamma_2(x)(0) \mid \ x \in B_r\}$ is compact. Thus, the set $\{\Gamma_2(x)(t) \mid \ x \in B_r\}$ is relatively compact in X for all $t \in [0, \tau]$.

Claim 2 : We prove that $\Gamma_2(B_r)$ is equicontinuous.

Let $t_1, t_2 \in [0, \tau]$ such that $t_1 < t_2$. We have

$$\begin{aligned}
\Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) &= \int_0^{t_1} s^{\alpha-1} [T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) - T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right)] f(s, x(s)) ds \\
&\quad + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds \\
&= [T\left(\frac{t_2^\alpha - t_1^\alpha}{\alpha}\right) - I)] \int_0^{t_1} s^{\alpha-1} T\left(\frac{t_1^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds \\
&\quad + \int_{t_1}^{t_2} s^{\alpha-1} T\left(\frac{t_2^\alpha - s^\alpha}{\alpha}\right) f(s, x(s)) ds.
\end{aligned}$$

Then, by using a computation, one has

$$\| \Gamma_2(x)(t_2) - \Gamma_2(x)(t_1) \| \leq \frac{\sup_{t \in [0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \, | \mu_r |_{L^\infty([0,\tau],\mathbb{R}^+)}}{\alpha} [(t_2^\alpha - t_1^\alpha) + \tau^\alpha \mid T(\frac{t_2^\alpha - t_1^\alpha}{\alpha}) - I \mid].$$

Therefore, we deduce that $\Gamma_2(x)$, $x \in B_r$ are equicontinuous at $t \in [0, \tau]$. By using the Arzela-Ascoli theorem, we obtain that Γ_2 is compact. Finally, the Krasnoselskii fixed-point theorem allows us to conclude that the operator $\Gamma_1 + \Gamma_2$ has at least one fixed point in $\mathcal{C}$, which is a mild solution of equation (1).

To obtain the uniqueness of mild solution, we need the following assumption:

(H_4) There exists a constant $l_2 > 0$ such that $\parallel f(t,y) - f(t,x) \parallel \leq l_2 \parallel y - x \parallel$, for all $x, y \in X$ and $t \in [0,\tau]$.

Theorem 4 — *Assume that* $(H_2) - (H_4)$ *hold, then the Cauchy problem* (1) *has a unique mild solution, provided that*

$$\left(\frac{\tau^\alpha}{\alpha} l_2 + l_1\right) \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| < 1.$$

PROOF : Define the operator $\Gamma : \mathcal{C} \longrightarrow \mathcal{C}$ by:

$$\Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds.$$

For $x, y \in \mathcal{C}$, we have

$$\begin{aligned}\Gamma(y)(t) - \Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] \\ + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s,y(s)) - f(s,x(s))]ds.\end{aligned}$$

Then, we obtain

$$\parallel \Gamma(y)(t) - \Gamma(x)(t) \parallel \leq \left(\frac{\tau^\alpha}{\alpha} l_2 + l_1\right) \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \mid y - x \mid .$$

Taking the supremum, we get

$$\mid \Gamma(y) - \Gamma(x) \mid \leq (\frac{\tau^\alpha}{\alpha} l_2 + l_1) \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \mid y - x \mid .$$

Hence, by using the Banach contraction principle, we conclude that the operator Γ has a unique fixed point in $\mathcal{C}$, which is the mild solution of the conformable fractional Cauchy problem (1).

3.2 *Stability of the mild solution*

Now, we will give the results concerning the continuous dependence of the mild solution to the initial condition.

***Theorem* 5** — *Assume that the conditions of Theorem 4 are satisfied. Let x_0, $y_0 \in X$ and denote by x and y the solutions associated with x_0 and y_0, respectively. Then, we have the following estimate*

$$| y - x | \leq \frac{\alpha \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|}{\alpha - \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|(\alpha l_1 + l_2\tau^\alpha)} \parallel y_0 - x_0 \parallel .$$

PROOF : For $t \in [0,\tau]$, we have

$$\begin{aligned} y(t) - x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[y_0 - x_0 + g(y) - g(x)] \\ + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s,y(s)) - f(s,x(s))]ds. \end{aligned}$$

Then, we obtain

$$\parallel y(t) - x(t) \parallel \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|[\parallel y_0 - x_0 \parallel + (l_1 + \frac{l_2\tau^\alpha}{\alpha}) | y - x |].$$

Therefore, we get

$$| y - x | \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|[\parallel y_0 - x_0 \parallel + (l_1 + \frac{l_2\tau^\alpha}{\alpha}) | y - x |].$$

Thus, we deduce the desired estimate

$$| y - x | \leq \frac{\alpha \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|}{\alpha - \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|(\alpha l_1 + l_2\tau^\alpha)} \parallel y_0 - x_0 \parallel .$$

***Theorem* 6** — *Assume that the conditions of Theorem 4 are satisfied. Let x_0, $y_0 \in X$ and denote by x and y the solutions associated with x_0 and y_0, respectively. Then, we have the following estimate*

$$| y - x | \leq \frac{\sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \exp\left(\frac{l_2\tau^\alpha}{\alpha} \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|\right)}{1 - l_1 \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \exp\left(\frac{l_2\tau^\alpha}{\alpha} \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|\right)} \parallel y_0 - x_0 \parallel,$$

provided that

$$l_1 \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \exp\left(\frac{l_2\tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |T(\frac{t^\alpha}{\alpha})|\right) < 1.$$

PROOF : Let $t \in [0, \tau]$, we have

$$\begin{aligned} y(t) - x(t) = T\left(\frac{t^\alpha}{\alpha}\right)[y_0 - x_0 + g(y) - g(x)] \\ + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s, y(s)) - f(s, x(s))]ds. \end{aligned}$$

Then, one has

$$\| y(t) - x(t) \| \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|[\| y_0 - x_0 \| + l_1 \mid y - x \mid + l_2 \int_0^t s^{\alpha-1} \| y(s) - x(s) \| ds].$$

By using the Gronwall inequality, we get

$$\| y(t) - x(t) \| \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|[\| y_0 - x_0 \| + l_1 \mid y - x \mid] \exp\left(\frac{l_2 \tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|\right).$$

Hence, we deduce

$$\mid y - x \mid \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|[\| y_0 - x_0 \| + l_1 \mid y - x \mid] \exp\left(\frac{l_2 \tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |T(\frac{t^\alpha}{\alpha})|\right).$$

Thus, we obtain the following estimate

$$\mid y - x \mid \leq \frac{\sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \exp(\frac{l_2 \tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|)}{1 - l_1 \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \exp\left(\frac{l_2 \tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|\right)} \| y_0 - x_0 \|.$$

Remark 1 : We remark that:

$$\frac{\exp\left(\frac{l_2 \tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |T(\frac{t^\alpha}{\alpha})|\right)}{1 - l_1 \sup_{t\in[0,\tau]} |T(\frac{t^\alpha}{\alpha})| \exp\left(\frac{l_2 \tau^\alpha}{\alpha} \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|\right)} < \frac{\alpha}{\alpha - \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|(\alpha l_1 + l_2 \tau^\alpha)}.$$

Then, Theorem 6 is better than Theorem 5.

3.3 *Regularity of the mild solution*

Here, we need the following assumptions:

(H_5) The function f is (α)-differentiable with respect to the first variable and differentiable with respect to the second variable.

(H_6) $x_0 + g(x) \in D(A)$, for all $x \in \mathcal{C}$.

Definition 5 — A (α)-differentiable function x is called (α)-classical solution of equation (1), if the following assertions are true:

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f(t, x(t)),$$

$$x(0) = x_0 + g(x).$$

***Theorem* 7** — *Assume that $(H_3) - (H_6)$ hold, then the Cauchy problem (1) has a unique (α)-classical solution, provided that*

$$\left(\frac{\tau^\alpha}{\alpha} l_2 + l_1\right) \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| < 1.$$

PROOF : The conditions of Theorem 4 are hold. We denote by x the unique mild solution of equation (1). Next, let y be the continuously solution of the following integral equation:

$$y(t) = T\left(\frac{t^\alpha}{\alpha}\right)[A(x_0 + g(x)) + f(0, x(0))] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\frac{\partial^\alpha f}{\partial s^\alpha}(s, x(s))ds$$
$$+ \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)\frac{\partial f}{\partial x}(s, x(s))y(s)ds,\ t \in [0, \tau].$$

We have $\dfrac{x(t + \varepsilon t^{1-\alpha}) - x(t)}{\varepsilon} \longrightarrow y(t)$ as $\varepsilon \longrightarrow 0$ for $t > 0$. Then, we conclude that $t \longmapsto f(t, x(t))$ is continuously $(\alpha)-$differentiable. Hence, the following Cauchy problem:

$$\frac{d^\alpha z(t)}{dt^\alpha} = Az(t) + f(t, x(t)), \quad z(0) = x_0 + g(x),$$

has a unique $(\alpha)-$classical solution. Thus, by uniqueness of the mild solution of equation (1), we conclude that $z \equiv x$.

3.4 *Special case of nonlocal condition*

Now, we study one special case of nonlocal condition, that is, the function g is given by:

$$g(x) = \sum_{i=1}^n c_i x(t_i),$$

where c_i, $i = 1, 2, ..., n$, are given constants and $0 < t_1 < t_2 < ... < t_n < \tau$.

Proposition 2 — Assume that (H_2) and (H_4) hold, then the Cauchy problem (1) has a unique mild solution, provided that there exists $\varepsilon_0 \in]0,1[$ such that

$$\sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \sum_{i=1}^{n} | c_i | < \varepsilon_0.$$

PROOF : Define the operator $\Gamma : \mathcal{C} \longrightarrow \mathcal{C}$ by:

$$\Gamma(x)(t) = T\left(\frac{t^\alpha}{\alpha}\right)[x_0 + g(x)] + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right) f(s, x(s))ds, \ t \in [0,\tau].$$

Now, we define a new norm $| x |_\alpha$ in $\mathcal{C}$ by:

$$| x |_\alpha = | \exp(\frac{-\varepsilon(.)^\alpha}{\alpha}) x |,$$

where

$$\varepsilon = \frac{l_2 \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)|}{\varepsilon_0 - \sup\limits_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| \sum\limits_{i=1}^{n} | c_i |}.$$

Next, for $x, y \in \mathcal{C}$ and $t \in [0,\tau]$, we have

$$\begin{aligned} \Gamma(y)(t) - \Gamma(x)(t) &= T\left(\frac{t^\alpha}{\alpha}\right)[g(y) - g(x)] \\ &\quad + \int_0^t s^{\alpha-1} T\left(\frac{t^\alpha - s^\alpha}{\alpha}\right)[f(s, y(s)) - f(s, x(s))]ds. \end{aligned}$$

Then, one has

$$\begin{aligned} \| \Gamma(y)(t) - \Gamma(x)(t) \| \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| [\exp(\frac{\varepsilon t^\alpha}{\alpha}) \sum_{i=1}^{n} | c_i | \\ + l_2 \int_0^t s^{\alpha-1} \exp(\varepsilon \frac{s^\alpha}{\alpha}) ds] \, | y - x |_\alpha . \end{aligned}$$

Therefore, we obtain

$$| \Gamma(y) - \Gamma(x) |_\alpha \leq \sup_{t\in[0,\tau]} |T\left(\frac{t^\alpha}{\alpha}\right)| [\sum_{i=1}^{n} | c_i | + \frac{l_2}{\varepsilon}] \, | y - x |_\alpha .$$

Hence, we get

$$| \Gamma(y) - \Gamma(x) |_\alpha \leq \varepsilon_0 \, | y - x |_\alpha .$$

Thus, Γ is a contraction operator on the Banach space $(\mathcal{C}, | \ . \ |_\alpha)$. This shows that the operator Γ has a unique fixed point in $\mathcal{C}$, which is the mild solution of the conformable fractional Cauchy problem (1).

4. APPLICATION

Consider the nonlocal partial fractional-differential equation:

$$\begin{cases} \frac{\partial^{\frac{1}{2}} u(t,x)}{\partial t^{\frac{1}{2}}} = \frac{\partial^2 u(t,x)}{\partial x^2} + a(t)\frac{|u(t,x)|}{1+|u(t,x)|}, \quad (t,x) \in [0,1] \times]0,\pi[, \\ \\ u(t,0) = u(t,\pi) = 0, \qquad\qquad t \in [0,1], \\ \\ u(0,x) = \sum_{i=1}^{n} c_i u(t_i, x), \qquad\qquad x \in [0,\pi], \end{cases} \tag{2}$$

where $0 < t_1 < ... < t_n < 1$, $c_1, ..., c_n$ are given real constants and $a : [0,1] \longrightarrow \mathbb{R}$ is a continuous function.

Let $X = L^2([0,\pi])$ and define the operator A by:

$$A = \frac{\partial^2(.)}{\partial x^2},$$

$$D(A) = \{\psi \in X : \ \dot{\psi}, \ \ddot{\psi} \in X\}.$$

The operator A generates a strongly analytic semigroup $(T(t))_{t\geq 0}$ on X.

Let $v(t)(x) = u(t,x)$, $f(t,\varphi) = a(t)\frac{|\varphi|}{1+|\varphi|}$ and $g(v) = \sum_{i=1}^{n} c_i u(t_i, .)$. Then, the Cauchy problem (2) becomes as follows:

$$\begin{cases} \frac{d^{\frac{1}{2}} v(t)}{dt^{\frac{1}{2}}} = Av(t) + f(t, v(t)), \quad t \in [0,1], \\ \\ v(0) = g(v). \end{cases} \tag{3}$$

In this application, all the assumptions of Theorem 4 are satisfied, then the above Cauchy problem has a unique mild solution, provided that

$$\left[2 \sup_{t\in[0,1]} (a(t)) + \sum_{i=1}^{n} \mid c_i \mid \right] \sup_{t\in[0,1]} |T(2\sqrt{t})| < 1.$$

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