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LAZARSFELD-MUKAI BUNDLES OF RANK 2 ON A POLARIZED K3 SURFACE OF LOW GENUS

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Let X be a $K3$ surface and let H be a very ample line bundle on X of sectional genus $g \leq 9$. In this paper, we characterize the destabilizing sheaf of the Lazarsfeld-Mukai bundle $E_{C,Z}$ of rank 2 associated with a smooth curve $C \in |H|$ and a base point free divisor Z on C with $h^0(\mathcal{O}_C(Z)) = 2$, in the case where it is not H-slope stable.

Key words : $K3$ surface; line bundle; Lazarsfeld-Mukai bundle; slope stability; Brill-Noether theory.

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1. Introduction

In the past decade, several problems on the existence of linear systems on a $K3$ surface X have been studied by many people, via the construction of Lazarsfeld-Mukai bundles on X. The study of the moduli spaces of stable Lazarsfeld-Mukai bundles has some applications concerning to higher rank Brill-Noether theory of curves. For example, the restrictions to curves of stable Lazarsfeld-Mukai bundles on X give some counterexamples to the Mercat conjecture on the higher rank Clifford index of a curve on X (for example, see [4, 7] and so on).

We fix a smooth curve C on X of genus $g \geq 3$ which is not hyperelliptic. Let Z be a base point free divisor of degree d on C. Then the Lazarsfeld-Mukai bundle $E_{C,Z}$ associated with them is defined as the dual of the kernel $F_{C,Z}$ of the evaluation map

$$\mathrm{ev} : H^0(\mathcal{O}_C(Z)) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_C(Z).$$

We note that if Z forms a linear system of dimension r, then the rank of $E_{C,Z}$ is $r+1$, and moreover, if $\rho(g,r,d) = g-(r+1)(g-d+r) < 0$, then $E_{C,Z}$ is not simple. Assume that $|Z|$ is a pencil. Then Donagi and Morrison have proved the following assertion.

***Theorem* 1.1** — *([3], Lemma 4.4). Let the notations be as above. If $E_{C,Z}$ is not simple, then there exist two line bundles M and N on X and a 0-dimensional subscheme $Z^{'} \subset X$ of finite length such that*

(a) $h^0(M) \geq 2,\ h^0(N) \geq 2$;

(b) N *is base point free;*

(c) *There exists an exact sequence*

$$0 \longrightarrow M \longrightarrow E_{C,Z} \longrightarrow N \otimes \mathcal{J}_{Z'} \longrightarrow 0.$$

Moreover, if $h^0(M \otimes N^{\vee}) = 0$, then the length of $Z^{'}$ is zero. Here, $\mathcal{J}_{Z'}$ is the ideal sheaf of $Z^{'}$ in X.

We call the exact sequence as in Theorem 1.1 the Donagi-Morrison's extension. Moreover, since $E_{C,Z}$ splits in the case where $h^0(M \otimes N^{\vee}) = 0$, if $\rho(g,1,d) = 2d-g-2 < 0$ and C is very ample, $E_{C,Z}$ is not C-slope stable. Hence, conversely, it is natural and interesting to consider the problem of when $E_{C,Z}$ is not C-slope stable, in the case where $\rho(g,1,d) \geq 0$.

In this paper, we will investigate the destabilizing sheaf of a Lazarsfeld-Mukai bundle $E_{C,Z}$ on a $K3$ surface of rank two which is not C-slope stable, in the case where C is a very ample smooth curve of genus $g \leq 9$. Our main theorem is as follows.

***Theorem* 1.2** — *Let X be a K3 surface and let C be a very ample smooth curve on X of genus $g \leq 9$. Let $H = \mathcal{O}_X(C)$ and let Z be a base point free divisor on C with $h^0(\mathcal{O}_C(Z)) = 2$. If $E_{C,Z}$ is not H-slope stable, then a saturated sub-line bundle $L \subset E_{C,Z}$ (i.e., a sub-line bundle L of $E_{C,Z}$ such that $E_{C,Z}/L$ is a torsion free sheaf) with $L.H \geq g-1$ satisfies the following conditions.*

(a) $h^0(L) \geq 2$ *and* $h^0(H \otimes L^{\vee}) \geq 2$.

(b) $h^1(L) = h^1(H \otimes L^{\vee}) = 0$.

It is known that if d is the minimal gonality of a smooth curve in $|H|$, $\rho(g,1,d) < 0$, and $|Z| = g^1_d$, then there exists a sub-line bundle L of $E_{C,Z}$ which satisfies the conditions (a) and (b) in Theorem 1.2 (cf. [2], Proposition 2.3).

Our plan of this paper is as follows. In Section 2, we recall the notion of the Mumford-Takemoto stability for vector bundles on projective varieties, and recall some classical results about linear systems and line bundles on $K3$ surfaces. In Section 3, we prove our main theorem.

Notations and conventions

We work over the complex number field $\mathbb{C}$. A curve and a surface are smooth projective. For a surface Y, we denote by $|D|$ the linear system defined by a divisor D on Y, and denote $D_1 \sim D_2$ if D_1 and D_2 are linearly equivalent, for two divisors D_1 and D_2 on Y. For a curve C, we denote by K_C the canonical line bundle of C.

A regular surface (i.e., a surface X with $h^1(\mathcal{O}_X) = 0$) is called a $K3$ surface if the canonical bundle of it is trivial. For a vector bundle E, we denote by $E^\vee$ the dual of it, and denote by $\operatorname{rk} E$ the rank of E.

Remark 1.1 : Let X be a $K3$ surface of Picard number ρ. Then the Hodge index theorem implies that the signature of the Picard lattice of X is (1, $\rho - 1$). This means that any two divisors D_1 and D_2 on X with $D_1^2 > 0$ and $D_2^2 > 0$ satisfy $D_1^2 D_2^2 \leq (D_1.D_2)^2$. This form of the theorem is often used to prove our main theorem in this paper.

2. Preliminaries

In this section, we recall the definition of the Mumford-Takemoto stability (i.e., slope stability) of vector bundles with respect to a given polarization on a projective variety, and recall several basic facts about line bundles on $K3$ surfaces.

Definition 2.1 — Let X be a smooth projective variety, H be a very ample line bundle on X, and let E be a torsion free sheaf on X of rank r. Then the H-slope of E is defined as follows;

$$\mu_H(E) = \frac{c_1(E).H}{r}.$$

E is called μ_H-semistable (resp. μ_H-stable) if for any subsheaf $0 \neq F \subset E$ with $\operatorname{rk} F < \operatorname{rk} E$, we have $\mu_H(F) \leq \mu_H(E)$ (resp. $\mu_H(F) < \mu_H(E)$).

Throughout this paper, we call a μ_H-semistable (resp. μ_H-stable) bundle a H-slope semi stable (resp. H-slope stable) bundle. It is well known that for a vector bundle E on X there is a unique filtration called the *Harder-Narasimhan* (*HN for short*) *filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that E_i is locally free and E_i/E_{i-1} is a torsion free and μ_H-semistable sheaf, for $1 \leq i \leq n$, and $\mu_H(E_{i+1}/E_i) < \mu_H(E_i/E_{i-1})$, for $1 \leq i \leq n-1$. Moreover, such a filtration satisfies the following inequality

$$\mu_H(E_1) > \mu_H(E_2) > \cdots > \mu_H(E).$$

It is clear that if E is not H-slope semistable, then $n \geq 2$. Then we call the sheaf E_1 the maximal destabilizing sheaf of E. Moreover, if a vector bundle E is H-slope semistable, there exists a filtration called a *Jordan-Hölder* (*JH for short*) *filtration*

$$0 = JH_0(E) \subset JH_1(E) \subset \cdots \subset JH_m(E) = E,$$

such that $\mathrm{gr}_i(E) := JH_i(E)/JH_{i-1}(E)$ is a torsion free and μ_H-stable sheaf of slope $\mu_H(E)$ for $1 \leq i \leq m$.

From now on, we assume that X is a $K3$ surface. First of all, we recall the following assertion.

Proposition 2.1 — ([5], Lemma 3.2). Let E and Q be torsion free sheaves such that $\mathrm{rk}\, E \geq 2$. If E is globally generated off a finite number of points, $h^2(E) = 0$ and there exists a surjective morphism $\varphi : E \longrightarrow Q$, then $h^0(Q^{\vee\vee}) \geq 2$. In particular, if the rank of Q is one, then $Q^{\vee\vee}$ is a non-trivial and base point free line bundle.

It is well known that if C is a smooth curve on X of genus $g \geq 3$, and Z is a base point free divisor on C, then the Lazarsfeld-Mukai bundle $E_{C,Z}$ associated with them is globally generated off the base points of $|K_C \otimes \mathcal{O}_C(-Z)|$ (cf. [1], Proposition 2.1). In particular, if $\mathrm{rk}\, E_{C,Z} = 2$, that is, $|Z|$ is a pencil and $|K_C \otimes \mathcal{O}_C(-Z)| \neq \emptyset$, then, by Proposition 2.1, $(E_{C,Z}/L)^{\vee\vee}$ is a non-trivial and base point free line bundle, for a saturated sub-line bundle $L \subset E_{C,Z}$.

Finally, we recall the following classical results about the classification of line bundles and linear systems on $K3$ surfaces.

Remark 2.1 : Let D be a divisor on a $K3$ surface X. Then the self-intersection D^2 is an even integer. In particular, if D is a rational curve on X, then we get $D^2 = -2$, by the adjunction formula. Moreover, if D is the fixed component of a non-zero effective divisor on X, then the self-intersection of it is negative.

Proposition 2.2 — ([8], Proposition 2.6). Let L be a line bundle on a $K3$ surface X such that $|L| \neq \emptyset$. Assume that $|L|$ is base point free. Then one of the following cases occurs.

(i) $L^2 > 0$ and the general member of $|L|$ is a smooth irreducible curve of genus $\dfrac{L^2}{2} + 1$.

(ii) $L^2 = 0$ and $L \cong \mathcal{O}_X(kF)$, where $k \geq 1$ is an integer and F is a smooth curve of genus one. In this case, $h^1(L) = k - 1$ and $h^0(L) = k + 1 \geq 2$ by [3, subsection (2.1)].

We note that if C is an irreducible curve satisfying $C^2 > 0$, then the linear system $|C|$ is base point free. Hence, by Proposition 2.2, any line bundle L with $|L| \neq \emptyset$ on a $K3$ surface has no base point outside its fixed component.

At the end of this section, we recall some classical results about very ample line bundles on $K3$ surfaces. It is well known that an ample linear system on a $K3$ surface which is not very ample is hyperelliptic [8]. Hence, by the classification of hyperelliptic linear systems on $K3$ surfaces, we have the following proposition.

Proposition 2.3 — (cf. [6], and [8], Theorem 5.2). Let L be a numerically effective line bundle with $L^2 \geq 4$ on a $K3$ surface X. Then L is very ample if and only if the following conditions are satisfied.

(i) There is no irreducible curve E such that $E^2 = 0$ and $E.L = 1$ or 2.

(ii) There is no irreducible curve E such that $E^2 = 2$ and $L \cong \mathcal{O}_X(2E)$.

(iii) There is no irreducible curve E such that $E^2 = -2$ and $E.L = 0$.

3. Proof of The Main Theorem

In this section, we prove Theorem 1.2. First of all, we show the following lemmas.

Lemma 3.1 — Let X be a $K3$ surface, C be a very ample smooth curve of genus g on X, and Z be a divisor on C such that $|Z|$ is a base point free pencil. Moreover, we set $H = \mathcal{O}_X(C)$. If $|K_C \otimes \mathcal{O}_C(-Z)| = \emptyset$, then $E_{C,Z}$ is H-slope stable.

Proof : Since $|Z|$ is a pencil on C, by the assumption and the Riemann-Roch theorem, we have $\deg Z = g + 1$. Assume that $E_{C,Z}$ is not H-slope stable, and let $L \subset E_{C,Z}$ be a saturated sub-line bundle with $L.H \geq g-1$. Then $E_{C,Z}/L \cong H \otimes L^\vee \otimes \mathcal{J}_{Z'}$ for a 0-dimensional subscheme Z' in X. Since

$$L.(H \otimes L^\vee) + \text{length}\, Z' = c_2(E_{C,Z}) = \deg Z,$$

we have $L.(H \otimes L^\vee) \leq g+1$. Since $g - 1 \leq L.H \leq g + 1 + L^2$, we have $L^2 \geq -2$ and hence, we have $|L| \neq \emptyset$. By the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_C(Z))^\vee \otimes L^\vee \longrightarrow L^\vee \otimes E_{C,Z} \longrightarrow H \otimes L^\vee \otimes \mathcal{O}_C(-Z) \longrightarrow 0,$$

we have

$$h^0(H \otimes L^\vee \otimes \mathcal{O}_C(-Z)) \geq h^0(L^\vee \otimes E_{C,Z}) > 0.$$

Since $\deg(H \otimes L^\vee \otimes \mathcal{O}_C(-Z)) \geq 0$, we have

$$H.(H \otimes L^\vee) \geq \deg Z. \tag{1}$$

On the other hand, since $h^0(L) \le h^0(E_{C,Z}) = 2$, we have $L^2 \le 0$. Moreover, since $H.L \ge g-1$, we have $(H \otimes L^\vee)^2 \le L^2$. Assume that $(H \otimes L^\vee)^2 = 0$. Then, by the same reason as above, we have

$$H.(H \otimes L^\vee) = L.(H \otimes L^\vee) \le \deg Z.$$

By the inequality (1), we have $H.(H \otimes L^\vee) = \deg Z$. Since

$$h^0(H \otimes L^\vee \otimes \mathcal{O}_C(-Z)) > 0,$$

this implies that $H \otimes L^\vee|_C = \mathcal{O}_C(Z)$. Hence, we have the exact sequence

$$0 \longrightarrow L^\vee \longrightarrow H \otimes L^\vee \longrightarrow \mathcal{O}_C(Z) \longrightarrow 0. \tag{2}$$

Since $|Z|$ is a pencil, we have $h^0(H \otimes L^\vee) \le h^0(\mathcal{O}_C(Z)) = 2$. Since $(H \otimes L^\vee)^2 = 0$, we have $h^0(H \otimes L^\vee) = 2$. Since $h^1(H \otimes L^\vee) = -\chi(H \otimes L^\vee) + 2 = 0$, by using the exact sequence (2) again, we have $h^1(L) = 0$. Since $L.(H \otimes L^\vee) = \deg Z$, we have the exact sequence

$$0 \longrightarrow L \longrightarrow E_{C,Z} \longrightarrow H \otimes L^\vee \longrightarrow 0.$$

Since $L^2 = 0$, this implies that

$$h^0(E_{C,Z}) = h^0(L) + h^0(H \otimes L^\vee) = 4.$$

However, this contradicts the fact that $h^0(E_{C,Z}) = 2$. Therefore, we have $(H \otimes L^\vee)^2 \le -2$. This implies that

$$H.(H \otimes L^\vee) \le L.(H \otimes L^\vee) - 2 \le \deg Z - 2.$$

However, this contradicts the inequality (1). Hence, $E_{C,Z}$ is H-slope stable. □

Lemma 3.2 — Let the notations be as in Lemma 3.1, and assume that $|K_C \otimes \mathcal{O}_C(-Z)| \neq \emptyset$. If a line bundle L is a saturated sub-line bundle of $E_{C,Z}$ with $L.H \ge g-1$ and $L^{\otimes 2} \neq H$, then the following statements hold.

(i) $h^2(L^{\otimes 2} \otimes H^\vee) = 0$ and $\mathcal{O}_C(Z) \subset H \otimes L^\vee|_C$.

(ii) If $(H \otimes L^\vee)^2 = 0$, then:

(a) $|H \otimes L^\vee|$ is an elliptic pencil with $H \otimes L^\vee|_C \cong \mathcal{O}_C(Z)$.

(b) $E_{C,Z}/L \cong H \otimes L^\vee$.

(c) $h^1(L) = 0$.

(iii) If $(H \otimes L^\vee)^2 = 2$ and $h^1(L) \neq 0$, then:

(a) $h^1(L)=1$, $h^0(H\otimes L^\vee\otimes\mathcal{O}_C(-Z))=1$, and $E_{C,Z}/L\cong H\otimes L^\vee$.

(b) $h^1(L^\vee\otimes E_{C,Z})\geq 2$.

(c) $h^1(L^{\otimes 2}\otimes H^\vee)\neq 0$.

PROOF : (i) Since $H.L\geq g-1$, we have $H.(H\otimes L^{\vee\otimes 2})\leq 0$. Since $L^{\otimes 2}\neq H$ by the ampleness of H, we have

$$h^2(L^{\otimes 2}\otimes H^\vee)=h^0(H\otimes L^{\vee\otimes 2})=0.$$

Moreover, since, by Proposition 2.1 and the assumption, $|H\otimes L^\vee|$ is base point free, we have $L^2\geq (H\otimes L^\vee)^2\geq 0$.

Hence, we have $h^0(L)\geq 2$. Since $L\subset E_{C,Z}$, by the exact sequence

$$0\longrightarrow H^0(\mathcal{O}_C(Z))^\vee\otimes L^\vee\longrightarrow L^\vee\otimes E_{C,Z}\longrightarrow H\otimes L^\vee\otimes\mathcal{O}_C(-Z)\longrightarrow 0, \tag{3}$$

we have

$$h^0(H\otimes L^\vee\otimes\mathcal{O}_C(-Z))\geq h^0(L^\vee\otimes E_{C,Z})>0. \tag{4}$$

Hence, we have $\mathcal{O}_C(Z)\subset H\otimes L^\vee|_C$.

(ii) Since $(H\otimes L^\vee)^2=0$, we have

$$\deg(L^\vee\otimes K_C\otimes\mathcal{O}_C(-Z))=L.(H\otimes L^\vee)-\deg Z\leq 0.$$

On the other hand, by the assertion of (i), we have $\deg(L^\vee\otimes K_C\otimes\mathcal{O}_C(-Z))\geq 0$. Hence, we have $H\otimes L^\vee|_C\cong\mathcal{O}_C(Z)$. This implies that $\deg Z=(H\otimes L^\vee).L$ and hence, $E_{C,Z}/L\cong H\otimes L^\vee$. By Proposition 2.1, $H\otimes L^\vee$ is base point free and not trivial. Hence, by Proposition 2.2 (ii), there exist an elliptic pencil F on X and an integer $r\geq 1$ such that $H\otimes L^\vee\cong F^{\otimes r}$. By the exact sequence

$$0\longrightarrow L^\vee\longrightarrow H\otimes L^\vee\longrightarrow\mathcal{O}_C(Z)\longrightarrow 0, \tag{5}$$

we have

$$r+1=h^0(H\otimes L^\vee)\leq h^0(\mathcal{O}_C(Z))=2.$$

Hence, we have $r=1$. Therefore, $|H\otimes L^\vee|$ is an elliptic pencil. By the exact sequence (5), we also have $h^1(L)=0$.

(iii) Since $H\otimes L^\vee$ is base point free and $(H\otimes L^\vee)^2=2$, by the theorem of Bertini, we have $h^1(H\otimes L^\vee)=0$. Hence, by the exact sequence

$$0\longrightarrow L^\vee\longrightarrow H\otimes L^\vee\longrightarrow H\otimes L^\vee|_C\longrightarrow 0,$$

we have

$$h^0(H \otimes L^\vee|_C) = h^1(L) + h^0(H \otimes L^\vee) = h^1(L) + 3.$$

Since $|Z|$ is a pencil on C and, by the statement of (i), $\mathcal{O}_C(Z) \subset H \otimes L^\vee|_C$, we have

$$H.(H \otimes L^\vee) - \deg Z \geq h^0(H \otimes L^\vee|_C) - h^0(\mathcal{O}_C(Z)) = h^1(L) + 1. \tag{6}$$

On the other hand, since $(H \otimes L^\vee)^2 = 2$, we have

$$\deg Z \geq L.(H \otimes L^\vee) = H.(H \otimes L^\vee) - 2.$$

Since $h^1(L) \neq 0$, by the inequality (6), we have $h^1(L) = 1$. This implies that $\deg Z = L.(H \otimes L^\vee)$. Hence, we have

$$E_{C,Z}/L \cong H \otimes L^\vee. \tag{7}$$

Since

$$\deg(H \otimes L^\vee \otimes \mathcal{O}_C(-Z)) = H.(H \otimes L^\vee) - \deg Z = 2,$$

and H is not hyperelliptic, the inequality (4) implies $h^0(H \otimes L^\vee \otimes \mathcal{O}_C(-Z)) = 1$, and hence, we have $h^0(L^\vee \otimes E_{C,Z}) = 1$. By the exact sequence (3), we have

$$h^1(L^\vee \otimes E_{C,Z}) \geq h^1(H^0(\mathcal{O}_C(Z))^\vee \otimes L^\vee) = 2.$$

If $h^1(L^{\otimes 2} \otimes H^\vee) = 0$, we have $\mathrm{Ext}^1(H \otimes L^\vee, L) = 0$. Hence, by (7), we have $E_{C,Z} \cong L \oplus (H \otimes L^\vee)$. Since $L^2 \geq (H \otimes L^\vee)^2 = 2$, by Proposition 2.1 and the theorem of Bertini, we have $h^1(L) = 0$. However, this contradicts the fact that $h^1(L) = 1$. Hence, we have $h^1(L^{\otimes 2} \otimes H^\vee) \neq 0$. □

PROOF OF THEOREM 1.2 : Assume that $E_{C,Z}$ is not H-slope stable. Then, by Lemma 3.1, we have $|K_C \otimes \mathcal{O}_C(-Z)| \neq \emptyset$. Let L be a saturated sub-line bundle of $E_{C,Z}$ such that $L.H \geq g - 1$. If $L^{\otimes 2} \cong H$, by the Kodaira vanishing theorem, we have $h^1(L) = 0$, and hence, $h^0(L) \geq 2$. Therefore, we assume that $L^{\otimes 2} \neq H$. Note that, by Proposition 2.1, $H \otimes L^\vee$ is base point free, and hence, $(H \otimes L^\vee)^2 \geq 0$. If $(H \otimes L^\vee)^2 = 0$, by Lemma 3.2 (ii), we have $h^0(L) \geq 2$, $h^0(H \otimes L^\vee) \geq 2$, and $h^1(L) = h^1(H \otimes L^\vee) = 0$. Therefore, we assume that $(H \otimes L^\vee)^2 \geq 2$. By the theorem of Bertini, we have $h^1(H \otimes L^\vee) = 0$. Since $L^2 \geq (H \otimes L^\vee)^2$, it is sufficient to show that $h^1(L) = 0$.

From now on, we assume that $g \leq 9$. First of all, we consider the case where $(H \otimes L^\vee)^2 \geq 4$. Since $L^2 \geq 4$, we have

$$L.(H \otimes L^\vee) = \frac{1}{2}(H^2 - L^2 - (H \otimes L^\vee)^2) \leq g - 5 \leq 4.$$

By the Hodge index theorem, we have $g = 9$ and $L^2 = (H \otimes L^\vee)^2 = 4$. Since $(H \otimes L^{\vee \otimes 2})^2 = 0$ and $H.(H \otimes L^{\vee \otimes 2}) = 0$, by the ampleness of H, we have $H \cong L^{\otimes 2}$. Hence, we have $h^1(L) = h^1(H \otimes L^\vee) = 0$.

Next, we consider the case where $(H \otimes L^\vee)^2 = 2$. Then we note that

$$L.(H \otimes L^\vee) = \frac{1}{2}(H^2 - L^2 - 2) = g - 2 - \frac{L^2}{2}. \tag{8}$$

Hence, we have $2 \leq L^2 \leq 6$. In fact, if $L^2 \geq 8$, by the equality (8), we have $L.(H \otimes L^\vee) \leq g - 6 \leq 3$. However, this contradicts the Hodge index theorem.

Assume that $L^2 = 6$. By the equality (8), we have $L.(H \otimes L^\vee) = g - 5$. By the Hodge index theorem, we have $g = 9$ and $L.(H \otimes L^\vee) = 4$. Hence, we have $(L^{\otimes 2} \otimes H^\vee)^2 = 0$ and $H.(L^{\otimes 2} \otimes H^\vee) = 4$. Then we have $h^1(L^{\otimes 2} \otimes H^\vee) = 0$. In fact, the movable part of $|L^{\otimes 2} \otimes H^\vee|$ is not empty. We denote it by D. If $D^2 \geq 2$, then we have the contradiction

$$\sqrt{32} \leq H.D \leq H.(L^{\otimes 2} \otimes H^\vee) = 4,$$

by the ampleness of H and the Hodge index theorem. Hence, we have $D^2 = 0$. By Proposition 2.2 (ii), there exist an elliptic curve F on X and a positive integer r such that $D \sim rF$. By Proposition 2.3 (i), we have $r = 1$. Since $h^0(L^{\otimes 2} \otimes H^\vee) = h^0(\mathcal{O}_X(D)) = 2$ and $(L^{\otimes 2} \otimes H^\vee)^2 = 0$, we have $h^1(L^{\otimes 2} \otimes H^\vee) = 0$. If $h^1(L) \neq 0$, this contradicts Lemma 3.2 (iii) (c). Hence, we have $h^1(L) = 0$.

Assume that $L^2 = 4$. Since $(H \otimes L^\vee)^2 = 2$, we have $(L^{\otimes 2} \otimes H^\vee)^2 = 14 - 2g$. Since $H.(L^{\otimes 2} \otimes H^\vee) = 2$, by Proposition 2.2 and Proposition 2.3 (i), $|L^{\otimes 2} \otimes H^\vee|$ has no movable part. Therefore, we have $g = 8$ or 9. Assume that $h^1(L) \neq 0$. Note that, by Lemma 3.2 (iii) (a), we have the exact sequence

$$0 \longrightarrow L \longrightarrow E_{C,Z} \longrightarrow H \otimes L^\vee \longrightarrow 0. \tag{9}$$

If $g = 8$, since $(L^{\otimes 2} \otimes H^\vee)^2 = -2$, we have $h^0(L^{\otimes 2} \otimes H^\vee) = 1$. Hence, we obtain $h^1(L^{\otimes 2} \otimes H^\vee) = 0$. This contradicts Lemma 3.2 (iii) (c). Hence, we have $g = 9$. By Lemma 3.2 (i), we have $h^2(L^{\otimes 2} \otimes H^\vee) = 0$. Since $|L^{\otimes 2} \otimes H^\vee|$ has no movable part and $(L^{\otimes 2} \otimes H^\vee)^2 = -4$, by Lemma 3.2 (iii) (c), we have

$$h^1(L^{\otimes 2} \otimes H^\vee) = h^0(L^{\otimes 2} \otimes H^\vee) = 1.$$

Applying $\otimes L^\vee$ to the exact sequence (9), we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L^\vee \otimes E_{C,Z} \longrightarrow H \otimes L^{\vee \otimes 2} \longrightarrow 0.$$

Since $h^1(\mathcal{O}_X) = 0$, we have $h^1(L^\vee \otimes E_{C,Z}) \leq h^1(H \otimes L^{\vee \otimes 2}) = 1$. This contradicts Lemma 3.2 (iii) (b). Hence, we have $h^1(L) = 0$.

Assume that $L^2 = 2$ and $h^1(L) \neq 0$. Since $(H \otimes L^\vee)^2 = 2$, we have $(H \otimes L^{\vee \otimes 2})^2 = 10 - 2g$. Then we have $g = 8$ or 9. In fact, if $g \leq 6$, we have $(H \otimes L^{\vee \otimes 2})^2 \geq -2$. However, since $H.(H \otimes L^{\vee \otimes 2}) = 0$, by the ampleness of H and the assumption that $L^{\otimes 2} \neq H$, this case does not occur. If $g = 7$, we have $(H \otimes L^{\vee \otimes 2})^2 = -4$. Since

$$|H \otimes L^{\vee \otimes 2}| = |L^{\otimes 2} \otimes H^\vee| = \emptyset,$$

we have $h^1(L^{\otimes 2} \otimes H^\vee) = 0$. This contradicts Lemma 3.2 (iii) (c). By Lemma 3.2 (iii) (a), we have $E_{C,Z}/L \cong H \otimes L^\vee$. Hence, we get the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L^\vee \otimes E_{C,Z} \longrightarrow H \otimes L^{\vee \otimes 2} \longrightarrow 0. \tag{10}$$

Here, we note that $h^2(L^\vee \otimes E_{C,Z}) = 0$. In fact, assume that $h^2(L^\vee \otimes E_{C,Z}) \neq 0$. Since $h^0(E_{C,Z}^\vee \otimes L) \neq 0$, there exists a non-zero morphism $E_{C,Z} \longrightarrow L$. If we denote the image of it by $N \otimes \mathcal{J}_W$, where N is a line bundle on X, W is a 0-dimensional sub-scheme on X, and $\mathcal{J}_W$ is the ideal sheaf of W, we get the exact sequence

$$0 \longrightarrow H \otimes N^\vee \longrightarrow E_{C,Z} \longrightarrow N \otimes \mathcal{J}_W \longrightarrow 0. \tag{11}$$

We set $M = L \otimes N^\vee$. Then we have $(H \otimes L^\vee \otimes N^\vee).H = M.H \geq 0$. Since $h^2(N^\vee \otimes E_{C,Z}) = h^0(E_{C,Z}^\vee \otimes N) \neq 0$, by the exact sequence

$$0 \longrightarrow M \longrightarrow N^\vee \otimes E_{C,Z} \longrightarrow H \otimes L^\vee \otimes N^\vee \longrightarrow 0$$

which is given by applying $\otimes M$ to the exact sequence (10), we have $M \cong \mathcal{O}_X$. In fact, if M is not trivial, $(H \otimes L^\vee \otimes N^\vee).H = M.H > 0$. This implies that

$$h^2(M) = h^2(H \otimes L^\vee \otimes N^\vee) = 0,$$

and hence, $h^2(N^\vee \otimes E_{C,Z}) = 0$. This is a contradiction. Therefore, $N \cong L$. Hence, by the exact sequence (11) and Proposition 2,1, L is base point free. Since $L^2 = 2$, by the theorem of Bertini, we have $h^1(L) = 0$. However, this contradicts the assumption that $h^1(L) \neq 0$. Hence, by the exact sequence (10), we have

$$h^1(L^\vee \otimes E_{C,Z}) = h^1(H \otimes L^{\vee \otimes 2}) - h^2(\mathcal{O}_X) = -\chi(H \otimes L^{\vee \otimes 2}) - 1 = g - 8 \leq 1.$$

However, this contradicts Lemma 3.2 (iii) (b). Hence, we have $h^1(L) = 0$. □

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