Indian J. Pure Appl. Math., **51**(1): 39-54, March 2020

DOI: 10.1007/s13226-020-0383-y

LOCALLY COMPACT HYPERGROUPOIDS

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(*Received* 26 *May* 2018; *after final revision* 28 *October* 2018; *accepted* 13 *December* 2018)

We give a set of axioms for the notion of locally compact hypergroupoids, as an extension of both groupoids and hypergroups, and study their basic properties. We show that, adding a natural condition on the continuity of support, one of the axioms assumed by Renault on the left Haar system automatically follows. We show that an irreducible representation of a compact hypergroupoid is fiberwise finite dimensional.

Key words : Hypergroups; groupoids; hypergroupoids; Haar system; irreducible representations.

2010 Mathematics Subject Classification : 43A62, 22A22

1. Introduction

The main focus of abstract harmonic analysis is the study of topological groups. There are however more general structures which have been the subject of intensive research in recent years. Two of these structures, which are particularly interesting because of their applications, are topological *groupoids* and topological *hypergroups*.

Roughly speaking, a topological hypergroup is a topological space with a compatible algebra structure on its measure space. In a hypergroup K, the set-valued continuous mapping $(x, y) \mapsto$ supp$(\delta_x * \delta_y)$ from $K \times K$ into the family of non-empty compact subsets of K is a replacement for the group multiplication. There are important examples of hypergroups, including double-coset hypergroups, polynomial hypergroups, and orbit hypergroups. For precise definition and basic properties of hypergroups as well as more examples, we refer to the monograph [3].

On the other hand, groupoids, first studied by Brandt in 1927, play a key role in harmonic analysis and mathematical physics. In algebraic geometry, groupoids were used by Grothendieck to investigate moduli spaces. In Crystallography, they are used to study microscopic symmetry via screw operators [9]. Shortly, a groupoid G is a small category with invertible morphisms. We refer to the monograph [9] for more details.

Renault in [12] introduced hypergroupoids, as an extension of both hypergroups and groupoids, and briefly studied the C^*-algebras of hypergroupoids (see also [11]). In his definition, he follows the axioms given by [7] (see also [13]). One limitation of this definition is that the continuity of the mapping $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ is not assumed in [12]. Adding this condition, and using an idea of Jewett [6], one could omit one of the axioms in the definition of the left Haar system given in [12, Definition 4.3] (c.f. Theorem 2.9). In Section 2, we introduce and study *hypergroupoids*, with the continuity of support added. We show how to get some groupoid structures from a hypergroupoid. In Section 3, as an extension of the hypergroup case [14], we show that every irreducible representation of a compact hypergroupoid is finite dimensional (see also [5, Theorem 5.2] for the group case). Section 4 is devoted to examples of hypergroupoids.

2. Locally Compact Hypergroupoids

For a locally compact Hausdorff space X, the space of all complex Radon measures of X is denoted by $\mathcal{M}(X)$, and the set of all positive compactly supported measures in $\mathcal{M}(X)$ is denoted by $\mathcal{M}_c^+(X)$. We denote the support of a measure $\mu \in \mathcal{M}(X)$ by $\text{supp}(\mu)$, and the Dirac mass at $x \in X$ by δ_x.

Also $C_c(X)$ denotes the space of compactly supported continuous functions on X. Unlike where groupoids are not necessarily Hausdorff, the theory of hypergroupoids in is restricted to the Hausdorff and second countable case. It is however noted that most of the theory goes through in the non-Hausdorff case. Here we also work in the Hausdorff case.

Definition 2.1 — Let H be a locally compact Hausdorff space and $H^{(0)} \subseteq H$. Let r and s be continuous open mappings from H onto $H^{(0)}$ such that $r(u) = s(u) = u$, for all $u \in H^{(0)}$. Let $x \mapsto x^-$ be a continuous map from H onto H such that $(x^-)^- = x$ and $r(x^-) = s(x)$, for all $x \in H$. We write

$$H^{(2)} := \{(x, y) \in H \times H : r(y) = s(x)\},$$

and

$$H^{(3)} := \{(x, y, z) \in H \times H \times H : (x, y), (y, z) \in H^{(2)}\},$$

and

$$H^u := \{x \in H : r(x) = u\}, \quad H_u := \{x \in H : s(x) = u\} \quad \text{and} \quad H_u^v := H_u \cap H^v,$$

for $u, v \in H^{(0)}$. Let $(x, y) \mapsto \delta_x * \delta_y$ be a continuous map from $H^{(2)}$ into $\mathcal{M}(H)$ such that

(1) for each $(x, y) \in H^{(2)}$, $\delta_x * \delta_y$ is a probability measure with compact support and $\mathrm{supp}(\delta_x * \delta_y) \subseteq H_{s(y)}^{r(x)}$;

(2) for each $(x, y, z) \in H^{(3)}$,

$$\int_H (\delta_x * \delta_t)\, d(\delta_y * \delta_z)(t) = \int_H (\delta_t * \delta_z)\, d(\delta_x * \delta_y)(t);$$

(3) for each $x \in H$, $\delta_{r(x)} * \delta_x = \delta_x = \delta_x * \delta_{s(x)}$;

(4) the mapping $(x, y) \mapsto$ supp$(\delta_x * \delta_y)$, from $H^{(2)}$ to $\mathscr{C}(H)$ is continuous, where $\mathscr{C}(H)$ is the family of all nonempty compact subsets of H equipped with Michael topology (see [8] and [6]);

(5) for each $(x, y) \in H^{(2)}$, $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$, i.e.

$$\int_H f(t^-)\, d(\delta_x * \delta_y)(t) = \int_H f(t)\, d(\delta_{y^-} * \delta_{x^-})(t) \qquad (f \in C_c(H));$$

(6) for each $\varepsilon > 0$ and $f \in C_c(H)$, there exists an open set U in H containing $H^{(0)}$ such that for all $(x, y) \in H^{(2)}$, if $(\delta_x * \delta_y)(U) > 0$, then $|f(x^-) - f(y)| < \varepsilon$;

(7) for each $x \in H$ we have $L_x\left(C_c(H^{s(x)})\right) \subseteq C_c(H^{r(x)})$, where for each $f \in C_c(H^{s(x)})$,

$$(L_x(f))(y) = f(x^- * y) := \int_H f\, d(\delta_{x^-} * \delta_y), \qquad \left(y \in H^{r(x)}\right).$$

Then $(H, H^{(0)}, r, s, ^-, *)$ (or simply, H) is called a *locally compact hypergroupoid*.

Locally compact Hausdorff groupoids and locally compact hypergroups are hypergroupoids. Any disjoint union of discrete hypergroupoids has also hypergroupoids structure. In the last section, we give some concrete examples of hypergroupoids.

A hypergroupoid H is called *principal* (*transitive*, respectively) if the mapping $x \mapsto (r(x), s(x))$ from H into $H^{(0)} \times H^{(0)}$ is injective (surjective, respectively).

Hausdorff groupoids are exactly the principal hypergroupoids: Let H be a principal hypergroupoid. For each $(x,y) \in H^{(2)}$ there exists a unique element $z \in H$ such that $s(z) = s(y)$ and $r(z) = r(x)$. Taking $xy := z$, one can easily see that H is a groupoid. The converse is trivial. Locally compact hypergroups are hypergroupoids with singleton unit space. More generally, hypergroups could be extracted as the isotropy spaces of hypergroupoids.

Lemma 2.2 — Let H be a hypergroupoid and $\mu, \nu \in M_c^+(H)$ such that $\mathrm{supp}(\mu) \times \mathrm{supp}(\nu) \subseteq H^{(2)}$. Then,

$$\text{supp}(\mu * \nu) \subseteq \text{supp}(\mu) * \text{supp}(\nu).$$

PROOF : Since $\mathrm{supp}(\mu) \times \mathrm{supp}(\nu) \subseteq H^{(2)}$, by axiom (1) of Definition 2.1 we have,

$$\Omega := \{\text{supp}(\delta_x * \delta_y) : (x,y) \in \mathrm{supp}(\mu) \times \mathrm{supp}(\nu)\} \subseteq \mathscr{C}(H).$$

Since $(x,y) \mapsto \mathrm{supp}(\delta_x * \delta_y)$ is a continuous mapping from $H^{(2)}$ into $\mathscr{C}(H)$ and $\mathrm{supp}(\mu) \times \mathrm{supp}(\nu)$ is compact, Ω is a compact subset of $\mathscr{C}(H)$, and so by [8, Theorem 2.5],

$$\text{supp}(\mu) * \text{supp}(\nu) = \bigcup_{A \in \Omega} A,$$

is compact (see also [6, 2.5F]). Now, we show that

$$\text{supp}(\mu * \nu) \subseteq \text{supp}(\mu) * \text{supp}(\nu).$$

Suppose that $t \in$ supp$(\mu * \nu)$ and $t \notin$ supp$(\mu) *$ supp$(\nu) =: V$. Then, there is an open neighborhood U of t such that

$$U \cap V = \bigcup \Big\{ U \bigcap \mathrm{supp}(\delta_x * \delta_y) : (x,y) \in \mathrm{supp}(\mu) \times \mathrm{supp}(\nu) \Big\} = \varnothing,$$

and since $U \bigcap$ supp$(\mu * \nu) \neq \varnothing$, there is a function $f \in C_c^+(H)$ such that $\mathrm{supp} f \subseteq U$ and $0 < \mu * \nu(f)$. This is a contradiction as,

$$\mu * \nu(f) = \int \int (\delta_x * \delta_y)(f) d\mu(x) d\nu(y) = 0,$$

since for each $(x,y) \in \mathrm{supp}(\mu) \times \mathrm{supp}(\nu)$, $\mathrm{supp}(f) \bigcap \mathrm{supp}(\delta_x * \delta_y) = \varnothing$. □

In the next result, the notion of semi-groupoid is the natural relaxation of the axioms on the existence of inverse, just as semigroups generalize the notion of groups.

***Theorem* 2.3** — *Let $(H, H^{(0)}, *)$ be a hypergroupoid. Put $\mathcal{H} := M_c^+(H)$ and*

$$\mathcal{H}^{(2)} := \{(\mu,\nu) \in \mathcal{H} \times \mathcal{H} : \mathrm{supp}(\mu) \times \mathrm{supp}(\nu) \subseteq H^{(2)}\}.$$

Then, $(\mathcal{H}, \mathcal{H}^{(2)})$ *is a semi-groupoid with the convolution*

$$\mu * \nu := \int_H \int_H (\delta_x * \delta_y)\, d\mu(x) d\nu(y) \qquad ((\mu, \nu) \in \mathcal{H}^{(2)}),$$

and the involution $\mu \longmapsto \mu^-$, *where*

$$\mu^-(f) := \int_H f(x^-)\, d\mu(x), \qquad (f \in C_c(H)).$$

PROOF : Let $(\mu, \nu, \kappa) \in \mathcal{H}^{(3)}$. Then, $(\mu, \nu), (\nu, \kappa) \in \mathcal{H}^{(2)}$ and we have supp$(\mu) \times$ supp$(\nu) \times$ supp$(\kappa) \subseteq H^{(3)}$. Now, if $(x, y) \in \mathrm{supp}(\mu) \times \mathrm{supp}(\nu * \kappa)$, then by Lemma 2.2, there are $y_1 \in \mathrm{supp}(\nu)$ and $y_2 \in \mathrm{supp}(\kappa)$ such that $y \in \mathrm{supp}(\delta_{y_1} * \delta_{y_2})$ and $r(y) = r(y_1)$ and $s(y) = s(y_2)$. On the other hand, $y_1 \in \mathrm{supp}(\nu)$ implies that $r(y_1) = s(x)$ and $r(y) = s(x)$. This shows that $(x, y) \in H^{(2)}$, thus $(\mu, \nu * \kappa) \in \mathcal{H}^{(2)}$. Similarly, $(\mu * \nu, \kappa) \in \mathcal{H}^{(2)}$. Next, for each $(\mu_1, \mu_2) \in M(H)^{(2)}$,

$$\mu_1 * \mu_2 = \int (\mu_1 * \delta_y) d\mu_2(y).$$

Therefore,

$$\begin{aligned}
(\mu * \nu) * \kappa &= \int\int\int [(\delta_x * \delta_y) * \delta_z] d\mu(x) d\nu(y) d\kappa(z) \\
&= \int\int\int\int (\delta_t * \delta_z) d(\delta_x * \delta_y)(t) d\mu(x) d\nu(y) d\kappa(z) \\
&= \int\int\int\int (\delta_x * \delta_t) d(\delta_y * \delta_z)(t) d\mu(x) d\nu(y) d\kappa(z) \\
&= \int\int\int [\delta_x * (\delta_y * \delta_z)] d\nu(y) d\kappa(z) d\mu(x) \\
&= \int [\delta_x * (\nu * \kappa)] d\mu(x) = \mu * (\nu * \kappa). \square
\end{aligned}$$

Definition 2.4 — A left Haar system on a locally compact hypergroupoid H is a system $\{\lambda^u\}_{u \in H^{(0)}}$, where for every $u \in H^{(0)}$, λ^u is a positive Radon measure on H^u, such that for each $f, g \in C_c(H)$,

(i) $\mathrm{supp}(\lambda^u) = H^u$;

(ii) the mapping $u \mapsto \int_{H^u} f d\lambda^u$ from $H^{(0)}$ into $\mathbb{C}$ is continuous;

(iii) for each $x \in H$,
$$\int_{H^{s(x)}} f(x * y) d\lambda^{s(x)}(y) = \int_{H^{r(x)}} f(y) d\lambda^{r(x)}(y);$$

(iv) the mapping $x \mapsto \int f(x * y) g(y) d\lambda^{s(x)}(y)$ is a compact supported continuous function from H into $\mathbb{C}$, We define the subset $S(H)$ of $\mathcal{M}(H)$ by

$$S(H) := \{\mu \in \mathcal{M}_c^+(H) : \text{ for each } x, y \in \text{supp}(\mu), s(x) = s(y) \text{ and } r(x) = r(y)\}.$$

The rang and source of each $\mu \in S(H)$ are defined by $r(\mu) := \delta_{r(x)}$ and $s(\mu) := \delta_{s(x)}$, where $x \in \text{supp}(\mu)$. We put $S(H)^{(0)} := \{\delta_u : u \in H^{(0)}\}$ and $S(H)^{(2)} := \{(\mu, \nu) \in S(H) \times S(H) : s(\mu) = r(\nu)\}$. □

Proposition 2.5 — Let $(H, H^{(0)}, *)$ be a hypergroupoid. Then, with above notations,

(1) for all $\mu \in S(H)$, $(\mu, \mu^-), (\mu^-, \mu) \in S(H)^{(2)}$;

(2) if $(\mu, \nu, \kappa) \in S(H)^{(3)}$, then $(\mu * \nu, \kappa), (\mu, \nu * \kappa) \in S(H)^{(2)}$ and $(\mu * \nu) * \kappa = \mu * (\nu * \kappa)$;

(3) for all $\mu \in S(H)$, $r(\mu) * \mu = \mu * s(\mu) = \mu$.

PROOF : (1) is obvious.

(2) follows from Theorem 2.3.

(3) Let $\mu \in S(H)$. Then, for each $x \in \text{supp}(\mu)$ we have $s(\mu) = \delta_{s(x)}$ and $r(\mu) = \delta_{r(x)}$. By Definition 2.1, $\delta_x * \delta_{s(x)} = \delta_x = \delta_{r(x)} * \delta_x$, and so for every $f \in C_c(H)$,

$$\begin{aligned}(\mu * s(\mu))(f) &= \int_{\text{supp}\mu} (\delta_x * \delta_{s(x)})(f) d\mu(x) \\ &= \int_{\text{supp}\mu} \delta_x(f) d\mu(x) \\ &= \int f(x) d\mu(x) = \mu(f).\end{aligned}$$

Similarly, $r(\mu) * \mu = \mu$.

Let $\mu \in S(H)$ and $\text{supp}(\mu) \subseteq H_u^v$, for some $u, v \in H^{(0)}$. Then, for each $f \in C(H^u)$ we define $\mu * f : H^v \to \mathbb{C}$ by

$$(\mu * f)(y) := \int_{H_u^v} f(x^- * y) d\mu(x).$$

In particular, for each $x \in H$ and $f \in C(H^{s(x)})$, we have

$$(\delta_x * f) : H^{r(x)} \to \mathbb{C}, \qquad (\delta_x * f)(y) = f(x^- * y).$$

The next lemma follows immediately from the definition.

Lemma 2.6 — Let $\mu \in S(H)$, supp$(\mu) \subseteq H^v_u$ and $f \in C^+_c(H^u)$. Then,

$$\int_{H^v} (\mu * f) d\lambda^v = \int_{H^u} f d(\mu^- * \lambda^v).$$

Note that supp$\mu^- \subseteq H^u_v$ and supp$(\mu^- * \lambda^v) \subseteq H^u$.

Lemma 2.7 — Let $\{h_\alpha\}$ be a net in $C^+_c(H^v_v)$ such that $\int_{H^v_v} h_\alpha d\lambda^v = 1$, and $\mathrm{supp}(h_\alpha) \to \{v\}$ in $\mathscr{C}(H)$ with Michael topology. Then, for each $f \in C^+_c(H^u)$,

$$\lim_\alpha \|(f\lambda^u * h^-_\alpha) - f\|_{\sup} = 0,$$

in which λ^u is restricted to H^u_v.

PROOF : Let $\epsilon > 0$ and U be as in Definition (2.1)(6). Then, U is a neighborhood of v, and so $\mathscr{C}_H(U) := \{E \in \mathscr{C}(H) : E \subseteq U\}$ is an open neighborhood of $\{v\}$ in $\mathscr{C}(H)$. Since supp$(h_\alpha) \to \{v\}$, there is α_0 such that supp$(h_\alpha) \in \mathscr{C}_H(U)$, for $\alpha \geq \alpha_0$. Now, if $x, y \in H^u_v$, for $\alpha \geq \alpha_0$, $h_\alpha(x^- * y) > 0$, thus

$$\mathrm{supp}(\delta_{x^-} * \delta_y) \bigcap \mathrm{supp}(h_\alpha) \neq \varnothing,$$

and so, supp$(\delta_{x^-} * \delta_y) \bigcap U \neq \varnothing$, which means, $\mid f(x) - f(y) \mid < \epsilon$.

Since $\{\lambda^u\}$ is a left Haar system,

$$\int_{H^u_v} h_\alpha(x^- * y) d\lambda^u(y) = \int_{H^v_v} h_\alpha(y) d\lambda^v(y) = 1.$$

Thus, for $\alpha \geq \alpha_0$ and $x \in H^u_v$,

$$\begin{aligned}
\left|(f\lambda^u * h^-_\alpha)(x) - f(x)\right| &= \left| \int_{H^u_v} h^-_\alpha(y^- * x) d(f\lambda^u)(y) - f(x) \int_{H^u_v} h_\alpha(x^- * y) d\lambda^u(y) \right| \\
&\leq \int_{H^u_v} |f(y) - f(x)|\, h_\alpha(x^- * y) d\lambda^u(y) \\
&< \epsilon \int_{H^u_v} h_\alpha(x^- * y) d\lambda^u(y) = \epsilon. \square
\end{aligned}$$

Lemma 2.8 — Let $f \in C^+_c(H^u)$ and $x \in H^v_u$. Then $\delta_x * (f\lambda^u) = (\delta_x * f)\lambda^v$. In particular, for each $x \in H$ and $f \in C^+_c(H^{s(x)})$,

$$\delta_x * (f\lambda^{s(x)}) = (\delta_x * f)\lambda^{r(x)}.$$

PROOF : Let $\{h_\alpha\}$ be a net in $C_c^+(H_v^v)$ such that $\int_{H_v^v} h_\alpha d\lambda^v = 1$ and $\mathrm{supp}(h_\alpha) \to \{v\}$ in $\mathscr{C}(H)$. By Lemmas 2.7 and [6, Theorem 2.2C], for each $x \in H_u^v$ and $g \in C_c^+(H^v)$,

$$\begin{aligned}
\int_{H^v} gd[(\delta_x * f)\lambda^v] &= \int_{H^v} (\delta_x * f)d(g\lambda^v)\\
&= \int_{H^u} fd(\delta_{x^-} * g\lambda^v)\\
&= \lim_\alpha \int_{H^u} (f\lambda^u * h_\alpha)d(\delta_{x^-} * g\lambda^v)\\
&= \lim_\alpha \int_{H^v} (\delta_x * f\lambda^u * h_\alpha)d(g\lambda^v)\\
&= \lim_\alpha \int_{H^v} gd((\delta_x * f\lambda^u * h_\alpha)\lambda^v)\\
&= \int_{H^v} gd((\delta_x * f\lambda^u).\square
\end{aligned}$$

The next result shows that the condition (ii) in [12, Definition 4.3] follows from other conditions in Definition 4.3.

***Theorem* 2.9** — *Let H be a hypergroupoid with a left Haar system $\{\lambda^u\}$. Then, for each $f \in C_c^+\left(H^{r(x)}\right)$ and $g \in C_c^+\left(H^{s(x)}\right)$,*

$$\int f(x * y)g(y)d\lambda^{s(x)}(y) = \int f(y)g(x^- * y)d\lambda^{r(x)}(y).$$

PROOF : By the above lemma,

$$\begin{aligned}
\int_{H^{s(x)}} f(x * y)g(y)d\lambda^{s(x)}(y) &= \int_{H^{s(x)}} f(x * y)d(g\lambda^{s(x)})(y)\\
&= \int_{H^{s(x)}} (\delta_{x^-} * f)d(g\lambda^{s(x)})\\
&= \int_{H^{r(x)}} fd(\delta_x * g\lambda^{s(x)})\\
&= \int_{H^{r(x)}} fd[(\delta_x * g)\lambda^{r(x)}]\\
&= \int_{H^{r(x)}} f(\delta_x * g)d\lambda^{r(x)}\\
&= \int_{H^{r(x)}} f(y)g(x^- * y)d\lambda^{r(x)}(y).\square
\end{aligned}$$

3. REPRESENTATIONS OF HYPERGROUPOIDS

In this section we study representations of hypergroupoids. The goal is to show that irreducible representations are fiberwise finite dimensional in the compact case.

Let $(H, H^{(0)})$ be a hypergroupoid with a left Haar system $\lambda = \{\lambda^u\}_{u \in H^{(0)}}$, and $\{\mathcal{H}_u\}_{u \in H^{(0)}}$ be a bundle of Hilbert spaces. Let

$$\pi \in \prod_{\mu \in S(H)} B(\mathcal{H}_{u_\mu}, \mathcal{H}_{v_\mu}),$$

where, for every $\mu \in S(H)$, $s(\mu) = \delta_{u_\mu}$ and $r(\mu) = \delta_{v_\mu}$. Then π is called a *representation* of $(H, H^{(0)})$ on $\{\mathcal{H}_u\}_{u \in H^{(0)}}$ if,

(1) for each $u \in H^{(0)}$, $\pi(\delta_u)$ is the identity map on H_u;

(2) for each $(\mu, \nu) \in S(H)^{(2)}$, $\pi(\mu * \nu) = \pi(\mu)\pi(\nu)$;

(3) for each $\mu \in S(H)$, $\pi(\mu^-) = \pi(\mu)^*$;

(4) the mapping $\mu \mapsto \langle \pi(\mu)\xi, \eta \rangle$ from $S(H)$ into $\mathbb{C}$ is continuous, for $\xi \in \mathcal{H}_{u_\mu}$ and $\eta \in \mathcal{H}_{v_\nu}$.

(5) for each $\mu \in S(H)$, $\|\pi(\mu)\| \leq \|\mu\|$.

We have

$$\langle \pi(\mu)\xi, \eta \rangle = \int_{H_u^v} \langle \pi(\delta_x)\xi, \eta \rangle \, d\mu(x),$$

when $\text{supp}(\mu) \subseteq H_u^v$.

Let $(H, H^{(0)})$ be a hypergroupoid. For each $u \in H^{(0)}$, put $\mathcal{H}_u := \mathbb{C}$. Then, the mapping π defined by $\pi(\mu) := \text{id}_{\mathbb{C}}$, $\mu \in S(H)$, is a representation, called the *trivial representation*. The next lemma is immediate.

Lemma 3.1 — Let $\mu \in S(H)$ such that $\text{supp}(\mu) \subseteq H_u^v$ for some $u, v \in H^{(0)}$, and $f \in L^2(H^u)$. Then, we have $\mu * f \in L^2(H^v)$ and $\|\mu * f\|_2 \leq \|\mu\|_{\text{sup}} \|f\|_2$.

Example 3.2 : Let $\mathcal{H}_w = L^2(H^w) := L^2(H^w, \lambda^w)$, and let the mapping

$$\rho \in \prod_{\mu \in S(H)} B(\mathcal{H}_{u_\mu}, \mathcal{H}_{v_\mu}),$$

be defined by

$$\rho(\mu) : L^2(H^{u_\mu}) \longrightarrow L^2(H^{v_\mu}), \quad \rho(\mu)(f) = \mu * f,$$

in which $s(\mu) = \delta_{u_\mu}$ and $r(\mu) = \delta_{v_\mu}$. Using Theorem 2.9 and Lemma 3.1, one can show that ρ is a representation of H on $\{L^2(H^u, \lambda^u)\}_{u \in H^{(0)}}$. In fact, for each $f \in L^2(H^u, \lambda^u)$ and $g \in L^2(H^v, \lambda^v)$ we have $\langle \mu * f, g \rangle = \langle f, \mu^- * g \rangle$. The other conditions can be easily obtained.

The representation ρ is called the *left regular representation* of H. One can see that ρ is faithful, in the sense that, for each $a_1, a_2 \in H_u^v$ with $a_1 \neq a_2$, we have $\rho(\delta_{a_1}) \neq \rho(\delta_{a_2})$.

A representation π of a hypergroupoid H is called *irreducible* if for each $\mu \in S(H)$, $\pi(\mu)T_u = T_v\pi(\mu)$ implies that $T_u = c_1 I_{\mathcal{H}_u}$ and $T_v = c_2 I_{\mathcal{H}_v}$, for some $c_1, c_2 \in \mathbb{C}$, for $T_u \in B(\mathcal{H}_u)$, $T_v \in B(\mathcal{H}_v)$ and $u, v \in H^{(0)}$ with $s(\mu) = \delta_u$ and $r(\mu) = \delta_v$.

As in the group case, irreducible representations of a compact hypergroups and compact groupoids are fiberwise finite dimensional [14]. However, there are compact groupoids with infinite dimensional irreducible representations. Let $G := [0,1] \times [0,1]$ and λ be the Lebesgue measure on $[0,1]$. Then G is a compact groupoid under product $(x,y)(y,z) := (x,z)$ and Haar system $\{\lambda^u\}$, where $\lambda^u = \delta_u \times \lambda$, and G has an irreducible representation which is not finite dimensional (c.f. [1, page 85]). We show that irreducible representations of compact hypergroupoids are fiberwise finite dimensional.

***Theorem* 3.3** — *Let $(H, H^{(0)})$ be a hypergroupoid and H be compact. Then, every irreducible representation π of H on a bundle $\{\mathcal{H}_u\}_{u \in H^{(0)}}$ is fiberwise finite-dimensional, i.e., for each $u \in H^{(0)}$, $\mathcal{H}_u$ is finite dimensional.*

PROOF : For each $u, v \in H^{(0)}$, since $r^{-1}(\{v\}) = H^v$ and $s^{-1}(\{u\}) = H_u$ are closed (and so compact) subsets of H. We may suppose that $\{\lambda_u\}$ and $\{\lambda^v\}$ are normalized, where $\{\lambda^v\}$ is a left Haar system for H, and for each $u \in H^{(0)}$, λ_u is a measure on H_u defined by $\lambda_u(E) := \lambda^u(E^-)$. For simplicity, we put $\pi_x := \pi(\delta_x)$, for all $x \in H$. For each $u, v \in H^{(0)}$ and $\eta, \zeta \in \mathcal{H}_u$, we define the linear operators $T_{\eta,\zeta}$ and $S_{\eta,\zeta}$ by

$$\langle T_{\eta,\zeta}(f_1), f_2\rangle = \int_{H_u^u} \langle f_1, \pi_y \eta\rangle \overline{\langle f_2, \pi_y \zeta\rangle} d\lambda^u(y), \quad (f_1, f_2 \in \mathcal{H}_u)$$

and

$$\langle S_{\eta,\zeta}(g_1), g_2\rangle = \int_{H_u^v} \langle g_1, \pi_x \eta\rangle \overline{\langle g_2, \pi_x \zeta\rangle} d\lambda^v(x), \quad (g_1, g_2 \in \mathcal{H}_v).$$

Setting $T_1 := T_{\eta,\zeta}$ and $T_2 := S_{\eta,\zeta}$, we show that $T_2 \pi_t = \pi_t T_1$, for all $t \in H_u^v$. By Theorem 2.9, for each $f_1 \in \mathcal{H}_u$ and $g_2 \in \mathcal{H}_v$,

$$\begin{aligned}
\langle T_2 \pi_t(f_1), g_2\rangle &= \int_{H_u^v} \langle \pi_t(f_1), \pi_x \eta\rangle \overline{\langle g_2, \pi_x \zeta\rangle} d\lambda^v(x) \\
&= \int_{H_u^v} \langle f_1, \pi(\delta_{t^-} * \delta_x)\eta\rangle \overline{\langle g_2, \pi_x \zeta\rangle} d\lambda^v(x) \\
&= \int_{H_u^v} \int_H \langle f_1, \pi_z \eta\rangle \overline{\langle g_2, \pi_x \zeta\rangle}\, d(\delta_{t^-} * \delta_x)(z) d\lambda^v(x) \\
&= \int_{H_u^v} F(t^- * x) G(x) d\lambda^v(x) \\
&= \int_{H_u^u} F(x) G(t * x) d\lambda^v(x)
\end{aligned}$$

$$= \int_{H_u^u} \int_H \langle f_1, \pi_x \eta \rangle \overline{\langle g_2, \pi_z \zeta \rangle} \, d(\delta_t * \delta_x)(z) d\lambda^u(x)$$
$$= \int_{H_u^u} \langle f_1, \pi_x \eta \rangle \overline{\langle g_2, \pi(\delta_t * \delta_x)\zeta \rangle} \, d\lambda^u(x)$$
$$= \int_{H_u^u} \langle f_1, \pi_x \eta \rangle \overline{\langle \pi_{t^-}(g_2), \pi_x \zeta \rangle} d\lambda^u(x)$$
$$= \langle T_1(f_1), \pi_{t^-}(g_2) \rangle = \langle \pi_t T_1(f_1), g_2 \rangle,$$

in which, $F(z) := \langle f_1, \pi_z \eta \rangle$ and $G(x) := \overline{\langle g_2, \pi_x \zeta \rangle}$. Hence, $T_2 \pi_t = \pi_t T_1$. Since π is irreducible, there are $c_{\eta,\zeta}, d_{\eta,\zeta} \in \mathbb{C}$ such that

$$T_1 = c_{\eta,\zeta} I_{\mathcal{H}_u} \quad \text{and} \quad T_2 = d_{\eta,\zeta} I_{\mathcal{H}_v}.$$

Thus,

$$\langle T_1(f_1), f_2 \rangle = c_{\eta,\zeta} \langle f_1, f_2 \rangle.$$

Similarly, for given elements $f_1, f_2 \in \mathcal{H}_u$, if we define the linear mappings T_3, T_4 by

$$\langle T_3(\eta), \zeta \rangle = \int_{H_u^u} \langle \eta, \pi_y f_1 \rangle \overline{\langle \zeta, \pi_y f_2 \rangle} d\lambda_u(y), \quad (\eta, \zeta \in \mathcal{H}_u),$$

$$\langle T_4(g_1), g_2 \rangle = \int_{H_u^v} \langle g_1, \pi_x f_1 \rangle \overline{\langle g_2, \pi_x f_2 \rangle} d\lambda_u(x), \quad (g_1, g_2 \in \mathcal{H}_v),$$

for each $t \in H_u^v$, we get $T_4 \pi_t = \pi_t T_3$. By irreducibility of π, there are $r_{f_1,f_2}, s_{f_1,f_2} \in \mathbb{C}$ such that,

$$T_3 = r_{f_1,f_2} I_{\mathcal{H}_u} \quad \text{and} \quad T_4 = s_{f_1,f_2} I_{\mathcal{H}_v}.$$

By the invariance of left Haar system, we have

$$\langle T_1(f_1), f_2 \rangle = \int_{H_u^u} \langle \pi_{y^-}(f_1), \eta \rangle \overline{\langle \pi_{y^-}(f_2), \zeta \rangle} d\lambda^u(y)$$
$$= \overline{\int_{H_u^u} \langle \eta, \pi_{y^-}(f_1) \rangle \overline{\langle \zeta, \pi_{y^-}(f_2) \rangle} d\lambda^u(y)}$$
$$= \overline{\int_{H_u^u} \langle \eta, \pi_y(f_1) \rangle \overline{\langle \zeta, \pi_y(f_2) \rangle} d\lambda_u(y)}$$
$$= \overline{\langle T_3(\eta), \zeta \rangle}.$$

Therefore, $\langle T_1(f_1), f_2 \rangle = \overline{\langle T_3(\eta), \zeta \rangle}$, thus,

$$c_{\eta,\zeta} \langle f_1, f_2 \rangle = \langle T_1(f_1), f_2 \rangle = \overline{\langle T_3(\eta), \zeta \rangle} = \overline{r_{f_1,f_2} \langle \eta, \zeta \rangle}.$$

Hence, there is $\alpha \in \mathbb{C}$ such that, for each $\eta, \zeta \in \mathcal{H}_u$, $c_{\eta,\zeta} = \alpha\overline{\langle \eta, \zeta\rangle}$, and so,

$$\int_{H_u^u} \langle f_1, \pi_y \eta\rangle \overline{\langle f_2, \pi_y \zeta\rangle} d\lambda^u(y) = \alpha \overline{\langle \eta, \zeta\rangle}\langle f_1, f_2\rangle.$$

In the above equality, for unit vectors e and f in the Hilbert space $\mathcal{H}_u$, putting $f_1 = f_2 = f$ and $\eta = \zeta = e$,

$$\int_{H_u^u} |\langle f, \pi_x(e)\rangle|^2 \, d\lambda^u(x) = \int_{H_u^u} \langle f, \pi_y e\rangle \overline{\langle f, \pi_y e\rangle} \lambda^u(y) = \alpha \overline{\langle e, e\rangle}\langle f, f\rangle = \alpha.$$

Since $u \in H_u^u$, $|\langle \pi_u(e), e\rangle| = 1$. By the continuity of the function $x \mapsto |\langle \pi_x(e), e\rangle|$, we have $0 < \alpha$. Now let $\{e_1, e_2, ...e_n\}$ be a finite set of orthonormal vectors in $\mathcal{H}_u$. By Bessel's inequality [3],

$$\begin{aligned} n\alpha = \sum_{i=1}^{n} \alpha = \sum_{i=1}^{n} \int_{H_u^u} |\langle e_i, \pi_x e\rangle|^2 \, d\lambda^u(x) &= \int_{H_u^u} \sum_{i=1}^{n} |\langle e_i, \pi_x e\rangle|^2 \, d\lambda^u(x) \\ &\leq \int_{H_u^u} \|\pi_x(e)\|^2 d\lambda^u(x) \leq \int_{H_u^u} \|\pi_x\|^2 \|e\|^2 d\lambda^u(x) \\ &\leq \int_{H_u^u} d\lambda^u(x) = 1. \end{aligned}$$

This implies that $n \leq 1/\alpha$, and the proof is complete. □

4. EXAMPLES

In this section we give a list of concrete examples of hypergroupoids.

Example 4.1 : Let $H := \{a, b, c, e_1, e_2\}$ be equipped with the discrete topology and $H^{(0)} := \{e_1, e_2\}$. Put $s(a) := e_1, s(b) = s(c) := e_2, s(e_1) := e_1, s(e_2) := e_2$ and $r(a) = r(b) := e_2, r(c) := e_1, r(e_1) := e_1, r(e_2) := e_2$. Then, $H^{e_1} = \{e_1, c\}$, $H^{e_2} = \{a, b, e_2\}$ and $H_{e_1} = \{e_1, a\}$, $H_{e_2} = \{b, c, e_2\}$. We define the involution by

$$a^- := c,\ b^- := b,\ e_1^- := e_1,\ e_2^- := e_2,$$

and the convolution by

$$\delta_a * \delta_c := \frac{1}{2}\delta_{e_2} + \frac{1}{2}\delta_b, \quad \delta_b * \delta_a := \delta_a,$$
$$\delta_c * \delta_b := \delta_c, \quad \delta_c * \delta_a := \delta_{e_1}, \quad \delta_b * \delta_b := \delta_{e_2}.$$

Then, $(H, H^{(0)})$ is a hypergroupoid. For each positive real number α, the set of measures

$$\lambda^{e_1} := \delta_{e_1} + 2\alpha\delta_c,$$

and

$$\lambda^{e_2} := \delta_a + \alpha\delta_b + \alpha\delta_{e_2}.$$

is a left Haar system for H.

In the next example, note that in discrete case, Definition 2.1 is the same as [12, Definition 4.1].

Example 4.2 : Suppose that $(G, G^{(0)}, r, s)$ is a discrete groupoid and X is a discrete G-space with a moment map $r_X : X \longrightarrow G^{(0)}$ such that the mapping $(g, x) \mapsto (gx, x)$ from $G * X$ to $X \times X$ is proper, where

$$G * X := \{(g, x) \in G \times X : s(g) = r_X(x)\}.$$

Put

$$Y = X * X := \{(x, y) \in X \times X : r_X(x) = r_X(y)\},$$

and

$$(G, Y) := \{(g, (x, y)) : g \in G, (x, y) \in Y, s(g) = r_X(x)\},$$

and define $(g, (x, y)) \mapsto (g.x, g.y)$ from (G, Y) to Y. For each $v, w \in Y$, let $v \sim w$, if and only if, for some $g \in G$, $g.v = w$. This is an equivalence relation. Let $[x, y]$ denote the equivalence class of $(x, y) \in Y$. By [12, Theorem 4.3], the quotient space $H := Y/G$ has a hypergroupoid structure with $H^{(0)} = \{[x, x] : x \in X\}$, and the source and rang maps defined by $s([x, y]) := [y, y]$ and $r([x, y]) := [x, x]$, the involution $[x, y]^- = [y, x]$, and the convolution

$$\delta_{[x,y]} * \delta_{[y,z]} = \int_{G(y)} \delta_{[g.x,z]}\, d\nu_y(g),$$

where $x, y, z \in X$, $G(y) := \{g \in G : g.y = y\}$, and ν_y is normalized Haar measure on $G(y)$.

Let $\mathbb{F}_{23}$ be a field with 23 elements and

$$G := \{A \in M_{n\times n}(\mathbb{F}_{23}) :\ \det(A) \neq 0, n \in \mathbb{N}\},$$

equipped with the discrete topology. Then, G is a groupoid with $G^{(0)} = \{I_n : n \in \mathbb{N}\}$. Put

$$X := \bigcup_{m,n\in\mathbb{N}} M_{m\times n}(\mathbb{F}_{23}).$$

and define

$$r_X : X \longrightarrow G^{(0)}, \quad r_X(A_{m\times n}) := I_m.$$

Setting $Y := X * X$, by the above argument, the space $Y/G = \{[A_{m\times n}, B_{m\times p}] : m, n, p \in \mathbb{N}\}$ is a hypergroupoid but the mapping $([x, y], [y, z]) \mapsto [x, z]$ is not well-defined, and so Y could not be a groupoid. In fact, with

$$x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad y = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{bmatrix}, \qquad z = \begin{bmatrix} 4 \\ 6 \end{bmatrix},$$

$$x' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad y' = \begin{bmatrix} 22 & 21 & 22 \\ 1 & 2 & 1 \end{bmatrix}, \qquad z' = \begin{bmatrix} 1 \\ 10 \end{bmatrix},$$

$$g_1 = \begin{bmatrix} 3 & 21 \\ 22 & 1 \end{bmatrix}, \qquad g_2 = \begin{bmatrix} 4 & 9 \\ 7 & 20 \end{bmatrix},$$

We have $g_1.x = x'$, $g_1.y = y'$, $g_2.y = y'$ and $g_2.z = z'$. Therefore, $[x, y] = [x', y']$ and $[y, z] = [y', z']$, while $[x, z] \neq [x', z']$.

Example 4.3 : Let $\mathbb{Z}_+$ be the set of all non-negative integers equipped with the discrete topology, $\mathbb{Z}_+^\infty$ be its one-point compactification, and p be a fixed prime number. For distinct $m, n \in \mathbb{Z}_+$, put

$$\delta_m * \delta_n := \delta_{\max\{m,n\}},$$

$$\delta_n * \delta_n := \frac{1}{p^{n-1}(p-1)}\delta_0 + \sum_{k=1}^{n-1} p^{k-n}\delta_k + \frac{p-2}{p-1}\delta_n.$$

Then, $\mathbb{Z}_+$ with the above convolution and the identity mapping on $\mathbb{Z}_+$ as involution, is a hypergroup [4]. This hypergroup is the dual of a compact countable strong hypergroup introduced by Dunkl and Ramirez in [4]. The measure m defined by

$$m(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ (p-1)p^{k-1}, & \text{if } k \geq 1, \end{cases}$$

is a Haar measure for $\mathbb{Z}_+$. Let $\{p_1, p_2, \ldots\}$ be the set of all prime numbers. We denote the above hypergroup corresponding to p_n by $\mathbb{Z}_{+,p_n}^\infty$, and put $K_n := \mathbb{Z}_{+,p_n}^\infty \times \{n\}$, for $n \in \mathbb{N}$. Then, K_n's are disjoint hypergroups and so $H := \bigcup_n K_n$ is a hypergroupoid with $H^{(0)} = \{(1, n) : (1, n) = 1_{K_n}, n \in \mathbb{N}\}$ and $H^{(2)} = \{(x, y) : x, y \in K_n \text{ for some } n \in \mathbb{N}\}$. Also, a Haar system is obtained by $\{\lambda^n\}$, where λ^n is the Haar measure on K_n.

The next example is adapted form of [10, Lemma 2.2].

Example 4.4 : A separable complete metrizable topological space is called a Polish space. Let X be a Polish space and K be a hypergroup which is also a Polish space. Consider $H := X \times K \times X$ equipped with the product topology. We show that H is a hypergroupoid with

$$H^{(0)} := \{(x, e, x) : x \in X\},\ H^{(2)} = \{((x, g, y), (y, h, z)) : x, y, z \in X, g, h \in K\},$$

$$s(x,g,y)=(y,e,y),\quad r(x,h,y)=(x,e,x),\quad (x,h,y)^{-}=(y,h^{-},x),$$

and

$$(\delta_{(x,g,y)}*\delta_{(y,h,z)})(f)=\int_K f(x,t,z)\,d(\delta_g*\delta_h)(t),\quad (f\in C_c(H)).$$

First observe that since K is a hypergroup, $\delta_{(x,g,y)}*\delta_{(y,h,z)}$ is a probability measure with compact support, since for each $f\in C_c(H)$,

$$\left(\delta_{(x,g,y)}*\delta_{(y,h,z)}\right)(f)=(\delta_g*\delta_h)\,(f_0),$$

where $f_0(t):=f(x,t,z)$. Also,

$$\operatorname{supp}\left(\delta_{(x,g,y)}*\delta_{(y,h,z)}\right)\subseteq H^{(x,e,x)}_{(z,e,z)},$$

since if $(u,g_0,v)\in\operatorname{supp}\left(\delta_{(x,g,y)}*\delta_{(y,h,z)}\right)$ and $(u,g_0,v)\notin H^{(x,e,x)}_{(z,e,z)}$, then there is a non-negative continuous function f on H with $f(u,g_0,v)\neq 0$ and $f=0$ on $H^{(x,e,x)}_{(z,e,z)}$, yielding a contradiction. For $((x,g_1,y),(y,g_2,z),(z,g_3,t))\in H^{(3)}$,

$$\begin{aligned}\int_H(\delta_{(x,g_1,y)}*\delta_{(y,h,z)})&d(\delta_{(y,g_2,z)}*\delta_{(z,g_3,t)})((y,h,z))\\ &=\int_k(\delta_{g_1}*\delta_h)d(\delta_{g_2}*\delta_{g_3})(h)\\ &=\int_k(\delta_h*\delta_{g_3})d(\delta_{g_1}*\delta_{g_2})(h)\\ &=\int_H(\delta_{(y,h,z)}*\delta_{(z,g_3,t)})d(\delta_{(x,g_1,y)}*\delta_{(y,g_2,z)})(h),\end{aligned}$$

and for each $(x,g,y)\in H$, $\delta_{(x,e,x)}*\delta_{(x,g,y)}=\delta_{(x,g,y)}=\delta_{(x,g,y)}*\delta_{(y,e,y)}$.

Finally, for $((x,g_1,y),(y,g_2,z))\in H^{(2)}$,

$$\begin{aligned}(\delta_{(x,g_1,y)}*\delta_{(y,g_2,z)})^{-}(f)&=\int_K f^{-}(x,t,z)(\delta_{g_1}*\delta_{g_2})(t)\\ &=\int_K f(z,t^{-},x)(\delta_{g_1}*\delta_{g_2})(t)\\ &=\int_K f(z,t,x)(\delta_{g_2^{-}}*\delta_{g_1^{-}})(t^{-})\\ &=\int_K f(z,t,x)(\delta_{g_2^{-}}*\delta_{g_1^{-}})(t)\\ &=(\delta_{(z,g_2^{-},y)}*\delta_{(y,g_1^{-},x)})(f)\\ &=(\delta_{(y,g_2,z)^{-}}*\delta_{(x,g_1,y)^{-}})(f),\end{aligned}$$

for $f \in C_c(H)$. The mappings $((x, g, y), (y, h, z)) \mapsto (g, h)$ and $(g, h) \mapsto$ supp$(\delta_g * \delta_h)$, respectively from $H^{(2)}$ to $K \times K$ and from $K \times K$ to $\mathscr{C}(K)$, are continuous. Also, we have supp$(\delta_{(x,g,y)} * \delta_{(y,h,z)}) = \{x\} \times$ supp$(\delta_g * \delta_h) \times \{z\}$. Thus, the property (4) in Definition 2.1 holds. Let $\varepsilon > 0$ and $f \in C_c(H)$. There exist open sets $U_1 \subseteq X$ and $U_2 \subseteq K$ such that $e \in U_2$, and for each $x_1, x_2 \in U_1$ and $g, h \in U_2$, $|f(x, g, x_1) - f(x, g, x_2)| < \frac{\varepsilon}{2}$ and $|f(x, g^-, y) - f(x, h, y)| < \frac{\varepsilon}{2}$. Setting $U := U_1 \times U_2 \times U_1$, if $(\delta_{(x,g,y)} * \delta_{(y,h,z)}(U) > 0$, then

$$|f((x, g, y)^-) - f(y, h, z)| \leq |f(y, g^-, x) - f(y, h, x)| + |f(y, h, x) - f(y, h, z)| < \varepsilon.$$

which verifies the last needed condition.

ACKNOWLEDGEMENT

We would like to thank the referee(s) of this paper for a careful reading and helpful suggestions.

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