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REVISIT TO RAMANUJAN'S MODULAR EQUATIONS OF DEGREE 21

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S. Ramanujan recorded six modular equations of degree 21 in his notebooks without recording proofs. B. C. Berndt proved all these modular equations by using the theory of modular forms. Recently Vasuki and Sharath proved two of them by using tools known to Ramanujan [5]. In this paper, we provide classical proof of remaining four identities.

Key words : Dedekind eta-function; modular equation.

1. Introduction

The complete elliptic integral $K = K(k)$ of first kind is defined by

$$K(k) = \int_{\theta=0}^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad \text{where} \quad -1 < k < 1.$$

The number k is called the modulus of K. The complementary modulus k' is defined by $k' = \sqrt{1-k^2}$. Set $K' = K(k')$. Let L and L' denote the complete elliptic integrals of first kind associated with the moduli l and l' respectively. Suppose the equality

$$n\frac{K'}{K} = \frac{L'}{L}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by the above equation. Ramanujan writes his modular equation in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. The multiplier m of modular equation of degree n is defined by

$$m = \frac{z_1}{z_n},$$

where $z_1 = K(k)$ and $z_n = L(k)$.

Let K, K', L_1, L'_1, L_2, L'_2, L_3, and L'_3 denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L'}{L}, \qquad n_2 \frac{K'}{K} = \frac{L'_2}{L_2} \qquad \text{and} \qquad n_3 \frac{K'}{K} = \frac{L'_3}{L_3} \tag{1.1}$$

hold. Then a modular equation of degree n_3 is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.1). We define the multiplier m and m' by

$$m = \frac{z_1}{z_{n_1}} = \frac{K}{L_1} \qquad \text{and} \qquad m' = \frac{z_{n_2}}{z_{n_3}} = \frac{L_2}{L_3}.$$

In Chapter 20 of his second notebook [5, p. 248], Ramanujan recorded six modular equations of degree 21. The proofs of all the modular equations via the theory of modular forms can be found in [2, pp. 401-408]. Recently Vasuki and Sharath [7] have proved two of these modular equations by using theta function identities recorded by Ramanujan, whose proofs are free from the theory of modular forms. In this paper, we provide an elementary proof of the remaining four modular equations (1.2)-(1.5) stated below. Our proofs are motivated by the comments made by Berndt in [2, p. 326].

***Theorem* 1.1** — *Let α, β, γ, δ be of first, third, seventh and twenty first degrees respectively. Let m denote the multiplier connecting α and β, and let m' be the multiplier relating γ and δ. Then*

$$\begin{aligned} &\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8} - \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/8} \\ &\qquad -2\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/12} = \left(\frac{z_1 z_3}{z_7 z_{21}}\right)^{1/2}, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\left(\frac{\alpha\beta}{\gamma\delta}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{1/8} - \left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8} \\ &\qquad -2\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/12} = 7\left(\frac{z_7 z_{21}}{z_1 z_3}\right)^{1/2}, \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4} + \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} \\ &- 2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8} \left\{1 + \left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\} = mm' \end{aligned} \tag{1.4}$$

and

$$\begin{aligned}&\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4}+\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4}\\&-2\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}\right\}=\frac{9}{mm'}.\end{aligned} \tag{1.5}$$

This work is organized as follows, in Section 2, we recall certain theta-function identities required to prove our main result. In Section 3, we prove (1.2)-(1.5).

2. Preliminary Results

Let τ be a complex number satisfying $Im(\tau) > 0$ and let $q = e^{2\pi i\tau}$. The Dedekind eta-function is defined by

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Let

$$f(-q) = q^{-1/24}\eta(\tau)$$

and

$$f_n = f(-q^n).$$

Throughout this paper we shall use the following notations

$$u = \frac{qf_7f_{21}}{f_1f_3}, \qquad v = \frac{q^{1/2}f_1f_{21}}{f_3f_7}, \qquad w = \frac{q^{2/3}f_3f_{21}}{f_1f_7}.$$

$$u_1 = \frac{q^2f_{14}f_{42}}{f_2f_6}, \qquad v_1 = \frac{qf_2f_{42}}{f_6f_{14}}, \quad \text{and} \quad w_1 = \frac{q^{4/3}f_6f_{42}}{f_2f_{14}}.$$

In [1], Baruah obtained following two identities by employing known theta-function identities found in Ramanujan's second notebook [5]:

$$3qf_3f_{21} = \frac{f_2^2f_3f_7^2}{f_1f_6f_{14}} - \frac{f_1^2f_{14}^2f_{21}}{f_2f_7f_{42}}$$

and

$$f_1f_7 = \frac{f_1f_6^2f_{21}^2}{f_2f_3f_{42}} - q^2\frac{f_3^2f_7f_{42}^2}{f_6f_{14}f_{21}}.$$

The above two identities are also found in the list of theta-function identities obtained by Somos [6]. In fact he has used the computer and runs PARI/GP scripts to obtain these identities. Using the above two identities, Baruah [1] has proved the following identities:

***Theorem* 2.1** — [1]. *We have*

$$\frac{u}{u_1}+\frac{u_1}{u}=\left(\frac{v}{v_1}\right)^2+\left(\frac{v_1}{v}\right)^2+3. \tag{2.1}$$

***Theorem* 2.2** — [3, p. 209], [5, p. 327]. *Let*

$$P=\frac{f_1^2}{q^{1/2}f_7^2} \quad and \quad Q=\frac{f_2^2}{qf_{14}^2}\ ,$$

then

$$PQ+\frac{49}{PQ}=\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3-8\left[\frac{Q}{P}+\frac{P}{Q}\right]. \tag{2.2}$$

For an elementary proof of the above, one may refer Veeresha [8].

***Theorem* 2.3** — [1]. *Let α, β, γ, δ have degrees 1, 3, 7 and 21, respectively, and*

$$P=(256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta))^{1/48}$$

and

$$Q=\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/48}\ ,$$

then

$$\begin{aligned} Q^{12}+\frac{1}{Q^{12}}-18\left(Q^6+\frac{1}{Q^6}\right)+18\sqrt{2}\left(Q^3+\frac{1}{Q^3}\right)\left(P^3+\frac{1}{P^3}\right)\\ -8\left(P^6+\frac{1}{P^6}\right)-54=0. \end{aligned} \tag{2.3}$$

Corollary 2.4 — Let α, β, γ, δ have degrees 1, 3, 7, 21 respectively

$$\frac{X^2}{Y^2}+\frac{Y^2}{X^2}-6\left[\frac{X}{Y}+\frac{Y}{X}-X-\frac{1}{X}-Y-\frac{1}{Y}+1\right]-4\left[XY+\frac{1}{XY}\right]=0, \tag{2.4}$$

where $X=(\alpha\gamma)^{1/8}$ and $Y=(\beta\delta)^{1/8}$.

PROOF : From Entry 19(i) [2, p. 314], we have

$$(\alpha\gamma)^{1/8}+\{(1-\alpha)(1-\gamma)\}^{1/8}=1.$$

Which implies

$$\{(1-\alpha)(1-\gamma)\}^{1/8} = 1 - X \text{ and } \{(1-\beta)(1-\delta)\}^{1/8} = 1 - Y. \tag{2.5}$$

Using (2.5) in (2.3), we deduce that

$$A(X,Y)B(X,Y) = 0, \tag{2.6}$$

where

$$\begin{aligned} A(X,Y) =& X^4 + Y^4 + 6[X^3Y^2 + X^2Y^3] - 6[X^3Y + XY^3] + 6[X^2Y + XY^2] \\ &- 4[X^3Y^3 + XY] - 6X^2Y^2 \end{aligned}$$

and

$$\begin{aligned} B(X,Y) =& X^4 + Y^4 - 6[X^3Y^2 + X^2Y^3] + 6[X^3Y + XY^3] - 12[X^2Y + XY^2] \\ &- 4[X^3 + Y^3 + X + Y - X^3Y^3] + 6[X^2 + Y^2 - X^2Y^2] + 10XY + 1. \end{aligned}$$

Converting X and Y into theta functions via Entry 12[v, vi] [2, p. 124], one can easily see that

$$X = \frac{1}{2}\left[q + q^2 + 2q^3 + 3q^4 + \cdots\right]$$

and

$$Y = \frac{1}{2}\left[q^3 + q^6 + 2q^9 + 3q^{12} + \cdots\right].$$

This implies X and Y tends to 0 as $q \to 0$. Therefore $A(X,Y) \to 0$ as $q \to 0$, whereas $B(X,Y)$ doesn't. Thus

$$A(X,Y) = 0.$$

Dividing this throughout by $(XY)^2$, we obtain the required result. □

3. RAMANUJAN'S MODULAR EQUATION OF DEGREE 21

In this section we prove (1.2)-(1.5). First, we give an elementary proof of the following lemma found in [4], where the proof depends on the theory of modular forms.

Lemma 3.1 — We have

$$uu_1 + 7(uu_1)^2 = u^3 + {u_1}^3 - 2\left(u^2u_1 + u{u_1}^2\right). \tag{3.1}$$

PROOF : Set

$$\frac{v}{v_1} + \frac{v_1}{v} = t. \tag{3.2}$$

This implies

$$\left(\frac{v}{v_1}\right)^2 + \left(\frac{v_1}{v}\right)^2 = t^2 - 2. \tag{3.3}$$

Using identity (3.2) in identity (2.1), we obtain

$$t^2 = \frac{u_1}{u} + \frac{u}{u_1} - 1. \tag{3.4}$$

This implies

$$\left(\frac{u_1}{u}\right)^2 + \left(\frac{u}{u_1}\right)^2 = t^4 + 2t^2 - 1. \tag{3.5}$$

Replacing q by q^3 in (2.2) and multiplying the resulting identity with (2.2), then using (3.3) and (3.5) repeatedly and upon some simplification, we obtain

$$\frac{1}{(uu_1)^2} + 7^4(uu_1)^2 = t^{12} - 2t^{10} - 15t^8 + 12t^6 + 68t^4 + 32t^2 - 94 \ ,$$

this implies

$$\frac{1}{uu_1} + 49\ uu_1 = t^6 - t^4 - 8t^2 - 2.$$

Using identity (3.4) above, we find that

$$\frac{1}{uu_1} + 49\ uu_1 = \left(\frac{u_1}{u}\right)^3 + \left(\frac{u}{u_1}\right)^3 - 4\left[\left(\frac{u_1}{u}\right)^2 + \left(\frac{u}{u_1}\right)^2\right],$$

which implies

$$\left(uu_1 + 7(uu_1)^2\right)^2 = (u^3 + {u_1}^3 - 2(u^2u_1 + uu_1))^2.$$

So, either

$$uu_1 + 7(uu_1)^2 = u^3 + {u_1}^3 - 2(u^2u_1 + uu_1),$$

or

$$uu_1 + 7(uu_1)^2 = -[u^3 + {u_1}^3 - 2(u^2u_1 + uu_1)].$$

One can easily see that

$$u^3 + {u_1}^3 = \frac{1}{q^6} - \frac{3}{q^4} + \frac{1}{q^3} - \frac{3}{q^2} + 4 + \cdots$$

and

$$u^2 u_1 + u{u_1}^2 = \frac{1}{q^5} - \frac{5}{q^3} - \frac{1}{q^2} + \frac{5}{q} + 10 + \cdots .$$

From the above series expansion, we see that as $q \to 0$,

$$q^5(u^3 + {u_1}^3) > q^5(u^2 u_1 + u{u_1}^2).$$

Hence, we consider the former equation which is the required result. □

PROOF OF (1.2) AND (1.3) : Employing Entry 12(ii, iii and iv) of [2, Chap. 17], one can easily see that (1.2) and (1.3) are equivalent to

$$\frac{q^3 f_1 f_3 f_{14}^2 f_{42}^2}{f_7 f_{21} f_2^2 f_6^2} + \frac{f_4 f_{12} f_{14}^2 f_{42}^2}{f_{28} f_{84} f_2^2 f_6^2} - \frac{q^3 f_7 f_{21} f_2 f_6 f_{28} f_{84}}{f_1 f_3 f_{14} f_{42} f_4 f_{12}} - 2\frac{q f_{14} f_{42}}{f_2 f_6} = 1$$

and

$$\frac{f_7 f_{21} f_2^2 f_6^2}{q^3 f_1 f_3 f_{14}^2 f_{42}^2} + \frac{f_{28} f_{84} f_2^2 f_6^2}{f_4 f_{12} f_{14}^2 f_{42}^2} - \frac{f_1 f_3 f_{14} f_{42} f_4 f_{12}}{q^3 f_7 f_{21} f_2 f_6 f_{28} f_{84}} - 2\frac{f_2 f_6}{q f_{14} f_{42}} = 7.$$

Setting

$$u_2 = \frac{q^4 f_{28} f_{84}}{f_4 f_{12}} ,$$

the above two equations can be written as

$$ {u_1}^3 u_2 + u{u_1}^3 - u^2 {u_2}^2 - 2u{u_1}^2 u_2 - u u_1 u_2 = 0 \tag{3.6}$$

and

$$u^2 u_2 + u{u_2}^2 - {u_1}^3 - 2u u_1 u_2 - 7u{u_1}^2 u_2 = 0. \tag{3.7}$$

Replacing q by q^2 in (3.1), we obtain

$$u_1 u_2 + 7(u_1 u_2)^2 = {u_1}^3 + {u_2}^3 - 2\left({u_1}^2 u_2 + u_1 {u_2}^2\right) . \tag{3.8}$$

Multiplying (3.1) by ${u_2}^2$ and (3.8) by u^2, and then subtracting one from the other, we obtain (3.6), which is equivalent to (1.2). Multiplying (3.1) by u_2 and (3.8) by u, and then subtracting one from the other, we obtain (3.7), which is equivalent to (1.3). □

PROOF OF (1.4) AND (1.5): From Entry 5(iii) [2, p. 230], we have

$$m-1=2\left(\frac{\beta^3}{\alpha}\right)^{1/8} \tag{3.9}$$

and

$$\frac{3}{m}-1=2\left(\frac{(1-\alpha)^3}{(1-\beta)}\right)^{1/8}. \tag{3.10}$$

This implies

$$m'-1=2\left(\frac{\delta^3}{\gamma}\right)^{1/8} \tag{3.11}$$

and

$$\frac{3}{m'}-1=2\left(\frac{(1-\gamma)^3}{(1-\delta)}\right)^{1/8}. \tag{3.12}$$

Multiplying (3.9) with (3.11) and then using (3.9), we obtain

$$mm'+1-m-m'=4\frac{Y^3}{X}, \tag{3.13}$$

where X and Y are as in Corollary 2.8. Multiplying (3.10) with (3.12), we obtain

$$9+mm'-3(m+m')=4\frac{(1-X)^3}{1-Y}mm'. \tag{3.14}$$

From (3.13) and (3.14), we find that

$$mm'=\frac{3(1-Y)(X+2Y^3)}{X\{1-Y+2(1-X)^3\}}. \tag{3.15}$$

Multiplying identity (2.4) by $Y^2(Y+2X-3)$ and re-arranging terms, we deduce that

$$\left[\frac{Y^2}{X^2}+\frac{(1-Y)^2}{(1-X)^2}+\frac{Y^2(1-Y)^2}{X^2(1-X)^2}-2\frac{Y(1-Y)}{X(1-X)}\left\{1+\frac{Y}{X}+\frac{1-Y}{1-X}\right\}\right]$$
$$\times\left[(1-X)^2\{1-Y+2(1-X)^3\}\right]-\frac{3(1-Y)(X+2Y^3)(1-X)^2}{X}=0.$$

Dividing the above identity by $(1-X)^2(1-Y+2(1-X)^3)$ throughout, using (3.15) and the definition of X and Y, we obtain (1.4).

Multiplying identity (2.4) by $X^2(2Y+X)$ and re-arranging, we deduce that

$$\left[\frac{X^2}{Y^2}+\frac{(1-X)^2}{(1-Y)^2}+\frac{X^2(1-X)^2}{Y^2(1-Y)^2}-2\frac{X(1-X)}{Y(1-Y)}\left\{1+\frac{X}{Y}+\frac{1-X}{1-Y}\right\}\right]$$
$$\times\left\{(1-Y)^2(X+2Y^3)\right\}-3X(1-Y)\{1-Y+2(1-X)^3\}=0.$$

Dividing the above identity by $(1-Y)^2(X+2Y^3)$ throughout, then using (3.15) and the definition of X and Y, we obtain (1.5). $\square$

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