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ON 2-ABSORBING MULTIPLICATION MODULES OVER PULLBACK RINGS

Farkhondeh Farzalipour

Department of Mathematics, Payame Noor University (PNU),
P.O. Box 19395-3697 *Tehran, Iran*
e-mail: f_farzalipour@pnu.ac.ir

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In this article, we classify all those indecomposable 2-absorbing multiplication modules with finite-dimensional top over pullback of two Dedekind domains. We extend the definition and results given in [10] to a more general 2-absorbing multiplication modules case.

Key words : Dedekind domains; non-separated modules; Prüfer modules; pure-injective modules; pullback; separated modules; 2-absorbing multiplication modules.

1. INTRODUCTION

One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. The reader is referred to [1, 33, 34] for a detailed discussion problems, their representation types (finite, tame, or wild), and useful computational reduction procedures.

Let $v_1 : R_1 \longrightarrow \bar{R}$ and $v_2 : R_2 \longrightarrow \bar{R}$ be homomorphisms of two local Dedekind domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $Ker(R \longrightarrow \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \longrightarrow P_i \longrightarrow R \longrightarrow R_j \longrightarrow 0$ is a exact sequence of R-modules (see [22]).

We know that every module is an elementary substructure of a pure-injective module. In fact, there is a minimal pure-injective elementary extension of each module M, denoted by $h(M)$, called the pure-injective hull of M and it is unique up to isomorphism fixing M. The class of pure-injectives is closed under direct summands and finite direct sums, but an infinite direct sum of pure-injectives need not be pure-injective. Observe that any finite module is pure-injective. In a sense, then, pure-injective modules are model theoretically typical: for example, classification of the complete theories of R-modules reduces to classifying the (complete theories of) pure-injectives. Also, for some rings, "small" (finite-dimensional, finitely generated, $\ldots$) modules are classified and in many cases this classification can be extended to give a classification of (indecomposable) pure-injective modules. Indeed, there is sometimes a strong connection between infinitely generated pure-injective modules and families of finitely generated modules. Therefore, pure-injective modules are very important (see [20, 30, 37]. One point of this paper is to introduce a subclass of pure-injective modules.

Modules over pullback rings have been studied by several authors (see for example, [4, 15, 18, 19, 27, 38]. The important work of Levy [23] provides a classification of all finitely generated indecomposable modules over Dedekind-like rings. Klingler [19] extended this classification to lattices over certain non-commutative Dedekind-like rings, and Klingler and Haefner [16, 17] classified lattices over certain non-commutative pullback rings, which they called special quasi triads. Common to all these classifications is the reduction to a "matrix problem" over a division ring (see [29, 33, 35] for background on matrix problems and their applications).

In the present article, we introduce a new class of R-modules, called 2-absorbing multiplication modules (see Definition 2.1), and we study it in details from the classification problem point of view. We are mainly interested in case either R is a Dedekind domain or R is a pullback of two local Dedekind domains. First, we give a complete description of the 2-absorbing multiplication modules over a local Dedekind domain. Let R be a pullback of two local Dedekind domains over a common factor field. Next, the main purpose of this paper is to give a complete description of the indecomposable 2-absorbing multiplication R-modules with finite-dimensional top over $R/Rad(R)$ (for any module M we define its top as $M/Rad(R)M$). In fact, we extend the definition and results given in [10] to a more general 2-absorbing multiplication modules case, and the class of 2-absorbing multiplication modules contains the class of weak multiplication modules defined in [10].

The classification is divided into two stages: the description of all indecomposable separated 2-absorbing multiplication R-modules and then, using this list of separated 2-absorbing multiplication modules, we show that non-separated indecomposable 2-absorbing multiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable 2-absorbing multiplication R-modules. Then we use the classification of separated indecomposable 2-absorbing multiplication modules from Section 3, together with results of Levy [24] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable 2-absorbing multiplication modules M with finite-dimensional top (see Theorem

4.8). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable 2-absorbing multiplication modules (where infinite length 2-absorbing multiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated 2-absorbing multiplication modules.

For the sake of completeness, we state some definitions and notations used throughout. In this article all rings are commutative with identity and all modules unitary. Let R be the pullback ring as mentioned in the begining of introduction. An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$). Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \longrightarrow S/PS \longleftarrow S/P_1S)$ [22, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also, S is separated if and only if $P_1S \cap P_2S = 0$ [22, Lemma 2.9].

If R is a pullback ring, then every R-module is a epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi = (S \xrightarrow{f} S' \longrightarrow M)$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \longrightarrow M$ with S' separated, then f is one-to-one. The module $K = Ker(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [22, Proposition 2.3]. An exact sequence $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [22, Proposition 2.8]. Every module M has a separated representation, which is unique up to isomorphism [22, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [22, Theorem 2.6].

Definition 1.1 —

(a) If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M.

(b) A proper submodule N of a module M over a ring R is said to primary submodule (resp., prime submodule) if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some n (resp., $m \in N$ or $r \in (N : M)$), so $Rad(N : M) = P$ (resp., $(N : M) = P'$) is a prime ideal of R, and N is said to be P-primary (resp., P'-prime) submodule. The set of all primary submodules (resp., prime submodules) in an R-module M is denoted $pSpec(M)$ (resp., $Spec(M)$).

(c) A proper ideal I of a commutative ring R is said to be 2-absorbing, if $abc \in I$ where $a, b, c \in R$, then $ab \in I$ or $ac \in I$ or $bc \in I$ [3].

(d) A proper submodule N of an R-module M is called to be 2-absorbing, if for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $ab \in (N : M)$ or $am \in N$ or $bm \in N$. So $(N : M)$ is a 2-absorbing ideal of R. The set of all 2-absorbing submodules in an R-module M is denoted by $2 - abSpec(M)$ (see [39]).

(e) An R-module M is defined to be a multiplication module if for each submodule N of M, $N = IM$, for some ideal I of R. In this case, we can take $I = (N : M)$.

(f) An R-module M is defined to be a weak multiplication module if Spec$(M) = \emptyset$ or for every prime submodule N of M, $N = IM$, for some ideal I of R (see [10]).

(g) An R-module M is defined to be a primary multiplication module if pSpec$(M) = \emptyset$ or for every primary submodule N of M, $N = IM$, for some ideal I of R [13].

(h) An R-module M is defined to be a semiprime multiplication module if for every semiprime submodule N of M, $N = IM$, for some ideal I of R (see [14]).

(i) A submodule N of an R-module M is called pure submodule, if any finite system of equiations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called relatively divisible (or an RD-submodule) in M if $rN = N \cap rM$ for all $r \in R$ (see [30, 37]).

(j) A module M is pure-injective if it has the injective property relative to all pure exact sequences (see [30, 37])

(k) An R-module M is called to be 2-absorbing, if its zero submodule is a 2-absorbing submodule.

Remark 1.2 :

(i) An R-module is pure-injective if and only if it is algebraically compact (see [38]).

(ii) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M [37].

(iii) Let N be an R-submodule of M. It is clear that N is an RD-submodule of M if and only if for all $m \in M$ and $r \in R$, $rm \in N$ implies that $rm = rn$ for some $n \in N$. Furthermore, if M is torsion-free, then N is an RD-sumodule if and only if for all $m \in M$ and for all non-zero

$r \in R$, $rm \in N$ implies that $m \in N$. In this case, N is an RD-submodule if and only if N is a prime submodule.

(iv) It is easy to see that if N is a 2-absorbing submodule of an R-module M, then M/N is a 2-absorbing R-module.

2. Basic Properties of 2-Absorbing Multiplication Modules

The aim of this section is to classify 2-absorbing multiplication modules over a local Dedekind domain. First, we collect basic properties concerning 2-absorbing multiplication modules.

Definition 2.1 — Let R be a commutative ring. An R-module M is said to be a 2-absorbing multiplication module, if $2 - abSpec(M) = \emptyset$ or for every 2-absorbing submodule N of M, $N = IM$, for ideal I of R.

One can easily show that if M is a 2-absorbing multiplication module, then $N = (N : M)M$ for every 2-absorbing submodule N of M. Since every prime submodule is 2-absorbing submodule, we obtain that the class of 2-absorbing multiplication modules contains the class of weak multiplication modules. We need the following Lemma proved in [28, Theorem 2.3] and [39, Lemma 2.4].

Lemma 2.2 — (i) let N be a 2-absorbing submodule of M and suppose that I, J be ideals of R and K a submodule of M. If $IJK \subseteq N$, then $IK \subseteq N$ or $JK \subseteq N$ or $IJ \subseteq (N : M)$.

(ii) Let $K \subseteq N$ be submodules of an R-module M. Then N is a 2-absorbing submodule if and only if N/K is a 2-absorbing submodule of M/K.

Lemma 2.3 — Let M be an R-module, N a 2-absorbing submodule of M and I an ideal of R with $I \subset (0 : M)$. Then N is a 2-absorbing submodule of M as an R/I-module.

Proof : By Remark 1.2, M/N is a 2-absorbing R-module. Let $(a + I)(b + I)m \in N$ for some $m \in M$ and $a + I, b + I \in R/I$, so $ab(m + N) = 0$, hence $am \in N$ or $bm \in N$ or $ab \in (0 : M/N) = (N : M)$, as needed. □

Lemma 2.4 — Let M be a 2-absorbing multiplication module over a commutative ring R. Then the following hold:

(i) If I is an ideal of R and N a non-zero R-submodule of M with $I \subseteq (N : M)$, then M/N is a 2-absorbing multiplication R/I-module.

(ii) If N is a submodule of M, then M/N is a 2-absorbing multiplication R-module.

(iii) Every direct summand of M is a 2-absorbing multiplication module.

PROOF : (i) Let K/N be a 2-absorbing submodule of M/N, Then by Lemma 2.2, K is a 2-absorbing submodule of M, then $K = (K : M)M$. An inspection will show that $K/N = (K/N :_{R/I} M/N)M/N$.

(ii) Take $I = 0$ in (i).

(iii) Following from (ii). □

Lemma 2.5 — Let R and R' be any commutative rings, $f : R \to R'$ a surjective homomorphism and M an R'-module. Then the following hold:

(i) If M is a 2-absorbing as an R-module, then M is 2-absorbing as an R'-module.

(ii) If N is a 2-absorbing R-submodule of M, then N is a 2-absorbing R'-submodule of M.

(iii) If M is a 2-absorbing multiplication R-module, then M is a 2-absorbing multiplication R'-module.

PROOF : (i) is obvious.

(ii) Clearly, M/N is a 2-absorbing R-module, so M/N is a 2-absorbing R'-module by (i), hence N is a 2-absorbing R'-submodule of M.

(iii) Let N be a 2-absorbing R-submodule of M. Then N is a 2-absorbing R'-submodule of M by (i), so $N = I'M$ for some ideal I' of R'. Set $I = f^{-1}(I')$. Then I is an ideal of R and $f(I) = f(f^{-1}(I')) = I' \cap f(R) = I'$; hence $IM = f(I)M = N$. □

Proposition 2.6 — Let R be a local Dedekind domain with unique maximal ideal $P = Rp$. Then $E = E(R/P)$, the injective hull of R/P, and $Q(R)$, the field of fractions of R, are 2-absorbing multiplication modules.

PROOF : By [7, Lemma 2.6], every non-zero proper submodule L of E is of the form $L = A_n = (0 :_E P^n)$ $(n \geq 1)$, $L = A_n = Ra_n$ and $PA_{n+1} = A_n$. However, no A_n is a 2-absorbing submodule of E, for if n is a positive integer, then $P^3A_{n+3} = A_n$, but $P^2A_{n+3} = A_{n+1} \nsubseteq A_n$ and $PA_{n+3} = A_{n+2} \nsubseteq A_n$ and $P^3 \subseteq (A_n : E) = 0$. Now we conclude that $2 - abSpec(E) = \emptyset$. Thus E is 2-absorbing multiplication module.

Clearly, 0 is a 2-absorbing (prime) submodule of $Q(R)$. To show that 0 is the only 2-absorbing submodule of $Q(R)$, we assume the contrary and let N is a non-zero 2-absorbing submodule of $Q(R)$. Since N is a non-zero submodule, there exists $0/1 \neq a/b$, where $a, b \in R$, so that $a/b \in N$. Clearly, $1/b \notin N$ (otherwise, $b/b = 1/1 \in N$, which is a contradiction). Now we have $a^2(1/ab) \in N$, but

$a(1/ab) = 1/b \notin N$ and $a^2 \notin (N :_R Q(R)) = 0$. Thus $2 - abSpec(Q(R)) = \{0\}$, and hence $Q(R)$ is a 2-absorbing multiplication module. □

Proposition 2.7 — Let M be a 2-absorbing multiplication module over an integral domain R (which is not a field). Then M is either torsion or torsion-free.

PROOF : Assume that $T(M)$ is the torsion submodule of M, and let $T(M) \neq M$. Then $T(M)$ is a prime submodule (so a 2-absorbing submodule) of M with $(T(M) : M) = 0$ by [26, Lemma 3.8]. It follows that $T(M) = (T(M) : M)M = 0$. Thus M is a torsion-free module and this complete the proof. □

Example 2.8 : Let R be a domain and let $Q(R)$ be the field of quotient of R. By Proposition 2.6, the only 2-absorbing submodule of $Q(R)$ is 0; hence $Q(R)$ is a 2-absorbing multiplication does not need to be multiplication module, but we have the following result:

***Theorem* 2.9** — *Let R be a local Dedekind domain with a unique maximal ideal $P = Rp$. Then the following is a complete list, up to isomorphism, of the indecomposable 2-absorbing multiplication modules:*

(i) R;

(ii) R/P^n ($n \geq 1$) *the indecomposable torsion module;*

(iii) $E(R/P)$, *the injective hull of* R/P;

(iv) $Q(R)$, *the field of fractions of* R.

PROOF : First, we note that each of the preceding modules is indecomposable (by [6, Proposition 1.3]) and 2-absorbing multiplication module. In the cases of R and R/P^n these follows, because modules R and R/P^n are multiplication. Moreover, $Q(R)$ and $E(R/P)$ are 2-absorbing multiplication by Proposition 2.6.

Now let M be an indecomposable 2-absorbing multiplication, and choose any non-zero element $a \in M$. Let $h(a) = sup\{n \mid a \in P^n M\}$ (so $h(a)$ is a non-negative integer or ∞). Also, $(0 : a) = \{r \in R \mid ra = 0\}$, thus $(0 : a)$ is an ideal of the form P^n or 0. Because $(0 : a) = P^{m+1}$ implies that $P^m a \neq 0$ and $P(P^m a) = 0$, we can choose a so that $(0 : a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $(0 : a)$.

Case 1 : $2 - abSpec(M) = \emptyset$. Since $Spec(M) \subseteq 2 - abSpec(M)$, it follows from [28, Lemma 1.3, Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and M is not finitely generated. We may assume that $(0 : a) = P$. By an argument like that in [7, Proposition 2.7], $M \cong E(R/P)$. So we may assume that $2 - abSpec(M) \neq \emptyset$.

Case $2:$ $h(a) = n$, $(0 : a) = 0$. say $a = p^n b$. Then $rb = 0$ implies $ra = 0$ and so $r = 0$. Thus $Rb \cong R$. We also have that Rb is pure in M (see [9, Theorem 2.12]) (otherwise, we would have $h(a) > n$). As M is a torsion-free R-module by Proposition 2.7, we must have Rb is a prime submodule of M (see Remark 1.2) (so 2-absorbing submodule), and so by the hypothesis $R \cong Rb = P^t M$ for some t. Then there is an element $m \in M$ such that $b = p^t m$, whence $a = p^{n+t} m$. Therefore, $t = 0$, and $R \cong Rb = P^0 M = RM = M$.

Case $3:$ $h(a) = n$, $(0 : a) = P$. Say $a = p^n b$. Then we have $Rb \cong R/P^{n+1}$. Furthermore, Rb is pure in M. Now since Rb is a pure submodule of bonded order of M, we obtain Rb is a direct summand of M by [20, Theorem 5]; hence $M = Rb \cong R/P^{n+1}$.

Case $4:$ $h(a) = \infty$, $(0 : a) = P$. By argument like that in [12, Theorem 2.5], we get $M \cong E(R/P)$, hence $2 - abSpec(M) = \emptyset$ by Proposition 2.6, which is a contradiction.

Case $5:$ $h(a) = \infty$, $(0 : a) = 0$. By an argument like in [9, Theorem 2.12], we obtain $M \cong Q(R)$. □

Corollary 2.10 — Let $R \neq M$ be a 2-absorbing multiplication module over a local Dedekind domain with maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n $(n \geq 1)$ and K is a direct sum of copies of $E(R/P)$ and $Q(R)$. In particular, every 2-absorbing multiplication R-module which is not isomorphic with R is pure-injective.

PROOF : Let M_i denote the indecomposable summand of M. Then by Lemma 2.4(iii), M_i is an indecomposable 2-absorbing multiplication module. Now the assertion follows from Theorem 2.9 and [6, Proposition 1.3] □

3. THE SEPARATED CASE

Throughout this section, we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \qquad (1)$$

is the pullback of two local Dedekind domains R_1, R_2 with maximal ideals P_1, P_2 generated, respectively, by p_1, p_2, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$).

Remark 3.1 : ([11, Remark 3.1]) Let R be the pullback ring as in (1), and let T be an R-module of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \to S_i$. Set $T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T$ for some $t_2 \in S_2\}$ and $T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T$ for some $t_1 \in S_1\}$.

Then for each i, $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\acute{\pi_1} = \pi_1|T : T \to T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/(0 \oplus Ker(f_2) \cap T) \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $Ker(f_1) = P_1S_1$ and $Ker(f_2) = P_2S_2$).

Proposition 3.2 — Let $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ be a proper submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ over the pullback ring as in (1). Then the following hold:

(i) $Rad(T : S) = I \oplus J$ if and only if $Rad(T_1 : S_1) = I$ and $Rad(T_2 : S_2) = J$, where $I \neq 0$ and $J \neq 0$.

(ii) $Rad(T : S) = P_1 \oplus 0$ if and only if $Rad(T_1 : S_1) = P_1$ and $Rad(T_2 : S_2) = 0$.

(iii) $Rad(T : S) = 0 \oplus P_2$ if and only if $Rad(T_1 : S_1) = 0$ and $Rad(T_2 : S_2) = P_2$.

PROOF : (i) Let $(T : S) = I \oplus J$ and let $a \in I$. Let $s_1 \in S_1$. Then there exists $s_2 \in S_2$ such that $(s_1, s_2) \in S$. Since $(a, 0) \in I \oplus J$, so $(a, 0)^n(s_1, s_2) \in T$ for some n. Therefore, $a^n s_1 \in T_1$, and $a^n \in (T_1 : S_1)$, so $a \in Rad(T_1 : S_1)$. For the other containment, assume that $a \in Rad(T_1 : S_1) \subseteq P_1$. Then $a^m S_1 \subseteq T_1$ for some m. Let $(s_1, s_2) \in S$, hence $(a, 0)^m(s_1, s_2) \in T$. Thus $(a, 0) \in Rad(T : S) = I \oplus J$, so $a \in I$, and we have equality. Similarly, $Rad(T_2 : S_2) = J$. Conversely, let $(r_1, r_2) \in I \oplus J \subseteq P_1 \oplus P_2$ and let $(s_1, s_2) \in S$. Then $r_1 \in Rad(T_1 : S_1)$ and $r_2 \in Rad(T_2 : S_2)$. So $r_1^n \in (T_1 : S_1)$ and $r_2^m \in (T_2 : S_2)$. Set $k = max\{m, n\}$. So $r_1^k s_1 \in T_1 \cap P_1S_1$ and $r_2^k s_2 \in T_2 \cap P_2S_2$ and $f_1(r_1^k s_1) = 0 = f_2(r_2^k s_2)$ since $Ker(f_1) = P_1S_1$ and $Ker(f_2) = P_2S_2$, hence $(r_1, r_2)^k(s_1, s_2) \in T$. Therefore, $(r_1, r_2) \in Rad(T : S)$, hence $I \oplus J \subseteq Rad(T : S)$. For the reverse inclusion, suppose that $(r_1, r_2) \in Rad(T : S)$. Then $(r_1, r_2)^m S \subseteq T$. Let $s_1 \in S_1$. So there exists $s_2 \in S_2$ such that $(s_1, s_2) \in S$, and hence $(r_1, r_2)^m(s_1, s_2) \in T$. Thus $r_1 \in Rad(T_1 : S_1) = I$, and similarly, $r_2 \in J$, and this completes the proof.

(ii) By (i), it suffices to show that $Rad(T_2 : S_2) = 0$. Suppose not. Let $0 \neq r_2 \in Rad(T_2 : S_2)$, so $0 \neq r_2^n \in (T_2 : S_2)$ for some n. Then there exists $a_2 \in R_2$ and positive integer t such that $0 \neq r_2^n = a_2 p_2^t$. Let $(s_1, s_2) \in S$. Since R is Noetherian ring, so there exists a positive integer m such that $(P_1 \oplus 0)^m \subseteq (T : M)$. Then $(P_1^m \oplus a_2 P_2^t)(s_1, s_2) \subseteq T$, since $p_1^m s_1 \in T_1$ ($(P_1^m \oplus 0)S \subseteq T$) and $r_2^n s_2 = a_2 p_2^t s_2 \in T_2$ ($r_2^n S_2 \subseteq T_2$) and $f_1(p_1^m s_1) = 0 = f_2(r_2^n s_2)$. So $(P_1^m \oplus a_2 P_2^t)S \subseteq T$.

Therefore, $(P_1^m \oplus a_2P_2^t) \subseteq (T:S) \subseteq Rad(T:S) = P_1 \oplus 0$, which is a contradiction. The proof of (iii) is similar. □

We need the following Lemma proved in [3, Theorem 2.4].

Lemma 3.3 — Let I be a 2-absorbing ideal of R. Then one of the following statement must hold:

(i) $Rad(I) = P$ is a prime ideal of R such that $P^2 \in I$.

(ii) $Rad(I) = P_1 \cap P_2$, $P_1P_2 \subseteq I$, and $(Rad(I))^2 \subseteq I$ where P_1, P_2 are the only distinct prime ideals of R that are minimal over I.

Proposition 3.4 — Let R be the pullback ring as in (1) and let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be any separated R-module. Then the following hold:

(i) If S has a 2-absorbing submodule $T = (T_1 \longrightarrow \bar{T} \longleftarrow T_2)$ with $Rad(T:S) = P = P_1 \oplus P_2$ and $P^2 \subseteq (T:S)$, then T_1 is a 2-absorbing submodule of S_1 with $Rad(T_1:S_1) = P_1$ and T_2 is a 2-absorbing submodule of S_2 with $Rad(T_2:S_2) = P_2$.

(ii) If S has a 2-absorbing submodule T with $Rad(T:S) = P_1 \oplus 0$ and $(P_1 \oplus 0)^2 \subseteq (T:S)$, then T_1 is a 2-absorbing submodule of S_1 with $Rad(T_1:S_1) = P_1$ and T_2 is a 2-absorbing submodule of S_2 with $Rad(T_2:S_2) = 0$.

(iii) If S has a 2-absorbing submodule T with $Rad(T:S) = 0 \oplus P_2$ and $(0 \oplus P_2)^2 \subseteq (T:S)$, then T_1 is a 2-absorbing submodule of S_1 with $Rad(T_1:S_1) = 0$ and T_2 is a 2-absorbing submodule of S_2 with $Rad(T_2:S_2) = P_2$.

PROOF : (i) Let $a_1b_1s_1 \in T_1$ where $a_1, b_1 \in R_1$ and $s_1 \in S_1$. Then $v_1(a_1) = v_2(a_2)$, $v_1(b_1) = v_2(b_2)$ and $f_1(s_1) = f_2(s_2)$ for some $a_2, b_2 \in R_2$ and $s_2 \in S_2$. Hence $(a_1,a_2)(b_1,b_2)(s_1,s_2) \in P^2S \subseteq T$. Therefore, since $T \in$ is a 2-absorbing submodule of S, we have $(a_1,a_2)(s_1,s_2) \in T$ or $(b_1,b_2)(s_1,s_2) \in T$ or $(a_1,a_2)(b_1,b_2) \in (T:_R S)$. So $a_1s_1 \in T_1$ or $b_1s_1 \in T_1$ or $a_1b_1 \in (T_1:S_1)$. Similarly, T_2 is a 2-absorbing submodule of S_2.

(ii) Let $a_1b_1s_1 \in T_1$ where $a_1, b_1 \in R_1$ and $s_1 \in S_1$. Then $(a_1,0),(b_1,0) \in R$, and there exist $s_2 \in S_2$ such that $f_1(s_1) = f_2(s_2)$. So $(a_1,0)(b_1,0)(s_1,s_2) \in (P_1 \oplus 0)^2S \subseteq T$ by hypothesise. Hence $(a_1,0)(b_1,0)(s_1,s_2) \in P^2S \subseteq T$. Therefore, T 2-absorbing gives $(a_1,0)(s_1,s_2) \in T$ or $(b_1,0)(s_1,s_2) \in T$ or $(a_1,0)(b_1,0) \in (T:_R S)$. So $a_1s_1 \in T_1$ or $b_1s_1 \in T_1$ or $a_1b_1 \in (T_1 :_{R_1} S_1)$. Now we show that T_2 is a 2-absorbing submodule of S_2. Suppose that $a_2b_2s_2 \in T_2$ and $a_2b_2 \notin (T_2:S_2) = 0$ where $a_2, b_2 \in R_2$. There exists $s_2 \in S_2$ such that $(s_1,s_2) \in S$. Therefore,

$a_2b_2 \neq 0$, and so $a_2 \neq 0$ and $b_2 \neq 0$. Hence $(p_1, a_2)(p_2, b_2)(s_1, s_2) \in T$, because $p_1^2 s_1 \in T_1 \cap P_1S_1$, $a_2b_2s_2 \in T_2 \cap P_2S_2$ and $f_1(p_1^2 s_1) = 0 = f_2(a_2b_2s_2)$. Since T is a 2-absorbing submodule, then $(p_1, a_2)(s_1, s_2) \in T$ or $(p_1, b_2) \in T$. Thus $a_2s_2 \in T_2$ or $b_2s_2 \in T_2$, as required. $\square$

Proposition 3.5 — Let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be any separated module over the pullback ring as (1). Then $2 - abSpec(S) = \emptyset$ if only if $2 - abSpec(S_i) = \emptyset$ for $i = 1, 2$.

Proof : For the necessity, assume that $2 - abSpec(S) = \emptyset$ and let π be the projection map of R onto R_i. Suppose that $2 - abSpec(S_1) \neq \emptyset$ and let T_1 be a 2-absorbing submodule of S_1, so T_1 is a 2-absorbing R-submodule of $S/(0 \oplus P_2)S \cong S_1$, by Lemma 2.2; hence $2 - abSpec(S) \neq \emptyset$, which is a contradiction. Similarly, $2 - abSpec(S_2) = \emptyset$. The sufficiency by Proposition 3.4. $\square$

***Theorem* 3.6** — *Let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be any separated module over the pullback ring as (1). Then S is a 2-absorbing multiplication R-module if and only if S_i is a 2-absorbing multiplication R_i-module, $i = 1, 2$.*

PROOF : Note that by Proposition 3.5, $2 - abSpec(S) = \emptyset$ if only if $2 - abSpec(S_i) = \emptyset$ for $i = 1, 2$. So we may assume that $2 - abSpec(S) \neq \emptyset$. Let S is a separated 2-absorbing multiplication R-module. If $\bar{S} = 0$, then by [6, Lemma 2.7], $S = S_1 \oplus S_2$; hence for each i, S_i is a 2-absorbing multiplication by Lemma 2.4. So we may assume that $\bar{S} \neq 0$. Since $(0 \oplus P_2) \subseteq ((0 \oplus P_2)S : S)$, Lemma 2.4, show $S_1 \cong S/(0 \oplus P_2)S$ is a 2-absorbing multiplication $R/(0 \oplus P_2) \cong R_1$-module. Similarly, S_2 is a 2-absorbing multiplication R_2-module. Conversely, assume that each S_i is a 2-absorbing multiplication R_i-module, and let $T = (T_1 \to \bar{T} \leftarrow T_2)$ be a 2-absorbing submodule of S. We may assume that $(T : S) \neq 0$. Now we split the proof into two cases for $Rad(T : S)$.

Case 1 : $Rad(T : S) = P$. Then $S_i \neq 0$ for $i = 1, 2$. By Proposition 3.4, we have T_1 is a 2-absorbing submodule of S_1 and T_2 is a 2-absorbing submodule of S_2. Hence $T_1 = P_1^m S_1 \subseteq P_1S_1$ and $T_2 = P_2^n S_2 \subseteq P_1S_1$ since S_1, S_2 are 2-absorbing multiplication modules. Let $k = min\{m, n\}$. Therefore, $T \subseteq T_1 \oplus T_2 \subseteq P^{k-1}(P_1S_1 \oplus P_2S_2) \subseteq P^kS$. For the containment, assume that $s = (p_1^k, p_2^k)(s_1, s_2) = (p_1^k s_1, p_2^k s_2) \in P^kS$. Then $s \in T$ since $p_1^k s_1 \in T_1$, $p_2^k s_2 \in T_2$ and $f_1(p_1^k s_1) = 0 = f_2(p_2^k s_2)$ (note that $Ker(f_1) = P_1S_1$ and $Ker(f_2) = P_2S_2$).

Case 2 : $Rad(T : S) = P_1 \oplus 0$. So by Proposition 3.4, T_1 is a 2-absorbing submodule of S_1 with $Rad(T_1 : S_1) = P_1$ and T_2 is a 2-absorbing submodule of S_2 with $(T_2 : S_2) = 0$ ($Rad(T_2 : S_2) = 0$). So $T_2 = 0$ since S_2 is a 2-absorbing multiplication R_2-module and also, $T_1 = P_1^m S_1$. Therefore, $T \subseteq T_1 \oplus T_2 \subseteq (P_1 \oplus 0)^m(P_1S_1 + P_2S_2) = (P_1 \oplus 0)^m S$. Now let $s = (p_1^m, 0)(s_1, s_2) \in (P_1 \oplus 0)^m S$, then $s \in T$ since $p_1^m s_1 \in T_1$ and $f_1(p_1^m s_1) = 0 = f_2(0)$. Hence $T = (P_1 \oplus 0)^m S$, so S is a 2-

absorbing multiplication module. Similarly, if $Rad(T:S) = 0 \oplus P_2$, then we get S is a 2-absorbing multiplication module. $\square$

We need the following Lemma proved in [13, Lemma 4.3].

Lemma 3.7 — Let $R \neq S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be a separated module over the pullback ring as (1). If S_1 or S_2 is torsion-free, then $\bar{S} = 0$.

Lemma 3.8 — Let R be the pullback ring as in (1). The following separated R-modules are indecomposable and 2-absorbing multiplication:

(1) $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$;

(2) $S = (E(R_1/P_1) \longrightarrow 0 \longleftarrow 0)$, $(0 \longrightarrow 0 \longleftarrow E(R_2/P_2))$, where $E(R_i/P_i)$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;

(3) $S = (Q(R_1) \longrightarrow 0 \longleftarrow 0)$, $(0 \longrightarrow 0 \longleftarrow Q(R_2))$, where $Q(R_i)$ is the field of fractions of R_i for $i = 1, 2$; and for all positive integers m, n,

(4) $R = (R_1/P_1^n \longrightarrow \bar{R} \longleftarrow R_2/P_2^m)$.

PROOF : By [6, Lemma 2.8], these modules are indecomposable. 2-Absorbing multiplicativity follows from Theorem 2.9 and Theorem 3.6. $\square$

We refer to modules of type (2) in Lemma 3.8 as P_1-prüfer and P_2-prüfer, respectively.

Theorem 3.9 — *Let $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$ be an indecomposable separated 2-absorbing module over the pullback ring as (1). Then S is isomorphic to one of the modules listed in Lemma 3.8. In particular, every indecomposable separated 2-absorbing multiplication R-module which is not isomorphic with R is pure-injective.*

PROOF : We may assume that $S \neq R$. If $2 - abSpec(S) = \emptyset$. Then $2 - abSpec(S_i) = \emptyset$ by Proposition 3.5, so $S_i = P_iS_i$ for each $i = 1, 2$ by Theorem 2.9; hence $S = PS = P_1S_1 \oplus P_2S_2 = S_1 \oplus S_2$. Therefore, $S = S_1$ or S_2 and so S is of type (2) in the list of Lemma 3.8 by Theorem 2.9. Now we may assume that $2 - abSpec(S) \neq \emptyset$. Suppose that $PS = S$. Then by [6, Lemma 2.7], $S = S_1$ or S_2 and so S is an indecomposable 2-absorbing multiplication R_i-modules for some i, and, since $PS = S$, is type (3) in the list Lemma 3.8, by Theorem 2.9. So we may assume that $S/PS \neq 0$.

By Theorem 3.6, S_i is a 2-absorbing multiplication R_i-module, for each $i = 1, 2$. Now by Theorem 2.9, for each i, S_i is torsion and it is not divisible R_i-module. Hence by the structure of 2-absorbing multiplication modules over a local Dedekind domain (see Corollary 2.10), $S_i = M_i \oplus N_i$

where N_i is a direct sum of copies of R_i/P_i^i ($n \geq 1$) and M_i is a direct sum of copies $E(R_i/P_i)$ and $Q(R_i)$. Then we have $S = (N_1 \longrightarrow \bar{S} \longleftarrow N_2) \oplus (M_1 \longrightarrow 0 \longleftarrow 0) \oplus (0 \longrightarrow 0 \longleftarrow M_2)$. Since S is indecomposable and $S/PS \neq 0$ it follows that $S = (N_1 \longrightarrow \bar{S} \longleftarrow N_2)$. We will see that each $S_i(= N_i)$ is indecomposable. There exist positive integers m, n and k such that $P_1^m S_1 = 0$, $P_2^k S_2 = 0$ and $P^n S = 0$. Now choose $t \in S_1 \cup S_2$ with $\bar{t} \neq 0$ and such that $o(t)$ is maximal. There is a $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then $R_i t_i$ is pure in S_i for $i = 1, 2$ (see [6, Theorem 2.9]). Therefore, $R_1 t_1 \cong R_1/P_1^m$ (resp. $R_2 t_2 \cong R_2/P_2^k$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i t_i$ is pure-injective [6]. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 t_1 = M_1 \longrightarrow \bar{M} \longleftarrow M_2 = R_2 t_2)$. Then M is an R-submodule of S which is 2-absorbing multiplication by Lemma 3.8, and is a direct summand of S; this implies that $S = M$, and S is as in (4) (see [6, Theorem 2.9]). □

Corollary 3.10 — Let R be the pullback ring as in (1), and let $S \neq R$ be a separated 2-absorbing multiplication R-module. Then S is of the form $S = M \oplus N$, where M is a direct sum of copies of the modules as in (3), N is a direct sum of copies of the modules as in (1)-(2) of the Lemma 3.8. In particular, every separated 2-absorbing multiplication R-module which is not isomorphic with R is pure-injective.

PROOF : Apply Theorem 3.9 and [6, Lemma 2.9]. □

4. THE NON-SEPARATED CASE

We continue to use the notation already established, so R is the pullback ring as in (1). In this section, we find the indecomposable non-separated 2-absorbing multiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable 2-absorbing multiplication modules.

Proposition 4.1 — Let R be the pullback ring as in (1). Then $E(R/P)$ is a non-separated 2-absorbing multiplication R-module.

PROOF : It suffices to show that $2 - abSpec(E(R/P)) = \emptyset$. Assume that L is any submodule of $E(R/P)$ described in [10, Proposition 3.1]. However, no L, say $E_1 + A_n$ is a 2-absorbing submodule of $E(R/P)$, for if n is any positive integer, then $P^3(E_1 + A_{n+3}) = E_1 + A_n$, but $P^2(E_1 + A_{n+1}) = E_1 + A_{n+1} \nsubseteq E_1 + A_n$, $P(E_1 + A_{n+3}) = E_1 + A_{n+2} \nsubseteq E_1 + A_n$ and $P^3 \nsubseteq (E_1 + A_n : E(R/P)) = 0$. Therefore, $E(R/P)$ is a non-separated 2-absorbing multiplication R-module (see [6, p. 4053]). □

Proposition 4.2 — Let R be the pullback ring as in (1), and let M be any R-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then $2 - abSpec(S) = \emptyset$ if and only if

$2 - abSpec(M) = \emptyset$.

PROOF : First suppose that $2 - abSpec(S) = \emptyset$ and let $2 - abSpec(M) \neq \emptyset$. So $M \cong S/K$ has a 2-absorbing submodule, say T/K, where T is a 2-absorbing submodule of S by Lemma 2.2, which is a contradiction. Next suppose that $2 - abSpec(M) = \emptyset$ and let $2 - abSpec(S) \neq \emptyset$. Let T be a 2-absorbing submodule of S. Then by [11, Proposition 4.3], $K \subseteq T$; hence T/K is a 2-absorbing submodule of M, which is a contradiction. □

***Theorem* 4.3** — *Let R be the pullback ring as in (1), and let M be any non-separated R-module. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then S is 2-absorbing multiplication if and only if M is 2-absorbing multiplication.*

PROOF : By Proposition 4.2, we may assume that $2 - abSpec(S) \neq \emptyset$. Suppose that M is a 2-absorbing multiplication R-module and let T be a non-zero 2-absorbing submodule of S. Then by [11, Proposition 4.3], $K \subseteq T$ and so T/K is a 2-absorbing submodule of S/K. Since $M \cong S/K$ is 2-absorbing multiplication, we must have $T/K = P^n(S/K) = (P^nS + K)/K = (P^nS)/K$ (note that $K \subseteq P^nS$ by [11, Proposition 4.3]; hence $T = P^nS$. Thus S is 2-absorbing multiplication. Conversely, if S is 2-absorbing multiplication, then $M \cong S/K$ is 2-absorbing multiplication by Lemma 2.4, and this complete the proof. □

Proposition 4.4 — Let R be the pullback ring as in (1), and let M be an indecomposable 2-absorbing multiplication non-separated R-module with finite dimensional top over $\bar{R}$. Let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then S is pure-injective.

PROOF : By [6, Proposition 2.6(i)], $S/PS \cong M/PM$, so S has finite-dimensional top. Now the assertion follows from Theorem 4.3 and Corollary 3.10. □

Lemma 4.5 : Let R be the pullback ring as in (1), and let M be an indecomposable 2-absorbing multiplication non-separated R-module with finite dimensional top over $\bar{R}$, and let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then R do not occur among the direct summand of S.

PROOF : Let $S = R \oplus T$ for some submodule T of S. Then since $Soc(R) = 0$, we must have $K \subseteq T$. Therefore, $M \cong T/K \oplus R$, a contradiction since M is indecomposable and non-separated. □

Let R be the pullback ring as in (1), and let M be an indecomposable 2-absorbing multiplication non-separated R-module with finite-dimensional top over $\bar{R}$. Consider the separated representation $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$. By Proposition 4.4, S is pure-injective. So in the proofs of [6, Lemma 3.1, Proposition 3.2 and Proposition 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [6, Proposition 2.6(ii)]), we can replace the statement "M is an indecomposable 2-absorbing pure-injective non-separated R-module" by "M is an indecomposable 2-absorbing multiplication non-separated R-module", because the main key in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following result.

Corollary 4.6 — Let R be the pullback ring as in (1), and let M be an indecomposable 2-absorbing multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M. Then the following hold:

(i) The quotient fields $Q(R_1)$ and $Q(R_2)$ of R_1 and R_2 do not occur among the direct summand of S;

(ii) S is a direct sum of finitely many indecomposable 2-absorbing multiplication modules;

(iii) At most two copies of modules of finite length can occur among the indecomposable summands of S.

Recall that every indecomposable R-module of finite length is 2-absorbing multiplication since it is a quotient of a multiplication R-module. So by Corollary 4.6(iii), the infinite length non-separated indecomposable 2-absorbing multiplication modules are obtained in just the same way as the deleted cycle type indecomposable ones are, except that at least one of the two "end" modules must be a separated indecomposable 2-absorbing multiplication of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one can not have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules: we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer and a P_2-Prüfer module. If the P_2-Prüfer are direct summands of S then we will describe these modules as **doubly infinite**. Those where S has just one infinite length summand we will call **singly infinite** (the reader is referred to [6, 8, 10] for more details). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable 2-absorbing multiplication modules.

***Theorem* 4.7** — *Let $R = (R_1 \longrightarrow \bar{R} \longleftarrow R_2)$ be the pullback ring of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the class of indecomposable non-separated 2-absorbing multiplication modules with finite-dimensional top consists of the following:*

(i) The indecomposable modules of finite length (apart from R/P which is separated);

(ii) The doubly infinite 2-absorbing multiplication modules as described above;

(iii) *The singly infinite 2-absorbing multiplication modules as described above, except the two Prüfer modules (2) in Lemma 3.8.*

PROOF : Let M be an indecomposable non-separated 2-absorbing multiplication R-module with finite-dimensional top, and let $0 \longrightarrow K \longrightarrow S \longrightarrow M \longrightarrow 0$ be a separated representation of M.

(i) Clearly, M is a 2-absorbing multiplication R-module. The indecomposability follows from [22]. □

(ii) and (iii) (involing one or two Prüfer modules) M is 2-absorbing multiplication since they are a quotient of a 2-absorbing multiplication R-module. Finally, the indecomposability follows from [6, Theorem 3.5]. □

Corollary 4.8 — Let R be the pullback ring as described in Theorem 4.7. Then every indecomposable 2-absorbing multiplication R -module with finite-dimensional top is pure-injective.

PROOF : Apply [6, Theorem 3.5] and Theorem 4.7. □

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