

***K*-THEORY OF TORIC HYPERKÄHLER MANIFOLDS**

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Let X be a toric hyperKähler manifold. The purpose of this note is to describe the topological K -ring $K^*(X)$ of X . We give a presentation for the topological K -ring in terms of generators and relations similar to the known description of the cohomology ring of these manifolds.

Key words : Toric hyperKähler manifolds; hyperplane arrangements; K -theory; toric hyperKähler varieties; toric quiver varieties.

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1. INTRODUCTION

Toric hyperKähler manifolds were defined by Bielawski and Dancer [2] and have been widely studied recently. The cohomology ring of a toric hyperKähler manifold has been described by Konno [6]. In [4], Hausel and Sturmfels gave the algebraic geometric construction of toric hyperKähler varieties and its relation with toric quiver varieties. These varieties carry the underlying structure of a toric hyperKähler manifold (see [4, Section 5]).

In this paper we shall use the topological description of toric hyperKähler manifolds in [6]. In [6], Konno showed the existence of certain canonical complex line bundles on the toric hyperKähler manifolds and proved that their first Chern classes generate the integral cohomology ring. In [6, Theorem 3.2], he further gave the presentation of the cohomology ring in terms of the combinatorics of certain smooth hyperplane arrangement naturally associated to the toric hyperKähler manifold.

More recently topological toric hyperKähler manifolds have been studied by Kuroki [7] from the viewpoint of cohomological rigidity problem, where he has described the equivariant cohomology ring of these manifolds. In [4, Theorem 1.1], Hausel and Sturmfels gave a presentation of the cohomology ring of toric hyperKähler varieties with application to the presentation of the cohomology ring of toric quiver varieties.

Our main aim in this paper is to exploit the known description of the cohomology ring and its generators to describe the topological K -ring of these manifolds by applying the results in [1]. Our methods are similar to those used by Sankaran [8] in the description of the topological K -ring of smooth complete toric varieties and torus manifolds. We recall here that the toric hyperKähler manifold although non-compact in general (e.g $T^*(\mathbb{P}_{\mathbb{C}}^n)$) is homotopy equivalent to a finite CW complex namely its core and hence has the structure of a CW complex of finite type. The key tool is the application of the Atiyah-Hirzebruch spectral sequence which degenerates in this setting. Using this in Theorem 3.1 we show that the isomorphism classes of the canonical line bundles on the toric hyperKähler manifold defined by Konno generate the topological K -ring. The presentation of the K -ring follows from that of the cohomology ring as in the case of toric manifolds.

One of the motivations of this paper is to understand the topological K -ring of toric hyperKähler varieties and toric quiver varieties developed by Hausel and Sturmfels in [4]. Since toric hyperKähler varieties are biholomorphic to toric hyperKähler manifolds, the presentation of their topological K -rings will follow from our main theorem. However because of their additional algebraic geometric and combinatorial structure, we expect that the presentation of their K -ring gets a canonical interpretation in terms of matroid ideals and circuit ideals, similar to that of their cohomology ring described in [4, Theorem 1.1]. This is work in progress.

Let T' be the compact torus of one fourth the dimension of the toric hyperKähler manifold which acts canonically on it. We believe that we can give a description of the T' -equivariant K -ring of these manifolds using its GKM structure similar to the description of the equivariant cohomology ring by Kuroki in [7, Section 5]. This shall be taken up in future work.

There is also the simplicial analogue of toric hyperKähler orbifolds (see [2, 4, 6, 7]). They are associated to simple and not necessarily smooth hyperplane arrangement. In this paper we shall unless otherwise specified work with smooth toric hyperKähler manifolds.

Let X denote a $4n$ dimensional toric hyperKähler manifold equipped with the action of the n -dimensional compact torus $T' := (S^1)^n$ preserving the hyperKähler structure. Let H_i , $1 \leq i \leq m$ denote the associated smooth hyperplane arrangement in the dual of the Lie algebra $\mathfrak{t}^* \simeq \mathbb{R}^n$ of T' . By [6] we know that there exists m complex line bundles L_i , $1 \leq i \leq m$ on X such that $\tau_i := c_1(L_i) \in H^2(X; \mathbb{Z})$ generate $H^*(X; \mathbb{Z})$. Let \mathfrak{J}' denote the ideal in $\mathbb{Z}[x_1, \dots, x_m]$ defined by the following two sets of relations:

$$\prod_{i \in I} x_i \text{ whenever } \bigcap_{i \in I} H_i = \emptyset \text{ for } I \subseteq [1, m] \quad (1.1)$$

$$\prod_{j, \langle u, v_j \rangle > 0} (1 - x_j)^{\langle u, v_j \rangle} - \prod_{j, \langle u, v_j \rangle < 0} (1 - x_j)^{-\langle u, v_j \rangle} \text{ for } u \in \mathfrak{t}_{\mathbb{Z}}^*. \quad (1.2)$$

The following is our main theorem which describes the topological complex K -ring of X .

Main Theorem : The map from $\mathbb{Z}[x_1, \dots, x_m]$ to $K^*(X)$ which sends $x_j \mapsto 1 - [L_j]$ defines a ring homomorphism $\psi : \mathcal{R} := \mathbb{Z}[x_1, \dots, x_m]/\mathfrak{I}' \longrightarrow K^*(X)$. Moreover, ψ defines an isomorphism of \mathbb{Z} -algebras.

2. DEFINITION OF TORIC HYPERKÄHLER MANIFOLDS

We begin by briefly recalling the definition of toric hyperKähler manifold, and required terminologies and notations (see [2, Section 3] [6, 7] and [4, Section 5]).

Recall that multiplication by i (resp. j and k) defines three complex structures I (resp. J and K) on the m dimensional quaternionic vector space \mathbb{H}^m which satisfy the quaternionic relations. Consider the Euclidean metric g on $\mathbb{H}^m \simeq \mathbb{R}^{4m} \simeq \mathbb{R}^m \oplus i\mathbb{R}^m \oplus j\mathbb{R}^m \oplus k\mathbb{R}^m$. We define the Kähler forms on \mathbb{H}^m

$$\omega_I(X, Y) = g(IX, Y) \quad (2.3)$$

(and similarly ω_J, ω_K) where X and Y are tangent vectors at a point in \mathbb{H}^m . Then g is a *hyperKähler metric* that is a Kähler metric with respect to all the three complex structures.

The symplectic group $Sp(m) \subseteq SO(4m)$ consists of matrices which commute with I, J and K . Then $Sp(m)$ preserves the hyperKähler metric or equivalently preserves the Kähler forms ω_I, ω_J and ω_K . We consider the action of $Sp(m)$ on \mathbb{H}^m from the right.

We fix the identification $\mathbb{H}^m \rightarrow \mathbb{C}^m \times \mathbb{C}^m$ such that $\xi := (\xi_1, \dots, \xi_m) \mapsto ((z, w) := (z_1, \dots, z_m), (w_1, \dots, w_m))$ where $\xi_r = z_r + w_r J$ for $z_j, w_j \in \mathbb{R} + \mathbb{R} \cdot I \simeq \mathbb{C}$ and $1 \leq r \leq m$.

The diagonal subgroup $T = (S^1)^m \subseteq Sp(m)$ acts on \mathbb{H}^m as

$$e^{i\theta} \cdot (z_1 + w_1 J, \dots, z_m + w_m J) = ((z_1 + w_1 J) \cdot e^{i\theta_1}, \dots, (z_m + w_m J) \cdot e^{i\theta_m})$$

where $e^{i\theta} := (e^{i\theta_1}, \dots, e^{i\theta_m}) \in T$. Using the quaternionic relation $i \cdot j = -j \cdot i = k$ the action of T on \mathbb{H}^m can be rewritten as follows: $\xi \cdot e^{i\theta} := (z \cdot e^{i\theta}, w \cdot e^{-i\theta})$.

Let \mathfrak{t} denote the Lie algebra of T . Since the action preserves the Kähler structure, it gives the Kähler moment map

$$\mu = (\mu_I, \mu_J, \mu_K) = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) : \mathbb{H}^m \rightarrow \mathfrak{t}^* \otimes \mathbb{R}^3.$$

where μ_I, μ_J, μ_K are the Kähler moment maps with respect to $\omega_I, \omega_J, \omega_K$ respectively.

Let $\{e_1, \dots, e_m\}$ denote the standard basis of $\mathfrak{t} \simeq \mathbb{R}^m$ and $\{e_1^*, \dots, e_m^*\} \subseteq \mathfrak{t}^*$ be the dual basis. Let $\mathfrak{t}_{\mathbb{Z}} := \sum_{r=1}^m \mathbb{Z}e_r$, $\mathfrak{t}_{\mathbb{Z}}^* := \sum_{r=1}^m \mathbb{Z}e_r^*$. We have the Kähler moment maps μ_I, μ_J, μ_K with respect to $\omega_I, \omega_J, \omega_K$ respectively. We can write

$$\mu_{\mathbb{R}} := \mu_I(z, w) = \frac{1}{2} \sum_{r=1}^m (|z_r|^2 - |w_r|^2) e_r^* \in \mathfrak{t}^*$$

$$\mu_{\mathbb{C}} := (\mu_J + i\mu_K)(z, w) = \sum_{r=1}^m z_r w_r e_r^* \in \mathfrak{t}^* \otimes \mathbb{C} = \mathfrak{t}_{\mathbb{C}}^*.$$

Here $\mu_{\mathbb{C}}$ is the moment map for the I -holomorphic action of $T_{\mathbb{C}} \simeq (\mathbb{C}^*)^m$ on \mathbb{H}^m with respect to the holomorphic symplectic form $\omega_{\mathbb{C}} = \omega_J + i\omega_K$.

Let $H \simeq (S^1)^{m-n}$ denote the subtorus of T which is the kernel of a surjective map of tori $T \longrightarrow T'$ where $T' \simeq T/H$ is a torus of dimension n . We therefore get the following short exact sequence of compact tori

$$0 \longrightarrow H \xrightarrow{\iota} T \xrightarrow{\rho} T' \longrightarrow 0 \quad (2.4)$$

where $\iota : H \hookrightarrow T$ is the inclusion and $\rho : T \longrightarrow T'$ is the projection.

We also have the following short exact sequences of the corresponding Lie algebras and their duals

$$0 \longrightarrow \mathfrak{h} \xrightarrow{\iota^*} \mathfrak{t} \xrightarrow{\rho^*} \mathfrak{t}' \longrightarrow 0 \quad (2.5)$$

$$0 \longrightarrow \mathfrak{t}'^* \xrightarrow{\rho^*} \mathfrak{t}^* \xrightarrow{\iota^*} \mathfrak{h}^* \longrightarrow 0. \quad (2.6)$$

Let $\iota_{\mathbb{C}}^* : \mathfrak{t}_{\mathbb{C}}^* \longrightarrow \mathfrak{h}_{\mathbb{C}}^*$ denote the map induced on the complexification of the dual of the Lie algebras.

Since H acts on \mathbb{H}^m preserving its hyperKähler structure, we have the hyperKähler moment map $\mu_H := (\iota^* \oplus \iota_{\mathbb{C}}^*) \circ (\mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}}) : \mathbb{H}^m \rightarrow \mathfrak{h}^* \oplus \mathfrak{h}_{\mathbb{C}}^*$.

For $\alpha \neq 0$ in \mathfrak{h}^* , $(\alpha, 0) \in \mathfrak{h}^* \oplus \mathfrak{h}_{\mathbb{C}}^*$ is a regular value of μ_H . Let $\mathfrak{h}_{\mathbb{Z}} = \mathfrak{h} \cap \mathfrak{t}_{\mathbb{Z}}$ and $\mathfrak{t}'_{\mathbb{Z}} = \rho_*(\mathfrak{t}_{\mathbb{Z}})$. The necessary and sufficient condition for the action of H on $\mu_H^{-1}(\alpha, 0)$ to be free or equivalently for $\mu_H^{-1}(\alpha, 0)/H$ to be a smooth manifold is the following : for any $J \subseteq [1, m]$ such that $\{\iota^*(e_j^*) | j \in J\}$ forms a basis of \mathfrak{h}^* , $\mathfrak{t}_{\mathbb{Z}} = \mathfrak{h}_{\mathbb{Z}} + \sum_{j \in J^c} \mathbb{Z}e_j$ as a \mathbb{Z} -module (see [6, Proposition 2.2], [2] and [7, Proposition 2.1]). We call the manifold $X := X(\alpha, 0) = \mu_H^{-1}(\alpha, 0)/H$ of real dimension $4n$ a *toric hyperKähler manifold*. Since ι^* is surjective we fix an element $v \in \mathfrak{t}^*$ such that $\iota^*(v) = \alpha$. Moreover, by [6, Theorem 2.1] the diffeomorphism type of X is independent of the regular value chosen.

Example 2.1 : Let $T = (S^1)^{n+1}$, $T' = (S^1)^n$ and $\rho : T \longrightarrow T'$ be a map of tori defined by $(t_0, \dots, t_n) \mapsto (t_0 t_n^{-1}, \dots, t_{n-1} t_n^{-1})$. The subgroup $H = \ker(\rho) \simeq S^1$ is the diagonal subgroup

$(t, t, \dots, t) \in T$. The complex projective space $\mathbb{C}\mathbb{P}^n$ is the Kähler quotient of \mathbb{C}^{n+1} by the H action given by scalar multiplication and the Kähler form $\frac{i}{2} \sum_{r=1}^{n+1} dz_r \wedge d\bar{z}_r$. On $\mathbb{H}^{n+1} = \mathbb{C}^{n+1} + \bar{\mathbb{C}}^{n+1} = \mathbb{C}^{n+1} \oplus (\mathbb{C}^{n+1})^*$ we consider the action of H given by $(z, w) \cdot t := (z \cdot t, w \cdot t^{-1})$. The Kähler form for the complex structure on \mathbb{H}^{n+1} induced from the complex structure on \mathbb{C}^{n+1} is given by $\omega_I := \frac{i}{2} \sum_{r=1}^{n+1} dz_r \wedge d\bar{z}_r + dw_r \wedge d\bar{w}_r$. Also $\omega_{\mathbb{C}} = \omega_J + i\omega_K = \sum_{r=1}^{n+1} dz_r \wedge dw_r$ is the corresponding holomorphic symplectic form. The moment map μ_H is given by $\mu_H(z, w) = (\frac{1}{2}(|z|^2 - |w|^2), z \cdot w)$. Taking the regular value $(\frac{1}{2}, 0) \in \mathbb{R} \oplus \mathbb{C}$ the hyperKähler quotient $\mu_H^{-1}(\frac{1}{2}, 0)/H$ is the space

$$\{(z, w) \in \mathbb{C}^{n+1} \oplus (\mathbb{C}^{n+1})^* : z \neq 0, z \cdot w = 0\}.$$

This can be identified with the cotangent space of $\mathbb{C}\mathbb{P}^n$ namely $T_{[z]}^* \mathbb{C}\mathbb{P}^n := [z] \otimes (\mathbb{C}^{n+1}/[z])^*$.

2.1 Hyperplane arrangement and Line bundles on toric hyperKähler manifolds

For $i = 1, \dots, m$, let $H_i := \{p \in \mathfrak{t}^* \mid \langle \rho^* p + \mathbf{v}, e_i \rangle = 0\}$. Then each H_i as an affine hyperplane in \mathfrak{t}^* with a choice of nonzero oriented vector $v_i := \rho(e_i)$ for $i = 1, \dots, m$ (see [7, Remark 2.5]). Equivalently $H_i := \{p \in \mathfrak{t}^* \mid \langle p, v_i \rangle = -\langle \mathbf{v}, e_i \rangle\}$.

Let $\mathcal{H} := \{H_1, \dots, H_m\}$ denote the hyperplane arrangement associated with the toric hyperKähler manifold $X(\alpha, 0)$. Then \mathcal{H} is a *smooth* hyperplane arrangement that is (i) whenever $H_{i_1} \cap \dots \cap H_{i_l} \neq \emptyset$ the codimension of $H_{i_1} \cap \dots \cap H_{i_l}$ is l and (ii) the vectors v_{i_1}, \dots, v_{i_l} can be extended to a \mathbb{Z} basis of $\mathfrak{t}'_{\mathbb{Z}}$ (see [2, Section 3]).

We now recall the definition of characteristic submanifolds of the toric hyperKähler manifold from [7, Section 3.3].

Let X_i denote the invariant connected submanifold of $X(\alpha, 0)$ which is fixed by the circle subgroup T_i of T' obtained by the exponent of $v_i \in \mathfrak{t}'$. The circle subgroup is determined by the primitive vector v_i upto sign. Then T_i acts on the normal bundle N_i of X_i by right scalar multiplication on the fibers which are isomorphic to $\mathbb{H} = \mathbb{C} \oplus \bar{\mathbb{C}}$. Fixing a sign of v_i or equivalently an orientation of N_i is called an omniorientation of X_i for $1 \leq i \leq m$.

We can alternately construct the characteristic submanifold X_i as hyperKähler quotient of the restricted action of H on the $m - 1$ dimensional subspace Y_i of $Y = \mathbb{H}^m$ defined by the vanishing of the i th coordinate vector (see [7, Proposition 3.6]). Thus the real dimension of X_i is $4n - 4$ for $1 \leq i \leq m$. The total space of the normal bundle N_i of X_i is $E(N_i) = Y_i \times_H \mathbb{H}_i$ by [7, Proposition 3.6]. Here $\mathbb{H}_i \simeq \mathbb{H} \simeq \mathbb{R}^4$ denotes the 1-dimensional \mathbb{H} -vector space which is the representation of H where H acts by $\iota_i := p_i \circ \iota$, corresponding to the i th projection p_i of T onto its coordinate

circle subgroup $S_i \simeq S^1$. Thus the normal bundle is the restriction to X_i of the associated bundle $\mu_H^{-1}(\alpha, 0) \times_H \mathbb{H}_i$ on $X(\alpha, 0)$. By construction there is a canonical induced action of the torus $T' = T/H$ on $\mu_H^{-1}(\alpha, 0) \times_H \mathbb{H}_i$ which gives it the structure of a T' -equivariant bundle on $X(\alpha, 0)$ (see [7, Section 5.2] for details).

Now, $\mathbb{H} = \mathbb{C} \oplus \bar{\mathbb{C}}$ so that the bundle $\mu_H^{-1}(\alpha, 0) \times_H \mathbb{H}_i$ splits into the following line bundles $\mu_H^{-1}(\alpha, 0) \times_H (\mathbb{C}_i \oplus \bar{\mathbb{C}}_i)$ where \mathbb{C}_i denotes the 1-dimensional complex representation of H where H acts on \mathbb{C} via the character $\iota_i := p_i \circ \iota$. Let

$$L_i := \mu_H^{-1}(\alpha, 0) \times_H \mathbb{C}_i$$

denote the associated complex line bundle on $X(\alpha, 0)$. We recall from [6, Section 4] and [7, Section 5.2] that these line bundles are holomorphic with respect to the complex structure I on $X(\alpha, 0)$ and $\tau_i := c_1(L_i)$ for $1 \leq i \leq m$ generate the integral cohomology ring of $X(\alpha, 0)$.

2.2 Cohomology ring of toric hyperKähler manifolds

We recall below the presentation of the ordinary integral cohomology ring of the toric hyperKähler manifold X due to Konno.

Theorem 2.2 — [6, Theorem 3.2]. *Let $X := (X(\alpha, 0), T')$ be a toric hyperKähler manifold and $\mathcal{H} := \{H_1, \dots, H_m\}$ be the associated smooth hyperplane arrangement. Let \mathfrak{J} denote the ideal in $\mathbb{Z}[x_1, \dots, x_m]$ defined by the following two sets of relations:*

$$\prod_{i \in I} x_i \text{ whenever } \bigcap_{i \in I} H_i = \emptyset \tag{2.7}$$

for $I \subseteq [1, m]$, and

$$\sum_{j=1}^m \langle u, v_j \rangle x_j \tag{2.8}$$

for $u \in \mathfrak{t}_{\mathbb{Z}}^*$. Then the canonical map from $R := \mathbb{Z}[x_1, \dots, x_m]/\mathfrak{J}$ to $H^*(X(\alpha, 0); \mathbb{Z})$ which sends $x_j \mapsto c_1(L_j)$ is an isomorphism of \mathbb{Z} -algebras.

3. MAIN THEOREM

We now state and prove the main theorem of this paper which gives a presentation for the topological K -ring of the toric hyperKähler manifold $X = X(\alpha, 0)$.

Theorem 3.1 — *Let $X := (X(\alpha, 0), T')$ be a toric hyperKähler manifold and $\mathcal{H} := \{H_1, \dots, H_m\}$ be the associated smooth hyperplane arrangement. Let \mathfrak{J}' denote the ideal in $\mathbb{Z}[x_1, \dots, x_m]$ defined*

by the following two sets of relations:

$$\prod_{i \in I} x_i \text{ whenever } \bigcap_{i \in I} H_i = \emptyset \text{ for } I \subseteq [1, m] \quad (3.10)$$

$$\prod_{j, \langle u, v_j \rangle > 0} (1 - x_j)^{\langle u, v_j \rangle} - \prod_{j, \langle u, v_j \rangle < 0} (1 - x_j)^{-\langle u, v_j \rangle} \text{ for } u \in \mathfrak{t}_{\mathbb{Z}}^*. \quad (3.11)$$

Then the canonical map ψ from $\mathcal{R} := \mathbb{Z}[x_1, \dots, x_m]/\mathfrak{I}'$ to the topological K -ring $K^*(X)$ of X which sends $x_j \mapsto 1 - [L_j]$ is an isomorphism of \mathbb{Z} -algebras.

PROOF : The proof of this theorem follows along the lines of the proof [6, Theorem 2.2]. The relation (2.8) in $H^*(X(\alpha, 0); \mathbb{Z})$ implies that if $H_{i_1} \cap \dots \cap H_{i_k} = \emptyset$ then

$$c_k(L_{i_1} \oplus L_{i_2} \oplus \dots \oplus L_{i_k}) = c_1(L_{i_1}) \cdot c_1(L_{i_2}) \cdot \dots \cdot c_1(L_{i_k}) = 0.$$

Now, applying the γ -operation in K -theory (see [5, Proposition 7.4]) we get

$$\gamma^k([L_{i_1} \oplus \dots \oplus L_{i_k}] - k) = (-1)^k c_k(L_{i_1} \oplus \dots \oplus L_{i_k}) = 0.$$

Further, we have

$$\gamma^k([L_{i_1} \oplus \dots \oplus L_{i_k}] - k) = ([L_{i_1}] - 1) \cdot ([L_{i_2}] - 1) \cdot \dots \cdot ([L_{i_k}] - 1).$$

This proves the relation (3.10) in $K^*(X)$. For $u \in \mathfrak{t}_{\mathbb{Z}}^*$ let

$$L_u := \prod_{i=1}^m L_i^{\langle u, v_i \rangle}.$$

Then $c_1(L_u) = \sum_{i=1}^m \langle u, v_i \rangle c_1(L_i) = 0$ by (2.9). This implies that L_u is isomorphic to a trivial line bundle on X . Thus in $K^*(X)$ we have

$$\prod_{i=1}^m [L_i]^{\langle u, v_i \rangle} = 1.$$

proving relation (3.11). Hence ψ defines a ring homomorphism $\mathcal{R} \rightarrow K^*(X)$. Further, by Theorem 2.2, $c_1(L_i), 1 \leq i \leq m$ generate the cohomology of X . In particular, $H^*(X; \mathbb{Z})$ is generated by $H^2(X; \mathbb{Z})$ and vanishes in odd dimensions. Also X has the homotopy type of a finite CW complex since by [6, Section 6], the $\text{Core}(X)$, which is a finite union of compact toric submanifolds of X is a T' -equivariant deformation retract of X . We now proceed by the arguments similar to [8, Section 3] and [9, Lemma 4.1, Lemma 4.2]. Let $f_i : X \rightarrow \mathbb{P}^\infty$ be a classifying map for the bundle L_i

for $1 \leq i \leq m$. We consider the map $f : X \longrightarrow (\mathbb{P}^\infty)^m$ defined as $f(x) = (f_1(x), \dots, f_m(x))$. Since $c_1(L_i), 1 \leq i \leq m$ generate $H^*(X; \mathbb{Z})$ it follows that $f^* : H^*((\mathbb{P}^\infty)^m; \mathbb{Z}) \longrightarrow H^*(X; \mathbb{Z})$ is surjective. By the naturality of the Atiyah-Hirzebruch spectral sequence it follows that $K^*(X)$ is generated by $[L_i], 1 \leq i \leq m$. Thus we conclude that ψ is surjective. Furthermore, it can be shown that R and hence $H^*(X; \mathbb{Z})$ is a free abelian group of finite rank (see [6, Theorem 3.2] and [3, 3.8]) which is equal to the Euler characteristic $\mathcal{X}(X)$. Thus the collapsing of the Atiyah Hirzebruch spectral sequence implies that $K^*(X)$ is also a free abelian group of rank equal to $\mathcal{X}(X)$ (see [1, p. 19]).

We shall now show that ψ is injective by proving that \mathcal{R} is free abelian of rank $\mathcal{X}(X)$.

As in [8, Section 4] we construct a filtered ring $S = \mathbb{Z}[x_1, \dots, x_m]$ and let \mathfrak{J}' the ideal defined by the relations (3.10) and (3.11). The ring S is graded with $\deg(x_i) = 1$. The abelian group of all homogeneous polynomials of degree j is denoted by $S_{(j)}$. We then have a multiplicative filtration $S = S_0 \supseteq S_1 \supseteq \dots \supseteq S_r \supseteq \dots$ where $S_r = \bigoplus_{j \geq r} S_{(j)}$. Since \mathfrak{J}' is an ideal generated by elements with constant term zero, the above filtration is \mathfrak{J}' -stable. This induces a decreasing multiplicative filtration $\mathcal{R} = \mathcal{R}_0 \supseteq \mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \dots$ on $\mathcal{R} := S/\mathfrak{J}'$. Let $\text{gr}(\mathcal{R})$ denote the associated graded ring with respect to this filtration. For $f \in S_r$ we denote by $\text{in}(f)$ the initial form of f which is the homogeneous polynomial of degree $r \geq 1$ such that $f - \text{in}(f) \in S_{r+1}$.

If $z_u := \prod_{j, \langle u, v_j \rangle > 0} (1 - x_j)^{\langle u, v_j \rangle} - \prod_{j, \langle u, v_j \rangle < 0} (1 - x_j)^{-\langle u, v_j \rangle}$ for each $u \in \mathfrak{t}_{\mathbb{Z}}^*$. Then $h_u := \text{in}(z_u) = \sum_{i=1}^m \langle u, v_i \rangle x_i$. Thus the ideal \mathfrak{J} is generated by the relations (2.8) and h_u for $u \in \mathfrak{t}_{\mathbb{Z}}^*$. Since \mathfrak{J} is a graded ideal in S , $R = S/\mathfrak{J}$ is a graded abelian group. Moreover, by the arguments in [8, pages 462-463] we have a surjective homomorphism of graded abelian groups $\eta : \text{gr}(R) = R \longrightarrow \text{gr}(\mathcal{R})$. Since R is free abelian of rank $\mathcal{X}(X)$ it follows that $\text{gr}(\mathcal{R})$ is free abelian of rank at most $\mathcal{X}(X)$. Since \mathcal{R} and $\text{gr}(\mathcal{R})$ are free abelian of the same rank it follows that \mathcal{R} is free abelian of rank at most $\mathcal{X}(X)$. Combining with the surjectivity of ψ this implies that ψ must be an isomorphism.

We now illustrate Theorem 3.1 by describing the K -ring of the cotangent bundle of the complex projective space whose construction as a toric hyperKähler manifold was described in Example 2.1. (See [7, Example 2.4 and Example 3.4]).

Example 3.2 : From Example 2.1 we have the inclusion $H \subset T = (S^1)^{n+1}$ of the diagonal and $T' = T/H \simeq (S^1)^n$. This induces the following exact sequence of dual of the corresponding Lie

algebras

$$\mathfrak{t}'^* \xrightarrow{\rho^*} \mathfrak{t}^* \xrightarrow{\iota^*} \mathfrak{h}^*$$

where $\iota^*(\alpha_1, \dots, \alpha_{n+1}) = \alpha_1 + \dots + \alpha_{n+1} \in \mathbb{R} = \mathfrak{h}^*$ where $(\alpha_1, \dots, \alpha_{n+1}) \in \mathfrak{t}^* \simeq \mathbb{R}^{n+1}$. Further,

$$\begin{aligned} \rho^*(u) &= (\langle u, e_1 \rangle, \dots, \langle u, e_n \rangle, \langle u, -(e_1 + \dots + e_n) \rangle) \\ &= (a_1, \dots, a_n, -a_1 - a_2 - \dots - a_n) \end{aligned}$$

for $u = (a_1, \dots, a_n) \in \mathfrak{t}'^* \simeq \mathbb{R}^n$. For $\alpha = n + 1$ we choose the lift $(1, 1, \dots, 1) = v \in \mathfrak{t}^*$. Thus the associated hyperplane arrangement is given by

$$\begin{aligned} H_1 &= \{(a_1, \dots, a_n) \in \mathfrak{t}'^* \mid a_1 = -1\} \\ &\vdots \\ H_n &= \{(a_1, \dots, a_n) \in \mathfrak{t}'^* \mid a_n = -1\} \\ H_{n+1} &= \{(a_1, \dots, a_n) \in \mathfrak{t}'^* \mid a_1 + a_2 + \dots + a_n = 1\} \end{aligned}$$

Clearly $I = [1, n + 1]$ is the only subset of $[1, n + 1]$ such that $\bigcap_{i \in I} H_i = \emptyset$. Hence by Theorem 3.1 it follows that the topological K -ring of $T^*(\mathbb{C}\mathbb{P}^n)$ is isomorphic to $\mathcal{R} := \mathbb{Z}[x_1, \dots, x_{n+1}]/\mathfrak{J}'$ where \mathfrak{J}' is the ideal in $\mathbb{Z}[x_1, \dots, x_{n+1}]$ generated by the monomial $x_1 \cdot x_2 \cdots x_{n+1}$ and the following n relations

$$(1 - x_1) - (1 - x_{n+1}), (1 - x_2) - (1 - x_{n+1}), \dots, (1 - x_n) - (1 - x_{n+1})$$

corresponding to the basis $\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 1)\}$ of \mathfrak{t}'^* . Let L_{n+1} be the canonical line bundle on $T^*(\mathbb{C}\mathbb{P}^n)$ corresponding to the hyperplane H_{n+1} . After suitable change of variables in the ring \mathcal{R} it can be seen that the map $x \mapsto [L_{n+1}]$ defines an isomorphism of \mathbb{Z} -algebras from $\mathbb{Z}[x]/(1 - x)^{n+1}$ to $K^*(X)$.

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