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PROOFS OF SOME CONJECTURES OF Z. -H. SUN ON RELATIONS BETWEEN SUMS OF SQUARES AND SUMS OF TRIANGULAR NUMBERS

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Let $N(a,b,c,d;n)$ be the number of representations of n as $ax^2+by^2+cz^2+dw^2$ and $T(a,b,c,d;n)$ be the number of representations of n as $a\frac{X(X+1)}{2}+b\frac{Y(Y+1)}{2}+c\frac{Z(Z+1)}{2}+d\frac{W(W+1)}{2}$, where a,b,c,d are positive integers, n,X,Y,Z,W are nonnegative integers, and x,y,z,w are integers. Recently, Z.-H. Sun found many relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$ and conjectured 23 more relations. Yao proved five of Sun's conjectures by using (p,k)-parametrization of theta functions and stated that six more could be proved by using the same method. More recently, Sun himself confirmed two more conjectures by proving a general result whereas Xia and Zhong proved three more conjectures of Sun by employing theta function identities. In this paper, we prove the remaining seven conjectures. Six are proved by employing Ramanujan's theta function identities and one is proved by elementary techniques.

Key words : Sum of squares; sum of triangular numbers; Ramanujan's theta function; representation of quaternary quadratic forms.

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1. Introduction

Let $\mathbb{N}^+$, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of positive integers, the set of nonnegative integers, and the set of integers, respectively. Let $\mathbb{N}^4=\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $\mathbb{Z}^4=\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. For $a,b,c,d\in\mathbb{N}^+$ and

$n \in \mathbb{N}$, define

$$N(a,b,c,d;n) := \left|\{(x,y,z,w) \in \mathbb{Z}^4 : ax^2 + by^2 + cz^2 + dw^2 = n\}\right|$$

and

$$T(a,b,c,d;n) := \left|\left\{(x,y,z,w) \in \mathbb{N}^4 : a\frac{x(x+1)}{2} + b\frac{y(y+1)}{2} + c\frac{z(z+1)}{2} + d\frac{w(w+1)}{2} = n\right\}\right|,$$

where we take $N(a,b,c,d;0) = T(a,b,c,d;0) = 1$.

Jacobi and Legendre proved that

$$N(1,1,1,1;n) = 8\sum_{d|n,4\nmid d} d$$

and

$$T(1,1,1,1;n) = \sigma(2n+1),$$

respectively, where $\sigma(n) = \sum_{d|n} d$.

For further formulas for $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$ for certain values of $a,b,c,d \in \mathbb{N}^+$, we refer to Dickson's historical comments [11], Cooper's papers [12, 13], Alaca's papers [2, 3], papers [4-8] by Alaca, Alaca, Lemire and Williams, Williams' papers [20, 21] and book [22], and papers [18, 19] by Wang and Sun.

Finding relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$ is another interesting area of research. For $a,b,c,d \in \mathbb{N}^+$ with $5 \le a+b+c+d \le 8$, let

$$C(a,b,c,d) = 16 + 4i_1(i_1-1)i_2 + 8i_1i_3,$$

where i_j is the number of elements in $\{a,b,c,d\}$ which are equal to j. When $5 \le a+b+c+d \le 7$, Adiga, Cooper and Han [1] proved that

$$C(a,b,c,d)T(a,b,c,d;n) = N(a,b,c,d;8n+a+b+c+d).$$

When $a+b+c+d = 8$, Baruah, Cooper and Hirschhorn [9] proved that

$$C(a,b,c,d)T(a,b,c,d;n) = N(a,b,c,d;8n+8) - N(a,b,c,d;2n+2).$$

Wang and Sun [18, 19] and Sun [16] discovered several new relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$. In particular, in [16], Sun posed 23 conjectures (Conjecture 2.1-Conjecture 2.23) stating some relations between $N(a,b,c,d;n)$ and $T(a,b,c,d;n)$. Five of the conjectures (Conjectures

2.2, 2.3, 2.4, 2.7, and 2.11) were proved by Yao [24] by utilizing (p, k)-parametrization of theta functions. In Section 8 of [24], Yao also remarked that Conjectures 2.1, 2.14, 2.15, 2.19, 2.20, and 2.21 can also be proved in a similar way. In fact, in another recent paper, Yao [25] proved some general relations from which Conjectures 2.1, 2.15, 2.19, 2.20, and 2.21 follow as special cases. Recently, Sun [17] himself confirmed Conjectures 2.2 and 2.6-2.8 by proving the following general result.

Let $m \equiv 1 \pmod 4$ or $m \equiv 4 \pmod 8$. Suppose that there is an odd prime divisor p of m such that $\left(\frac{4n+5}{p}\right) = -1$, where $\left(\frac{a}{p}\right)$ is the Legendre symbol. Then

$$32T(1, 1, 8, m; n) = N(1, 1, 8, m; 8n + 10 + m).$$

Most recently, Xia and Zhong [23] proved Conjectures 2.18, 2.22, and 2.23 by using theta function identities. In this paper, we prove the remaining seven conjectures, namely, Conjectures 2.5, 2.9, 2.10, 2.12, 2.13, 2.16, and 2.17, of Sun [16]. The main results are presented in the following seven theorems.

Theorem 1.1 — *(Conjecture 2.9 in Sun [16]). Let $n \in \mathbb{N}^+$ with $n \equiv 0, 3, 5, 6, 7 \pmod{11}$. Then*

$$48T(1, 1, 4, 11; n) = N(1, 1, 4, 11; 8n + 17). \tag{1.1}$$

Theorem 1.2 — *(Conjecture 2.10 in Sun [16]). Let $n \in \mathbb{N}^+$ with $n \equiv 0, 1, 2, 4, 7 \pmod{11}$. Then*

$$48T(1, 1, 2, 22; n) = N(1, 1, 2, 22; 8n + 26). \tag{1.2}$$

Theorem 1.3 — *(Conjecture 2.12 in Sun [16]). Let $n \in \mathbb{N}^+$ with $n \equiv 1 \pmod 4$. Then*

$$32T(3, 5, 20, 32; n) = N(3, 5, 20, 32; 8n + 60) - 4N(3, 5, 20, 32; 2n + 15). \tag{1.3}$$

Theorem 1.4 — *(Conjecture 2.13 in Sun [16]). Let $n \in \mathbb{N}^+$ with $n \equiv 1 \pmod 4$. Then*

$$24T(1, 6, 15, 18; n) = N(1, 6, 15, 18; 8n + 40) - 3N(1, 6, 15, 18; 2n + 10). \tag{1.4}$$

Theorem 1.5 — *(Conjecture 2.16 in Sun [16]). Let $n \in \mathbb{N}^+$ with $n \equiv 0 \pmod 4$. Then*

$$24T(1, 7, 10, 30; n) = N(1, 7, 10, 30; 8n + 48) - 3N(1, 7, 10, 30; 2n + 12). \tag{1.5}$$

Theorem 1.6 — *(Conjecture 2.17 in Sun [16]). Let $n \in \mathbb{N}^+$ with $n \equiv 3 \pmod 4$. Then*

$$24T(1, 10, 15, 30; n) = N(1, 10, 15, 30; 8n + 56) - 3N(1, 10, 15, 30; 2n + 14). \tag{1.6}$$

***Theorem* 1.7** — *(Conjecture 2.5 in Sun [16]). Let* $n \in \mathbb{N}^+$ *with* $n \equiv 0, 2 \pmod 8$. *Then*

$$4T(1,2,4,17;n) = N(1,2,4,17;n+3), \tag{1.7}$$

We employ elementary dissections of Ramanujan's theta functions to prove the first six theorems. We could not effectively use that method to prove the last theorem. So we prove that theorem by employing some elementary techniques. It is to be noted that in [17], Sun also listed seven more conjectures. Five are on ternary quadratic forms and two are on quaternary quadratic forms. The five conjectures on the ternary case are proved in [15] by an elementary method. All the seven conjectures in [17] are proved in [10] by using Ramanujan's theta functions.

We organize the paper in the following way. In the next section, we present the background material on Ramanujan's theta functions and some useful lemmas. In Sections 3-9, we prove Theorems 1.1-1.7, respectively.

2. Ramanujan's Theta Functions and Some Useful Lemmas

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1.$$

Three special cases of $f(a,b)$ are

$$\phi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2},$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2},$$

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}.$$

It is clear from the definitions of ϕ and ψ that

$$\sum_{n=0}^{\infty} N(a,b,c,d;n) q^n = \phi(q^a)\phi(q^b)\phi(q^c)\phi(q^d)$$

and

$$\sum_{n=0}^{\infty} T(a,b,c,d;n) q^n = \psi(q^a)\psi(q^b)\psi(q^c)\psi(q^d).$$

In the following lemma, we record some well-known 2-dissections and identities from Berndt's book [11, pp. 39, 40, 49, 114, and 115]. The Identities (2.1), (2.2), (2.5)-(2.7) can also be found in a recent book by Hirschhorn [14, Eqs. (1.9.4), (1.10.1), (1.7.1), (10.7.3), and (10.7.6)], which contains many other interesting results.

Lemma 2.1 — We have

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \tag{2.1}$$

$$\phi(q)^2 = \phi(q^2)^2 + 4q\psi(q^4)^2, \tag{2.2}$$

$$\phi(q)\psi(q^2) = \psi(q)^2, \tag{2.3}$$

$$\psi(q) = f(q^6, q^{10}) + qf(q^2, q^{14}), \tag{2.4}$$

$$f(-q)^3 = \phi(-q)^2\psi(q) = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}, \tag{2.5}$$

$$\phi(-q)^2 f(-q) = \sum_{k=-\infty}^{\infty}(6k+1)q^{k(3k+1)/2}, \tag{2.6}$$

and

$$\psi(q^2)f(-q)^2 = \sum_{k=-\infty}^{\infty}(3k+1)q^{k(3k+2)}. \tag{2.7}$$

Some more useful 2-dissections are given in the following lemma.

Lemma 2.3 — We have

$$\phi(q)\phi(q^3) = \phi(q^4)\phi(q^{12}) + 2q\psi(q^2)\psi(q^6) + 4q^4\psi(q^8)\psi(q^{24}), \tag{2.8}$$

$$\psi(q)\psi(q^3) = \psi(q^4)\phi(q^6) + q\phi(q^2)\psi(q^{12}), \tag{2.9}$$

$$\psi(q)\psi(q^7) = \psi(q^8)\phi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\phi(q^4)\psi(q^{56}), \tag{2.10}$$

$$\psi(q^3)\psi(q^5) = \psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120}), \tag{2.11}$$

$$\phi(q^3)\phi(q^5) = \phi(-q^2)\phi(-q^{30}) + 2q^2\psi(q)\psi(q^{15}), \tag{2.12}$$

$$\begin{aligned}\phi(q)\phi(q^{15}) = {} & \phi(-q^6)\phi(-q^{10}) + 2q\psi(q^8)\phi(q^{60}) + 2q^4\psi(q^2)\psi(q^{30}) \\ & + 2q^{15}\phi(q^4)\psi(q^{120}).\end{aligned} \tag{2.13}$$

PROOF : Identity (2.8) follows by setting $(\mu, \nu) = (2, 1)$ in [11, p. 68, eq. (36.2)], and then employing (2.1), identities (2.9), (2.10) and (2.11) follow from (36.8) of [11, p. 69] by setting $(\mu, \nu) = (2, 1)$, $(\mu, \nu) = (4, 3)$ and $(\mu, \nu) = (4, 1)$, respectively. Identity (2.12) is in [11, p. 377, Entry 9(ii)]. Finally, identity (2.13) follows from [11, p. 377, Entry 9(ii)] and (2.11). □

Lemma 2.3 — We have

$$\phi(q^{22})\psi(q^4) + q^5\phi(q^2)\psi(q^{44}) = f(-q)f(-q^{11}) + q\psi(q)\psi(q^{11}). \tag{2.14}$$

PROOF : With the aid of (2.1) and the identity (see Berndt's book [11, p. 365, eq. (7.5)])

$$\phi(q)\phi(q^{11}) - \phi(-q)\phi(-q^{11}) = 4qf(-q^2)f(-q^{22}) + 4q^3\psi(q^2)\psi(q^{22}),$$

we have

$$\begin{aligned}\phi(q^{44})\psi(q^8) + q^{10}\phi(q^4)\psi(q^{88}) &= \left(\frac{\phi(q^{11}) + \phi(-q^{11})}{2}\right)\left(\frac{\phi(q) - \phi(-q)}{4q}\right) \\ &\quad + q^{10}\left(\frac{\phi(q) + \phi(-q)}{4q}\right)\left(\frac{\phi(q^{11}) - \phi(-q^{11})}{2}\right) \\ &= \frac{1}{4q}\left(\phi(q)\phi(q^{11}) - \phi(-q)\phi(-q^{11})\right) \\ &= \frac{1}{4q}\left(4qf(-q^2)f(-q^{22}) + 4q^3\psi(q^2)\psi(q^{22})\right) \\ &= f(-q^2)f(-q^{22}) + q^2\psi(q^2)\psi(q^{22}).\end{aligned}$$

Replacing q^2 by q we arrive at (2.14). □

Lemma 2.4 — We have

$$\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20}) - f(q,q^7)f(q^{45},q^{75}) - q^7f(q^3,q^5)f(q^{15},q^{105}) = 0. \tag{2.15}$$

PROOF : With the help of (2.4), we find that

$$\begin{aligned}&f(q^2,q^{14})f(q^{90},q^{150}) + q^{14}f(q^6,q^{10})f(q^{30},q^{210}) \\ &= \frac{1}{4q}\left(\psi(q) - \psi(-q)\right)\left(\psi(q^{15}) + \psi(-q^{15})\right) + \frac{1}{2q}\left(\psi(q) + \psi(-q)\right)\left(\psi(q^{15}) - \psi(-q^{15})\right) \\ &= \frac{1}{2q}\left(\psi(q)\psi(q^{15}) - \psi(-q)\psi(-q^{15})\right).\end{aligned} \tag{2.16}$$

On the other hand, with the aid of (2.1) and (2.12), we find that

$$\begin{aligned}&\phi(q^{20})\psi(q^{24}) + q^2\phi(q^{12})\psi(q^{40}) \\ &= \frac{1}{8q^3}\left(\phi(q^5) + \phi(-q^5)\right)\left(\phi(q^3) - \phi(-q^3)\right) + \frac{1}{8q^3}\left(\phi(q^3) + \phi(-q^3)\right)\left(\phi(q^5) - \phi(-q^5)\right) \\ &= \frac{1}{4q^3}\left(\phi(q^3)\phi(q^5) - \phi(-q^3)\phi(-q^5)\right) \\ &= \frac{1}{2q}\left(\psi(q)\psi(q^{15}) - \psi(-q)\psi(-q^{15})\right).\end{aligned} \tag{2.17}$$

From (2.16) and (2.17) , we arrive at

$$f(q^2,q^{14})f(q^{90},q^{150})+q^{14}f(q^6,q^{10})f(q^{30},q^{210}) =\phi(q^{20})\psi(q^{24})+q^2\phi(q^{12})\psi(q^{40}),$$

which is clearly equivalent to (2.15) with q^2 replaced by q. □

Lemma 2.5 —

$$\psi(q)\psi(q^{15})=\psi(q^6)\psi(q^{10})+q\phi(q^{20})\psi(q^{24})+q^3\phi(q^{12})\psi(q^{40}). \tag{2.18}$$

PROOF : Identity (2.18) easily follows from (2.17) and the identity

$$\psi(q)\psi(q^{15})+\psi(-q)\psi(-q^{15})=2\psi(q^6)\psi(q^{10}),$$

in [11, p. 377, Entry 9]. □

3. PROOF OF THEOREM 1.1

We have

$$\begin{aligned}\sum_{n=0}^{\infty}N(1,1,4,11;n)q^n &= \phi(q)^2\phi(q^4)\phi(q^{11})\\ &= \phi(q^4)\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\left(\phi(q^{44})+2q^{11}\psi(q^{88})\right),\end{aligned}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty}N(1,1,4,11;2n+1)q^n &= 4\phi(q^2)\phi(q^{22})\psi(q^2)^2+2q^5\phi(q^2)\phi(q)^2\psi(q^{44})\\ &= 4\phi(q^2)\phi(q^{22})\psi(q^2)^2+2q^5\phi(q^2)\psi(q^{44})\left(\phi(q^2)^2+4q\psi(q^4)^2\right),\end{aligned}$$

from which we further extract

$$\begin{aligned}\sum_{n=0}^{\infty}N(1,1,4,11;4n+1)q^n &= 4\psi(q)^2\phi(q)\phi(q^{11})+8q^3\psi(q^2)^2\phi(q)\psi(q^{22})\\ &= 4\phi(q)^2\phi(q^{11})\psi(q^2)+8q^3\phi(q)\psi(q^2)^2\psi(q^{22})\\ &= 4\left(\phi(q^2)^2+4q\psi(q^4)^2\right)\left(\phi(q^{44})+2q^{11}\psi(q^{88})\right)\psi(q^2)\\ &\quad +8q^3\left(\phi(q^4)+2q\psi(q^8)\right)\psi(q^2)^2\psi(q^{22}),\end{aligned}$$

where the second equality is due (2.3).

It follows that

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n = 4\psi(q)\phi(q)^2\phi(q^{22}) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44}) + 16q^2\psi(q)^2\psi(q^4)\psi(q^{11}),$$

which becomes

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 16\sum_{n=0}^{\infty} T(1,1,4,11;n)q^{n+2}\\
&= 4\phi(q)^2\phi(q^{22})\psi(q) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44})\\
&= 4\left(\phi(-q)^2 + 8q\psi(q^4)^2\right)\phi(q^{22})\psi(q) + 32q^6\psi(q)\psi(q^2)^2\psi(q^{44})\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q)\psi(q^4)^2 + q^5\psi(q)\psi(q^2)^2\psi(q^{44})\right)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q^4)^2 + q^5\phi(q^2)\psi(q^4)\psi(q^{44}))\right)\psi(q)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(\phi(q^{22})\psi(q^4) + q^5\phi(q^2)\psi(q^{44}))\right)\psi(q)\psi(q^4)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\left(f(-q)f(-q^{11}) + q\psi(q)\psi(q^{11})\right)\psi(q)\psi(q^4)\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\psi(q^4)\psi(q)f(-q)f(-q^{11}) + 32q^2\psi(q)^2\psi(q^4)\psi(q^{11}),
\end{aligned}$$

and so,

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 48q^2\sum_{n=0}^{\infty} T(1,1,4,11;n)q^n\\
&= 4\phi(-q)^2\phi(q^{22})\psi(q) + 32q\psi(q^4)f(-q^2)^2f(-q^{11})\\
&= 4\phi(q^{22})f(-q)^3 + 32qf(-q^{11})\psi(q^4)f(-q^2)^2\\
&= 4\phi(q^{22})\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2} + 32f(-q^{11})\sum_{n=-\infty}^{\infty}(3n+1)q^{6n^2+4n+1},
\end{aligned}$$

where the last equality is due to (2.5) and (2.7).

Now, it can be easily verified that $(n^2+n)/2 \equiv 0,1,3,4,6,$ or $10 \pmod{11}$ and $6n^2+4n+1 \equiv 0,1,3,4,6,$ or $10 \pmod{11}$. Therefore, extracting the terms involving q^n for $n \equiv 2,5,7,8,9 \pmod{11}$ in the above, we find that

$$\sum_{n=0}^{\infty} N(1,1,4,11;8n+1)q^n - 48\sum_{n=0}^{\infty} T(1,1,4,11;n)q^{n+2} = 0,$$

which readily implies that, for $n \equiv 0,3,5,6,7 \pmod{11}$,

$$N(1,1,4,11;8n+17) = 48T(1,1,4,11;n).$$

This completes the proof.

4. Proof of Theorem 1.2

We have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,2,22;n)q^n &= \phi(q)^2\phi(q^2)\phi(q^{22})\\ &= \left(\phi(q^2)^2+4q\psi(q^4)^2\right)\phi(q^2)\phi(q^{22}),\end{aligned}$$

from which we extract

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,2,22;2n)q^n &= \phi(q)^3\phi(q^{11})\\ &= \left(\phi(q^4)+2q\psi(q^8)\right)^3\left(\phi(q^{44})+2q^{11}\psi(q^{88})\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n &= 6\phi(q)^2\psi(q^2)\phi(q^{11})+24q^3\phi(q)\psi(q^2)^2\psi(q^{22})\\ &= 6\phi(q)\psi(q)^2\phi(q^{11})+24q^3\psi(q)^2\psi(q^2)\psi(q^{22}),\end{aligned}$$

which implies that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n-24\sum_{n=0}^{\infty}T(1,1,2,22;n)q^{n+3}\\ &=6\phi(q)\psi(q)^2\phi(q^{11})\\ &=6\psi(q)^2\phi(-q)\phi(-q^{11})+24q^3\psi(q)^2\psi(q^2)\psi(q^{22})+24q\psi(q)^2f(-q^2)f(-q^{22}),\end{aligned}$$

and so,

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n-48q^3\sum_{n=0}^{\infty}T(1,1,2,22;n)q^{n}\\ &=6\psi(q)^2\phi(-q)\phi(-q^{11})+24q\psi(q)^2f(-q^2)f(-q^{22})\\ &=6f(-q^2)^3\phi(-q^{11})+24q\psi(q^2)f(q)^2f(-q^{22})\\ &=6\phi(-q^{11})\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n^2+n}+24f(-q^{22})\sum_{n=-\infty}^{\infty}(3n+1)(-q)^{3n^2+2n+1}.\end{aligned}$$

Since $n^2+n\equiv 0,1,2,6,8,9 \pmod{11}$ and $3n^2+2n+1\equiv 0,1,2,6,8,9 \pmod{11}$, extracting the terms involving q^n for $n\equiv 3,4,5,7,10 \pmod{11}$ in the above, we find that

$$\sum_{n=0}^{\infty} N(1,1,2,22;8n+2)q^n - 48\sum_{n=0}^{\infty} T(1,1,2,22;n)q^{n+3} = 0.$$

Thus, for $n\equiv 0,1,2,4,7 \pmod{11}$, we have

$$N(1,1,2,22;8n+26) = 48T(1,1,2,22;n).$$

5. Proof of Theorem 1.3

We have

$$\begin{aligned}\sum_{n=0}^{\infty} N(3,5,20,32;n)q^n &= \phi(q^3)\phi(q^5)\phi(q^{20})\phi(q^{32})\\ &= \left(\phi(q^{12})+2q^3\psi(q^{24})\right)\left(\phi(q^{20})+2q^5\psi(q^{40})\right)\phi(q^{20})\phi(q^{32}).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} N(3,5,20,32;4n)q^n &= \left(\phi(q^3)\phi(q^5)+4q^2\psi(q^6)\psi(q^{10})\right)\phi(q^5)\phi(q^8)\\ &= \Big(\left(\phi(q^{12})+2q^3\psi(q^{24})\right)\left(\phi(q^{20})+2q^5\psi(q^{40})\right)+4q^2\psi(q^6)\psi(q^{10})\Big)\\ &\quad\times\left(\phi(q^{20})+2q^5\psi(q^{40})\right)\phi(q^8),\end{aligned}$$

from which we extract

$$\begin{aligned}&\sum_{n=0}^{\infty} N(3,5,20,32;8n+4)q^n\\ &= 2q\phi(q^4)\phi(q^{10})^2\psi(q^{12})+4q^2\phi(q^4)\phi(q^6)\phi(q^{10})\psi(q^{20})+8q^6\phi(q^4)\psi(q^{12})\psi(q^{20})^2\\ &\quad+8q^3\psi(q^3)\psi(q^5)\phi(q^4)\psi(q^{20})\\ &= 2q\phi(q^4)\phi(q^{10})^2\psi(q^{12})+4q^2\phi(q^4)\phi(q^6)\phi(q^{10})\psi(q^{20})+8q^6\phi(q^4)\psi(q^{12})\psi(q^{20})^2\\ &\quad+8q^3\left(\psi(q^8)\phi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\phi(q^4)\psi(q^{120})\right)\phi(q^4)\psi(q^{20}).\end{aligned}$$

We further extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(3,5,20,32;16n+4)q^n \\
&= 4q\phi(q^2)\phi(q^3)\phi(q^5)\psi(q^{10}) + 8q^3\phi(q^2)\psi(q^6)\psi(q^{10})^2 + 8q^3\phi(q^2)\psi(q^{10})\psi(q)\psi(q^{15}) \\
&= 4q\phi(q^2)\psi(q^{10})\left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\phi(q^{20}) + 2q^5\psi(q^{40})\right) + 8q^3\phi(q^2)\psi(q^6)\psi(q^{10})^2 \\
&\quad + 8q^3\phi(q^2)\psi(q^{10})\left(f(q^6,q^{10}) + qf(q^2,q^{14})\right)\left(f(q^{90},q^{150}) + q^{15}f(q^{30},q^{210})\right),
\end{aligned}$$

from which we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(3,5,20,32;32n+4)q^n \\
&= 8q^2\phi(q)\psi(q^5)\phi(q^{10})\psi(q^{12}) + 8q^3\phi(q)\psi(q^5)\phi(q^6)\psi(q^{20}) \\
&\quad + 8q^2\phi(q)\psi(q^5)f(q,q^7)f(q^{45},q^{75}) + 8q^9\phi(q)\psi(q^5)f(q^3,q^5)f(q^{15},q^{105}). \qquad (5.1)
\end{aligned}$$

Next, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N(3,5,20,32;4n+1)q^n &= 2q\phi(q^3)\psi(q^{10})\phi(q^5)\phi(q^8) \\
&= 2q\left(\phi(q^{12}) + 2q^3\psi(q^{24})\right)\left(\phi(q^{20}) + 2q^5\psi(q^{40})\right)\phi(q^8)\psi(q^{10}),
\end{aligned}$$

from which it follows that

$$\sum_{n=0}^{\infty} N(3,5,20,32;8n+1)q^n = 4q^2\phi(q^4)\psi(q^5)\psi(q^{12})\phi(q^{10}) + 4q^3\phi(q^4)\psi(q^5)\phi(q^6)\psi(q^{20}). \qquad (5.2)$$

Again,

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(3,5,20,32;n)q^n \\
&= \psi(q^3)\psi(q^5)\psi(q^{20})\psi(q^{32}) \\
&= \left(\psi(q^8)\phi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\phi(q^4)\psi(q^{120})\right)\psi(q^{20})\psi(q^{32}),
\end{aligned}$$

from which we extract

$$\begin{aligned}
\sum_{n=0}^{\infty} T(3,5,20,32;2n+1)q^n &= q\psi(q)\psi(q^{15})\psi(q^{10})\psi(q^{16}) \\
&= q\big(\phi(-q^6)\phi(-q^{10}) + 2q\psi(q^8)\phi(q^{60}) + 2q^4\psi(q^2)\psi(q^{30}) \\
&\quad + 2q^{15}\phi(q^4)\psi(q^{120})\big)\psi(q^{10})\psi(q^{16}),
\end{aligned}$$

from which we further extract

$$\sum_{n=0}^{\infty} T(3,5,20,32;4n+1)q^n = q\psi(q^5)\psi(q^8)f(q,q^7)f(q^{45},q^{75}) \\ + q^8\psi(q^5)\psi(q^8)f(q^3,q^5)f(q^{15},q^{105}). \tag{5.3}$$

From (5.1), (5.2) and (5.3), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(3,5,20,32;32n+4)q^n - 4\sum_{n=0}^{\infty} N(3,5,20,32;8n+1)q^n \\
&- 32\sum_{n=0}^{\infty} T(3,5,20,32;4n+1)q^{n+2} \\
&= 8q^2\phi(q)\psi(q^5)\phi(q^{10})\psi(q^{12}) + 8q^3\phi(q)\psi(q^5)\phi(q^6)\psi(q^{20}) \\
&\quad + 8q^2\phi(q)\psi(q^5)f(q,q^7)f(q^{45},q^{75}) + 8q^9\phi(q)\psi(q^5)f(q^3,q^5)f(q^{15},q^{105}) \\
&\quad - 16q^2\phi(q^4)\psi(q^5)\psi(q^{12})\phi(q^{10}) - 16q^3\phi(q^4)\psi(q^5)\phi(q^6)\psi(q^{20}) \\
&\quad - 32q^3\psi(q^5)\psi(q^8)f(q,q^7)f(q^{45},q^{75}) - 32q^{10}\psi(q^5)\psi(q^8)f(q^3,q^5)f(q^{15},q^{105}) \\
&= 8q^2\psi(q^5)\left(\phi(q) - 2\phi(q^4)\right)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right) \\
&\quad + 8q^2\psi(q^5)\left(\phi(q^4) - 2q\psi(q^8)\right)\left(f(q,q^7)f(q^{45},q^{75}) + q^7f(q^3,q^5)f(q^{15},q^{105})\right) \\
&= -8q^2\psi(q^5)\phi(-q)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right) \\
&\quad + 8q^2\psi(q^5)\phi(-q)\left(f(q,q^7)f(q^{45},q^{75}) + q^7f(q^3,q^5)f(q^{15},q^{105})\right) \\
&= -8q^2\psi(q^5)\phi(-q)\big(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20}) - f(q,q^7)f(q^{45},q^{75}) \\
&\quad - q^7f(q^3,q^5)f(q^{15},q^{105})\big) \\
&= 0,
\end{aligned}$$

where the last equality is due to (2.15).

It follows that

$$N(3,5,20,32;32n+68) - 4N(3,5,20,32;8n+17) = 32T(3,5,20,32;4n+1).$$

This completes the proof.

6. PROOF OF THEOREM 1.4

We have

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,6,15,18;n)q^n &= \phi(q)\cdot\phi(q^{15})\cdot\phi(q^6)\phi(q^{18})\\ &= \left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)\\ &\quad\times\left(\phi(q^{24})\phi(q^{72})+2q^6\psi(q^{12})\psi(q^{36})+4q^{24}\psi(q^{48})\psi(q^{144})\right),\end{aligned}$$

from which it follows that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,6,15,18;4n)q^n\\ &=\left(\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)+4q^4\psi(q^2)\psi(q^{30})\right)\\ &\quad\times\left(\phi(q^6)\phi(q^{18})+4q^6\psi(q^{12})\psi(q^{36})\right).\end{aligned}\tag{6.1}$$

We extract from here that

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,6,15,18;8n)q^n\\ &=\left(\phi(q^2)\phi(q^{30})+4q^8\psi(q^4)\psi(q^{60})+4q^2\psi(q)\psi(q^{15})\right)\left(\phi(q^3)\phi(q^9)+4q^3\psi(q^6)\psi(q^{18})\right)\\ &=\left(\phi(q^2)\phi(q^{30})+4q^8\psi(q^4)\psi(q^{60})+4q^2\left(\psi(q^6)\psi(q^{10})+q\phi(q^{20})\psi(q^{24})\right.\right.\\ &\quad\left.\left.+q^3\phi(q^{12})\psi(q^{40})\right)\right)\left(\phi(q^{12})\phi(q^{36})+6q^3\psi(q^6)\psi(q^{18})+4q^{12}\psi(q^{24})\psi(q^{72})\right),\end{aligned}$$

from which we extract

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,6,15,18;16n)q^n\\ &=\left(\phi(q^6)\phi(q^{18})+4q^6\psi(q^{12})\psi(q^{36})\right)\left(\phi(q)\phi(q^{15})+4q^4\psi(q^2)\psi(q^{30})+4q\psi(q^3)\psi(q^5)\right)\\ &\quad+24q^3\psi(q^3)\psi(q^9)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right)\\ &=\left(\phi(q^6)\phi(q^{18})+4q^6\psi(q^{12})\psi(q^{36})\right)\Bigg(\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{60})+2q^{15}\psi(q^{120})\right)\\ &\quad+4q^4\psi(q^2)\psi(q^{30})+4q\left(\psi(q^8)\phi(q^{60})+q^3\psi(q^2)\psi(q^{30})+q^{14}\phi(q^4)\psi(q^{120})\right)\Bigg)\\ &\quad+24q^3\left(\psi(q^{12})\phi(q^{18})+q^3\phi(q^6)\psi(q^{36})\right)\left(\phi(q^{10})\psi(q^{12})+q\phi(q^6)\psi(q^{20})\right),\end{aligned}$$

from which we further extract

$$\begin{aligned}&\sum_{n=0}^{\infty} N(1,6,15,18;32n+16)q^n\\ &=6\phi(q^3)\phi(q^9)\psi(q^4)\psi(q^{30})+6q^7\phi(q^3)\phi(q^9)\phi(q^2)\psi(q^{60})+24q^3\psi(q^6)\psi(q^{18})\psi(q^4)\psi(q^{30})\\ &\quad+24q^{10}\psi(q^6)\psi(q^{18})\phi(q^2)\psi(q^{60})+24q\phi(q^5)\psi(q^6)^2\phi(q^9)+24q^3\phi(q^3)^2\psi(q^{10})\psi(q^{18}).\end{aligned}\tag{6.2}$$

From (6.1), we also extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,6,15,18;8n+4)q^n \\
&= 2\phi(q^3)\phi(q^9)\psi(q^4)\psi(q^{30}) + 2q^7\phi(q^3)\phi(q^9)\phi(q^2)\psi(q^{60}) \\
&\quad + 8q^3\psi(q^6)\psi(q^{18})\psi(q^4)\psi(q^{30}) + 8q^{10}\psi(q^6)\psi(q^{18})\phi(q^2)\psi(q^{60}).
\end{aligned} \tag{6.3}$$

Again,

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,6,15,18;n)q^n \\
&= \psi(q)\psi(q^{15}) \cdot \psi(q^6)\psi(q^{18}) \\
&= \left(\psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24}) + q^3\phi(q^{12})\psi(q^{40})\right)\psi(q^6)\psi(q^{18}),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,6,15,18;2n+1)q^n \\
&= \psi(q^3)\psi(q^9)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right) \\
&= \left(\psi(q^{12})\phi(q^{18}) + q^3\phi(q^6)\psi(q^{36})\right)\left(\phi(q^{10})\psi(q^{12}) + q\phi(q^6)\psi(q^{20})\right),
\end{aligned}$$

from which it follows that

$$\sum_{n=0}^{\infty} T(1,6,15,18;4n+1)q^n = \psi(q^6)^2\phi(q^9)\phi(q^5) + q^2\phi(q^3)^2\psi(q^{18})\psi(q^{10}). \tag{6.4}$$

From (6.2), (6.3) and (6.4), we have

$$\begin{aligned}
&2\sum_{n=0}^{\infty} N(1,6,15,18;32n+16)q^n - 6\sum_{n=0}^{\infty} N(1,6,15,18;8n+4)q^n \\
&= 48\sum_{n=0}^{\infty} T(1,6,15,18;4n+1)q^{n+1},
\end{aligned}$$

and hence,

$$N(1,6,15,18;32n+48) - 3N(1,6,15,18;8n+12) = 24T(1,6,15,18;4n+1).$$

Thus we complete the proof.

7. Proof of Theorem 1.5

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;n)q^n \\
&= \phi(q)\phi(q^7)\phi(q^{10})\phi(q^{30}) \\
&= \left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{28})+2q^7\psi(q^{56})\right) \\
&\quad \times \left(\phi(q^{40})\phi(q^{120})+2q^{10}\psi(q^{20})\psi(q^{60})+4q^{40}\psi(q^{80})\psi(q^{240})\right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;4n)q^n \\
&= \left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q)\phi(q^7)+4q^2\psi(q^2)\psi(q^{14})\right) \\
&= \left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right) \\
&\quad \times \left(\left(\phi(q^4)+2q\psi(q^8)\right)\left(\phi(q^{28})+2q^7\psi(q^{56})\right)+4q^2\psi(q^2)\psi(q^{14})\right),
\end{aligned} \tag{7.1}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;8n)q^n \\
&= \left(\phi(q^5)\phi(q^{15})+4q^5\psi(q^{10})\psi(q^{30})\right)\left(\phi(q^2)\phi(q^{14})+4q^4\psi(q^4)\psi(q^{28})+4q\psi(q)\psi(q^7)\right) \\
&= \left(\phi(q^{20})\phi(q^{60})+6q^5\psi(q^{10})\psi(q^{30})+4q^{20}\psi(q^{40})\psi(q^{120})\right) \\
&\quad \times \left(\phi(q^2)\phi(q^{14})+4q^4\psi(q^4)\psi(q^{28})+4q\psi(q^8)\phi(q^{28})+4q^2\psi(q^2)\psi(q^{14})\right. \\
&\quad \left.+4q^7\phi(q^4)\psi(q^{56})\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;16n)q^n \\
&= \left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q)\phi(q^7)+4q^2\psi(q^2)\psi(q^{14})+4q\psi(q)\psi(q^7)\right) \\
&\quad +6\psi(q^5)\psi(q^{15})\left(4q^3\psi(q^4)\phi(q^{14})+4q^6\phi(q^2)\psi(q^{28})\right) \\
&= \left(\phi(q^{10})\phi(q^{30})+4q^{10}\psi(q^{20})\psi(q^{60})\right)\left(\phi(q^4)\phi(q^{28})+6q\psi(q^8)\phi(q^{28})\right. \\
&\quad \left.+6q^7\phi(q^4)\psi(q^{56})+4q^8\psi(q^8)\psi(q^{56})+8q^2\psi(q^2)\psi(q^{14})\right) \\
&\quad +24q^3\left(\psi(q^{20})\phi(q^{30})+q^5\phi(q^{10})\psi(q^{60})\right)\left(\psi(q^4)\phi(q^{14})+q^3\phi(q^2)\psi(q^{28})\right),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;32n+16)q^n \\
&= 6\phi(q^5)\psi(q^4)\phi(q^{14})\phi(q^{15}) + 6q^3\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{28}) \\
&\quad + 24q^5\psi(q^4)\psi(q^{10})\phi(q^{14})\psi(q^{30}) + 24q^8\phi(q^2)\psi(q^{10})\psi(q^{28})\psi(q^{30}) \\
&\quad + 24q\psi(q^2)\phi(q^7)\psi(q^{10})\phi(q^{15}) + 24q^5\phi(q)\phi(q^5)\psi(q^{14})\psi(q^{30}).
\end{aligned} \tag{7.2}$$

From (7.1), we also extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,7,10,30;8n+4)q^n \\
&= 2\psi(q^4)\phi(q^{14})\phi(q^5)\phi(q^{15}) + 2q^3\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{28}) \\
&\quad + 8q^5\psi(q^4)\psi(q^{10})\phi(q^{14})\psi(q^{30}) + 8q^8\phi(q^2)\psi(q^{10})\psi(q^{28})\psi(q^{30}).
\end{aligned} \tag{7.3}$$

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,7,10,30;n)q^n \\
&= \psi(q)\psi(q^7)\psi(q^{10})\psi(q^{30}) \\
&= \left(\psi(q^8)\phi(q^{28}) + q\psi(q^2)\psi(q^{14}) + q^6\phi(q^4)\psi(q^{56})\right)\left(\psi(q^4)\phi(q^6) + q\phi(q^2)\psi(q^{12})\right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} T(1,7,10,30;4n)q^n \\
&= \psi(q^2)\phi(q^7)\psi(q^{10})\phi(q^{15}) + q^4\phi(q)\phi(q^5)\psi(q^{14})\psi(q^{30}).
\end{aligned} \tag{7.4}$$

From (7.2), (7.3) and (7.4), we have

$$\begin{aligned}
&2\sum_{n=0}^{\infty} N(1,7,10,30;32n+16)q^n - 6\sum_{n=0}^{\infty} N(1,7,10,30;8n+4)q^n \\
&= 48\sum_{n=0}^{\infty} T(1,7,10,30;4n)q^{n+1},
\end{aligned}$$

which implies that

$$N(1,7,10,30;32n+48) - 3N(1,7,10,30;8n+12) = 24T(1,7,10,30;4n).$$

Thus we finish the proof.

8. Proof of Theorem 1.6

We have

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;n)q^n \\
&= \phi(q)\phi(q^{15}) \cdot \phi(q^{10})\phi(q^{30}) \\
&= \left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{60}) + 2q^{15}\psi(q^{120})\right) \\
&\quad \times \left(\phi(q^{40})\phi(q^{120}) + 2q^{10}\psi(q^{20})\psi(q^{60}) + 4q^{40}\psi(q^{80})\psi(q^{240})\right),
\end{aligned}$$

from which we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;4n)q^n \\
&= \left(\phi(q)\phi(q^{15}) + 4q^4\psi(q^2)\psi(q^{30})\right)\left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right) \\
&= \left(\left(\phi(q^4) + 2q\psi(q^8)\right)\left(\phi(q^{60}) + 2q^{15}\psi(q^{120})\right) + 4q^4\psi(q^2)\psi(q^{30})\right) \\
&\quad \times \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right),
\end{aligned} \tag{8.1}$$

from which it follows that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;8n)q^n \\
&= \left(\phi(q^2)\phi(q^{30}) + 4q^8\psi(q^4)\psi(q^{60}) + 4q^2\psi(q)\psi(q^{15})\right) \\
&\quad \times \left(\phi(q^5)\phi(q^{15}) + 4q^5\psi(q^{10})\psi(q^{30})\right) \\
&= \left(\phi(q^2)\phi(q^{30}) + 4q^8\psi(q^4)\psi(q^{60}) + 4q^2\psi(q^6)\psi(q^{10})\right. \\
&\quad \left. + 4q^3\phi(q^{20})\psi(q^{24}) + 4q^5\phi(q^{12})\psi(q^{40})\right) \\
&\quad \times \left(\phi(q^{20})\phi(q^{60}) + 6q^5\psi(q^{10})\psi(q^{30}) + 4q^{20}\psi(q^{40})\psi(q^{120})\right).
\end{aligned}$$

From the above we extract

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;16n)q^n \\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right) \\
&\quad \times \left(\phi(q)\phi(q^{15}) + 4q^4\psi(q^2)\psi(q^{30}) + 4q\psi(q^3)\psi(q^5)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + 6q^2\psi(q^5)\psi(q^{15})\left(4q^2\phi(q^{10})\psi(q^{12}) + 4q^3\phi(q^6)\psi(q^{20})\right)\\
&= \left(\phi(q^{10})\phi(q^{30}) + 4q^{10}\psi(q^{20})\psi(q^{60})\right)\\
&\quad \times \left(\phi(-q^6)\phi(-q^{10}) + 6q\psi(q^8)\phi(q^{60}) + 6q^4\psi(q^2)\psi(q^{30}) + 6q^{15}\phi(q^4)\psi(q^{120})\right.\\
&\quad \left. + 4q^4\psi(q^2)\psi(q^{30})\right) + 6q^2\left(\psi(q^{20})\phi(q^{30}) + q^5\phi(q^{10})\psi(q^{60})\right)\\
&\quad \times \left(4q^2\phi(q^{10})\psi(q^{12}) + 4q^3\phi(q^6)\psi(q^{20})\right),
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;32n+16)q^n\\
&= 6\psi(q^4)\phi(q^5)\phi(q^{15})\phi(q^{30}) + 6q^7\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{60})\\
&\quad + 24q^5\psi(q^4)\psi(q^{10})\phi(q^{30})\psi(q^{30}) + 24q^{12}\phi(q^2)\psi(q^{10})\psi(q^{30})\psi(q^{60})\\
&\quad + 24q^2\phi(q^3)\psi(q^{10})^2\phi(q^{15}) + 24q^4\phi(q^5)^2\psi(q^6)\psi(q^{30}).
\end{aligned} \tag{8.2}
$$

Now, from (8.1) we also extract

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,10,15,30;8n+4)q^n\\
&= \left(2\psi(q^4)\phi(q^{30}) + 2q^7\phi(q^2)\psi(q^{60})\right)\left(\phi(q^5)\phi(q^{15}) + 4q^5\psi(q^{10})\psi(q^{30})\right)\\
&= 2\psi(q^4)\phi(q^5)\phi(q^{15})\phi(q^{30}) + 2q^7\phi(q^2)\phi(q^5)\phi(q^{15})\psi(q^{60})\\
&\quad + 8q^5\psi(q^4)\psi(q^{10})\phi(q^{30})\psi(q^{30}) + 8q^{12}\phi(q^2)\psi(q^{10})\psi(q^{30})\psi(q^{60}).
\end{aligned} \tag{8.3}
$$

On the other hand, we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} T(1,10,15,30;n)q^n &= \psi(q)\psi(q^{15})\cdot\psi(q^{10})\psi(q^{30})\\
&= \left(\psi(q^6)\psi(q^{10}) + q\phi(q^{20})\psi(q^{24}) + q^3\phi(q^{12})\psi(q^{40})\right)\\
&\quad \times \left(\psi(q^{40})\phi(q^{60}) + q^{10}\phi(q^{20})\psi(q^{120})\right),
\end{aligned}
$$

from which we extract

$$
\sum_{n=0}^{\infty} T(1,10,15,30;4n+3)q^n = \phi(q^3)\psi(q^{10})^2\phi(q^{15}) + q^2\phi(q^5)^2\psi(q^6)\psi(q^{30}). \tag{8.4}
$$

From (8.2), (8.3) and (8.4), we have

$$\begin{aligned}&2\sum_{n=0}^{\infty}N(1,10,15,30;32n+16)q^n-6\sum_{n=0}^{\infty}N(1,10,15,30;8n+4)q^n\\&=48\sum_{n=0}^{\infty}T(1,10,15,30;4n+3)q^{n+2},\end{aligned}$$

which readily implies that

$$N(1,7,10,30;32n+80)-3N(1,7,10,30;8n+20)=24T(1,7,10,30;4n+3).$$

This completes the proof of Theorem 1.6.

9. PROOF OF THEOREM 1.7

Our proof of Theorem 1.7 is quite different from the proofs of Theorems 1.1-1.6. In fact, the method is quite similar to that of [15], which considers the ternary case of Sun's conjectures in [17].

For a quaternary quadratic form $f(x,y,z,w)$ and a positive integer n, we define

$$R(f,n)=\{(x,y,z,w)\in\mathbb{Z}^4 : f(x,y,z,w)=n\} \quad \text{and} \quad r(f,n)=|R(f,n)|.$$

At first, we state and prove a proposition.

Proposition 9.1 — For any positive integer $n\equiv 3,5 \pmod 8$, we have

$$r(x^2+2y^2+4z^2+17w^2,n)=r(2x^2+3y^2+4z^2+8w^2+2xy+2yz+2yw,n).$$

PROOF OF PROPOSITION 9.1 : Let

$$\begin{aligned}f&=f(x,y,z,w)=x^2+2y^2+4z^2+17w^2,\\g&=g(x,y,z,w)=2x^2+3y^2+4z^2+8w^2+2xy+2yz+2yw.\end{aligned}$$

First, we consider the case when n is a positive integer congruent to $3 \pmod 8$. Note that if $(x,y,z,w)\in R(f,n)$, then $x\not\equiv w \pmod 2$. Furthermore, one may easily show that if $(x,y,z,w)\in R(f,n)$ and $x\equiv 1 \pmod 2$, then $2x+2y-2z-3w\equiv 0 \pmod 4$. Since $f(-x,y,z,w)=f(x,y,z,w)$, the map $\eta_1 : R(f,n)\to R(f,n)$ defined by

$$\eta_1(x,y,z,w)=(-x,y,z,w),$$

is a well defined bijective map. Hence we have

$$
\begin{aligned}
&|\{(x,y,z,w)\in R(f,n): x\equiv 1 \ (\text{mod } 2), 2x+2y-2z-3w\equiv 0 \ (\text{mod } 8)\}| \\
&=|\{(x,y,z,w)\in R(f,n): x\equiv 1 \ (\text{mod } 2), 2x+2y-2z-3w\equiv 4 \ (\text{mod } 8)\}|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&|\{(x,y,z,w)\in R(f,n): x\equiv 1 \ (\text{mod } 2)\}| \\
&=2|\{(x,y,z,w)\in R(f,n): x\equiv 1 \ (\text{mod } 2), 2x+2y-2z-3w\equiv 0 \ (\text{mod } 8)\}|.
\end{aligned}
$$

Note that if $(x,y,z,w)\in R(f,n)$ and $x\equiv 0 \pmod 2$, then $x+6y-6z+6w\equiv 0 \pmod 4$. Since $f(x,y,z,-w)=f(x,y,z,w)$, the map $\eta_2: R(f,n)\to R(f,n)$ defined by

$$\eta_2(x,y,z,w)=(x,y,z,-w),$$

is a well defined bijective map. Hence we have

$$
\begin{aligned}
&|\{(x,y,z,w)\in R(f,n): x\equiv 0 \ (\text{mod } 2), x+6y-6z+6w\equiv 0 \ (\text{mod } 8)\}| \\
&=|\{(x,y,z,w)\in R(f,n): x\equiv 0 \ (\text{mod } 2), x+6y-6z+6w\equiv 4 \ (\text{mod } 8)\}|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&|\{(x,y,z,w)\in R(f,n): x\equiv 0 \ (\text{mod } 2)\}| \\
&=2|\{(x,y,z,w)\in R(f,n): x\equiv 0 \ (\text{mod } 2), x+6y-6z+6w\equiv 0 \ (\text{mod } 8)\}|.
\end{aligned}
$$

Now, if we define

$$
\begin{aligned}
F_1&=\{(x,y,z,w)\in R(f,n): x\equiv 1 \ (\text{mod } 2), 2x+2y-2z-3w\equiv 0 \ (\text{mod } 8)\}, \\
F_2&=\{(x,y,z,w)\in R(f,n): x\equiv 0 \ (\text{mod } 2), x+6y-6z+6w\equiv 0 \ (\text{mod } 16)\}, \\
F_3&=\{(x,y,z,w)\in R(f,n): x\equiv 0 \ (\text{mod } 2), x+6y-6z+6w\equiv 8 \ (\text{mod } 16)\},
\end{aligned}
$$

then we have

$$r(f,n)=2(|F_1|+|F_2|+|F_3|).$$

Now, we analyze the set $R(g,n)$. First, we note that $y\equiv 1 \pmod 2$ for any $(x,y,z,w)\in R(g,n)$. Since $g(x+y,-y,-z,-w)=g(x,y,z,w)$, the map $\eta_3: R(g,n)\to R(g,n)$ defined by

$$\eta_3(x,y,z,w)=(x+y,-y,-z,-w)$$

is a well defined bijective map. Therefore, we have

$$r(g,n) = 2|\{(x,y,z,w) \in R(g,n) : x \equiv 0 \pmod 2\}|.$$

One may easily check that for $(x,y,z,w) \in R(g,n)$, if $x \equiv 0 \pmod 2$, then $x - z + w \equiv 0 \pmod 4$. Furthermore, if $x - z + w \equiv 4 \pmod 8$, then $7x - 4y + 9z - w \equiv 0 \pmod 8$. Thus if we define

$$G_1 = \{(x,y,z,w) \in R(g,n) : x \equiv 0 \pmod 2,\ \ x - z + w \equiv 0 \pmod 8\},$$
$$G_2 = \left\{(x,y,z,w) \in R(g,n) : \begin{array}{r} x \equiv 0 \pmod 2,\ \ x - z + w \equiv 4 \pmod 8, \\ 7x - 4y + 9z - w \equiv 8 \pmod{16} \end{array}\right\},$$
$$G_3 = \left\{(x,y,z,w) \in R(g,n) : \begin{array}{r} x \equiv 0 \pmod 2,\ \ x - z + w \equiv 4 \pmod 8, \\ 7x - 4y + 9z - w \equiv 0 \pmod{16} \end{array}\right\},$$

then the set $\{(x,y,z,w) \in R(g,n) : x \equiv 0 \pmod 2\}$ is a disjoint union of G_1, G_2 and G_3. Hence we have

$$r(g,n) = 2(|G_1| + |G_2| + |G_3|).$$

Now, for $j = 1,2,3$, we define maps $\phi_j : G_j \to F_j$ by

$$\phi_1(x,y,z,w) = \frac{1}{8}\begin{pmatrix} 4 & 8 & -4 & -12 \\ -2 & -8 & -6 & -10 \\ -3 & 0 & -5 & 5 \\ -2 & 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$
$$\phi_2(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 2 & 24 & -2 & 18 \\ -10 & -8 & -6 & 22 \\ -3 & -4 & -13 & -11 \\ -4 & 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$
$$\phi_3(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 6 & -8 & -6 & -42 \\ 2 & -8 & 14 & 2 \\ 7 & 12 & 9 & -1 \\ 4 & 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

It is easy to check that all of them are well defined bijective maps. Therefore, we have

$$r(f,n) = 2(|F_1| + |F_2| + |F_3|) = 2(|G_1| + |G_2| + |G_3|) = r(g,n).$$

Next, we consider the case when n is a positive integer congruent to 5 (mod 8). Note that if $(x, y, z, w) \in R(f, n)$ and $x \equiv 1 \pmod 2$, then $2x + 2y - 2z + 5w \equiv 0 \pmod 4$. Since $f(-x, y, z, w) = f(x, y, z, w)$, we have

$$\begin{aligned} &|\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ \ 2x+2y-2z+5w \equiv 0 \pmod 8\}| \\ &= |\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ \ 2x+2y-2z+5w \equiv 4 \pmod 8\}|, \end{aligned}$$

which implies that

$$\begin{aligned} &|\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2\}| \\ &= 2|\{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ 2x+2y-2z+5w \equiv 0 \pmod 8\}|. \end{aligned}$$

For $(x, y, z, w) \in R(f, n)$, if $x \equiv 0 \pmod 2$, then we have $w \equiv 1 \pmod 2$ and $x + 6y - 6z + 6w \equiv 0 \pmod 4$. Since $f(x, y, z, -w) = f(x, y, z, w)$, we have

$$\begin{aligned} &|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 0 \pmod 8\}| \\ &= |\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 4 \pmod 8\}|, \end{aligned}$$

which implies that

$$\begin{aligned} &|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2\}| \\ &= 2|\{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 0 \pmod 8\}|. \end{aligned}$$

Thus, if we define

$$\begin{aligned} X_1 &= \{(x,y,z,w) \in R(f,n) : x \equiv 1 \pmod 2,\ 2x+2y-2z+5w \equiv 0 \pmod 8\}, \\ X_2 &= \{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 0 \pmod{16}\}, \\ X_3 &= \{(x,y,z,w) \in R(f,n) : x \equiv 0 \pmod 2,\ x+6y-6z+6w \equiv 8 \pmod{16}\}, \end{aligned}$$

then we have

$$r(f,n) = 2(|X_1| + |X_2| + |X_3|).$$

Now, we analyze the set $R(g, n)$. One may check the followings;

(i) if $(x, y, z, w) \in R(g, n)$ and $x \equiv 0 \pmod 2$, then $x + y + z - w \equiv 0 \pmod 4$;

(ii) if $(x, y, z, w) \in R(g, n)$ and $x \equiv 1 \pmod 2$, then $x - z + w \equiv 0 \pmod 4$.

Since $g(x+y,-y,-z,-w)=g(x,y,z,w)$, we have

$$\begin{aligned}&|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\ &=|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 0 \pmod 8\}|\end{aligned}$$

and

$$\begin{aligned}&|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 4 \pmod 8\}|\\ &=|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8\}|.\end{aligned}$$

Therefore, we have

$$\begin{aligned}r(g,n)=&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2\}|\\ =&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\ &+2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 4 \pmod 8\}|\\ =&2|\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ x+y+z-w\equiv 0 \pmod 8\}|\\ &+2|\{(x,y,z,w)\in R(g,n): x\equiv 1 \pmod 2,\ x-z+w\equiv 4 \pmod 8\}|.\end{aligned}$$

One may easily show that for $(x,y,z,w)\in R(g,n)$, if $x\equiv 1 \pmod 2$ and $x-z+w\equiv 4 \pmod 8$, then $7x-4y+9z-w\equiv 0 \pmod 8$. Thus if we define

$$\begin{aligned}Y_1&=\{(x,y,z,w)\in R(g,n): x\equiv 0 \pmod 2,\ \ x+y+z-w\equiv 0 \pmod 8\},\\ Y_2&=\left\{(x,y,z,w)\in R(g,n):\begin{array}{r} x\equiv 1 \pmod 2,\ \ x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 8 \pmod{16}\end{array}\right\},\\ Y_3&=\left\{(x,y,z,w)\in R(g,n):\begin{array}{r} x\equiv 1 \pmod 2,\ \ x-z+w\equiv 4 \pmod 8,\\ 7x-4y+9z-w\equiv 0 \pmod{16}\end{array}\right\},\end{aligned}$$

then, we have

$$r(g,n)=2(|Y_1|+|Y_2|+|Y_3|).$$

For $j=1,2,3$, if we define maps $\psi_j: Y_j\to X_j$ by

$$\psi_1(x,y,z,w)=\frac{1}{8}\begin{pmatrix}4&-4&4&12\\-2&6&6&10\\-3&-3&5&-5\\-2&-2&-2&2\end{pmatrix}\begin{pmatrix}x\\y\\z\\w\end{pmatrix},$$

$$\psi_2(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 2 & 24 & -2 & 18 \\ -10 & -8 & -6 & 22 \\ -3 & -4 & -13 & -11 \\ -4 & 0 & 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

$$\psi_3(x,y,z,w) = \frac{1}{16}\begin{pmatrix} 6 & -8 & -6 & -42 \\ 2 & -8 & 14 & 2 \\ 7 & 12 & 9 & -1 \\ 4 & 0 & -4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

then one may check that they are all bijective. Therefore, we have

$$r(f,n) = 2(|X_1| + |X_2| + |X_3|) = 2(|Y_1| + |Y_2| + |Y_3|) = r(g,n),$$

which completes the proof. □

Now we are in a position to prove Theorem 1.7.

PROOF OF THEOREM 17 : Let

$$\begin{aligned} f &= f(x,y,z,w) = x^2 + 2y^2 + 4z^2 + 17w^2, \\ g &= g(x,y,z,w) = 2x^2 + 3y^2 + 4z^2 + 8w^2 + 2xy + 2yz + 2yw, \\ h_1 &= h_1(x,y,z,w) = 2x^2 + 4y^2 + 4z^2 + 6w^2 + 2xw + 2yw + 4zw, \\ h_2 &= h_2(x,y,z,w) = x^2 + 2y^2 + 2z^2 + 9w^2 + 2zw. \end{aligned}$$

First, note that

$$\begin{aligned} &|\{(x,y,z,w) \in R(h_1, 2n+6) : w \equiv 0 \pmod 2\}| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(x,y,z,2w) = 2n+6\right\}\right| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : 2\cdot h_2(x+w, z+w, y, w) = 2n+6\right\}\right| = r(h_2, n+3). \end{aligned}$$

Next, for $(x,y,z,w) \in R(h_1, 2n+6)$, if $w \equiv 1 \pmod 2$, then $y \equiv 0 \pmod 2$. Hence we have

$$\begin{aligned} &|\{(x,y,z,w) \in R(h_1, 2n+6) : x \equiv 0 \pmod 2,\ y \equiv 0 \pmod 2,\ w \equiv 1 \pmod 2\}| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(2x, 2y, z, w) = 2n+6\right\}\right| \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : 2\cdot g(z,w,x,y) = 2n+6\right\}\right| = r(g, n+3). \end{aligned}$$

Finally, since $h_1(w-2x, 2y, z, w) = 2 \cdot g(z, w, x-w, y)$, we have,

$$\begin{aligned}
&|\{(x,y,z,w) \in R(h_1, 2n+6) : x \equiv 1 \pmod 2,\ y \equiv 0 \pmod 2,\ w \equiv 1 \pmod 2\}| \\
&= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : h_1(w-2x, 2y, z, w) = 2n+6\right\}\right| \\
&= r(g, n+3).
\end{aligned}$$

Therefore, we have

$$r(h_1, 2n+6) = r(h_2, n+3) + 2r(g, n+3), \tag{9.1}$$

for any nonnegative even integer n.

By Proposition 9.1 and (9.1), we arrive at

$$2 \cdot r(f, n+3) = r(h_1, 2n+6) - r(h_2, n+3) \ \text{ for any } n \equiv 0, 2 \pmod 8. \tag{9.2}$$

Now, if $8x^2 + y^2 + 2z^2 + 9w^2 - 4xw \equiv 0 \pmod 4$, then $y \equiv z \equiv w \pmod 2$. Since $h_1(y, x, z, -w) = 4x^2 + 2y^2 + 4z^2 + 6w^2 - 2xw - 2yw - 4zw$, we have

$$\begin{aligned}
&|\{(x,y,z,w) \in R(f, 8n+24) : x \equiv w \pmod 4\}| \\
&= r((w-4x)^2 + 2y^2 + 4z^2 + 17w^2, 8n+24) \\
&= r(8x^2 + y^2 + 2z^2 + 9w^2 - 4xw, 4n+12) \\
&= r(8x^2 + (w-2y)^2 + 2(w-2z)^2 + 9w^2 - 4xw, 4n+12) \\
&= r(4x^2 + 2y^2 + 4z^2 + 6w^2 - 2xw - 2yw - 4zw, 2n+6) \\
&= r(h_1, 2n+6).
\end{aligned}$$

Now, if $x^2 + 8y^2 + 16z^2 + 17w^2 = 8n + 24$, then $x \equiv w \equiv 0 \pmod 2$. Since $2 \cdot h_2(y, z, x, -w) = (w-2x)^2 + 2y^2 + 4z^2 + 17w^2$, we see that

$$\begin{aligned}
&|\left\{(x,y,z,w) \in \mathbb{Z}^4 : x^2 + 8y^2 + 16z^2 + 17w^2 = 8n+24\right\}| \\
&= |\left\{(x,y,z,w) \in \mathbb{Z}^4 : 4x^2 + 8y^2 + 16z^2 + 68w^2 = 8n+24\right\}| \\
&= r(x^2 + 2y^2 + 4z^2 + 17w^2, 2n+6) = r((w-2x)^2 + 2y^2 + 4z^2 + 17w^2, 2n+6) \\
&= r(h_2, n+3).
\end{aligned}$$

From these equalities and (9.2), we have

$$\begin{aligned}
2 \cdot r(f, n+3) = &|\{(x,y,z,w) \in R(f, 8n+24) : x \equiv w \pmod 4\}| \\
&- |\left\{(x,y,z,w) \in \mathbb{Z}^4 : x^2 + 8y^2 + 16z^2 + 17w^2 = 8n+24\right\}|,
\end{aligned} \tag{9.3}$$

for any $n \equiv 0, 2 \pmod 8$.

Now, as $\dfrac{x(x-1)}{2} = \dfrac{(-x+1)(-x)}{2}$, we see that

$$\begin{aligned}
&16T(1,2,4,17;n)\\
&= 16\left|\left\{(x,y,z,w) \in \mathbb{N}^4 : \frac{x(x+1)}{2} + 2\frac{y(y+1)}{2} + 4\frac{z(z+1)}{2} + 17\frac{w(w+1)}{2} = n\right\}\right|\\
&= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 : \frac{x(x+1)}{2} + 2\frac{y(y+1)}{2} + 4\frac{z(z+1)}{2} + 17\frac{w(w+1)}{2} = n\right\}\right|\\
&= |\{(x,y,z,w) \in \mathbb{Z}^4 : (2x+1)^2 + 2(2y+1)^2 + 4(2z+1)^2 + 17(2w+1)^2 = 8n+24\}|\\
&= |\{(x,y,z,w) \in R(f, 8n+24) : xyzw \equiv 1 \text{ (mod 2)}\}|.
\end{aligned}$$

Note that if $x^2 + 2y^2 + 4z^2 + 17w^2 = 8n + 24$, then

$$(x^2, 2y^2, 4z^2, 17w^2) \equiv (1,2,4,1), (0,0,0,0), (4,0,0,4), (4,0,4,0) \text{ or } (0,0,4,4) \pmod 8.$$

From this and (9.3), we may easily deduce that

$$\begin{aligned}
8T(1,2,4,17;n) =& |\{(x,y,z,w) \in R(f, 8n+24) : x \equiv w \text{ (mod 4)},\ y \equiv z \equiv 1 \text{ (mod 2)}\}|\\
=& |\{(x,y,z,w) \in R(f, 8n+24) : x \equiv w \text{ (mod 4)}\}|\\
&- |\{(x,y,z,w) \in R(f, 8n+24) : y \equiv z \equiv 0 \text{ (mod 2)}\}|\\
=& |\{(x,y,z,w) \in R(f, 8n+24) : x \equiv w \text{ (mod 4)}\}|\\
&- |\{(x,y,z,w) \in \mathbb{Z}^4 : x^2 + 8y^2 + 16z^2 + 17w^2 = 8n+24\}|\\
=& 2 \cdot r(f, n+3),
\end{aligned}$$

which is equivalent to (1.7). This completes the proof. □

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