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CHARACTERIZATIONS OF LIE HIGHER DERIVATIONS ON TRIANGULAR ALGEBRAS

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In this paper we mainly characterize Lie higher derivations on triangular algebras by local action. Let $\mathcal{T} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{smallmatrix}\right]$ be a triangular algebra over a commutative ring $\mathcal{R}$ and $\mathcal{Z}(\mathcal{T})$ be the center of $\mathcal{T}$. Under some mild conditions on $\mathcal{T}$, we prove that if a family $\Delta = \{\delta_n\}_{n=0}^{\infty}$ of $\mathcal{R}$-linear mappings on $\mathcal{T}$ satisfies the condition

$$\delta_n([X, Y]) = \sum_{i+j=n} [\delta_i(X), \delta_j(Y)]$$

for any $X, Y \in \mathcal{T}$ with $XY = 0$ (resp. $XY = P$, where P is a fixed nontrivial idempotent of $\mathcal{T}$), then there exist a higher derivation $D = \{d_n\}_{n=0}^{\infty}$ and an $\mathcal{R}$-linear mapping $\tau_n : \mathcal{T} \longrightarrow \mathcal{Z}(\mathcal{T})$ vanishing on commutators $[X, Y]$ with $XY = 0$ (resp. $XY = P$) such that

$$\delta_n(X) = d_n(X) + \tau_n(X)$$

for all $X \in \mathcal{T}$.

Key words : Lie higher derivable mapping; triangular algebra.

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1. INTRODUCTION

Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ be an algebra over $\mathcal{R}$. Suppose that $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is an $\mathcal{R}$-linear mapping from $\mathcal{A}$ into itself. We say that δ is a *derivation* of $\mathcal{A}$ if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. δ is called a *Jordan derivation* of $\mathcal{A}$ if $\delta(xy+yx) = \delta(x)y+x\delta(y)+\delta(y)x+y\delta(x)$ for all $x, y \in \mathcal{A}$ and is said to be a *Lie derivation* of $\mathcal{A}$ if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$.

Clearly, each derivation is a Jordan derivation and each derivation is also a Lie derivation. But, the converse statements are in general not true. One interesting question is to investigate conditions under which derivations, Jordan derivations or Lie derivations on noncommutative algebras and operator algebras can be completely determined by the action on some sets of the given algebra. This is actually to study the “local behaviours” of linear mappings. There is a fairly substantial literature on so-called local mappings for operator algebras, starting with the papers of Larson and Sourour [24], Kadison [23] and Crist [11] which was then followed up by a large number of authors over the past 30 years. The vast majority mimicked the ideas of the original papers: take a special type of mapping, such as a derivation, a Jordan derivation or a Lie derivation and consider the pointwise or “local” case, where a mapping is equal at certain points to a derivation, a Jordan derivation, a Lie derivation or other type of mapping, and then decide if this forces the mapping to be a global version of the special mapping, or something nearly the case, see [1-3, 15, 19-22, 27-34, 39-44].

In general there are two directions in the study of the local actions of derivations (resp. Jordan derivations/Lie derivations) on noncommutative algebras and operator algebras. One is the well-known local derivation (resp. local Jordan derivations/local Lie derivations) problem. The other is to study conditions under which derivations (resp. Jordan derivations/Lie derivations) on noncommutative algebras and operator algebras can be completely determined by the action on some sets of the involved algebra.

Given a “pointwise” condition close to the definition of a particular type of mapping (automorphism, derivation, Jordan derivation, Lie derivation etc.), the problem of characterizing mappings, especially on noncommutative algebras and operator algebras, derivable at a certain point and the problem of finding full-derivable points have been studied by several authors. In [27], Lu and Jing proved that if a linear map $\delta\colon B(X) \longrightarrow B(X)$ (where $B(X)$ is the algebra of all bounded linear operators on a Banach space X of dimension greater than 2) satisfies that $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in B(X)$ with $xy = 0$ (resp. $xy = p$, where p is a fixed nontrivial idempotent), then $\delta = d + \tau$, where d is a derivation of $B(X)$ and $\tau\colon B(X) \longrightarrow \mathbb{C}I$ is a linear mapping vanishing at commutators $[x, y]$ with $xy = 0$ (resp. $xy = p$). Ji and Qi [21] investigated the same question on triangular algebras. Let $\mathcal{R}$ be a commutative ring with identity and let $\mathcal{T} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{smallmatrix}\right]$ be a triangular algebra over $\mathcal{R}$, where $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over $\mathcal{R}$ and ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. They studied $\mathcal{R}$-linear mappings $\delta\colon \mathcal{T} \longrightarrow \mathcal{T}$ that act like Lie derivations on certain subsets of $\mathcal{T}$: $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{T}$ with $xy = 0$ (resp. $xy = p$, where $p \in \mathcal{T}$ is the standard idempotent of $\mathcal{T}$). They showed that under certain conditions on a triangular algebra $\mathcal{T}$ any such linear mapping $\delta\colon \mathcal{T} \longrightarrow \mathcal{T}$ is the sum of a derivation $d\colon \mathcal{T} \longrightarrow \mathcal{T}$ and a linear central mapping

$\tau\colon \mathcal{T} \longrightarrow \mathcal{Z}(\mathcal{T})$ vanishing at all commutators $[x, y]$ with $xy = 0$ (resp. $xy = p$).

Higher derivations are also an active subject of research in algebras which may not be associative or commutative [3-9, 13, 14, 16-18, 25, 26, 36-43]. Let us recall some basic facts related to Lie higher derivations of associative algebras. Let $\mathcal{A}$ be an unital associative algebra over a commutative ring $\mathcal{R}$. Let $\mathbb{N}$ be the set of all non-negative integers and $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ be a family of $\mathcal{R}$-linear mappings on $\mathcal{A}$ such that $\delta_0 = id_{\mathcal{A}}$. D is called:

(a) a *higher derivation* if
$$\delta_n(xy) = \sum_{i+j=n} \delta_i(x)\delta_j(y)$$
for all $x, y \in \mathcal{A}$ and for each $n \in \mathbb{N}$;

(b) a *Lie higher derivation* if
$$\delta_n([x, y]) = \sum_{i+j=n} [\delta_i(x), \delta_j(y)]$$
for all $x, y \in \mathcal{A}$ and for each $n \in \mathbb{N}$.

Note that d_1 is always a Lie derivation if $D = \{d_n\}_{n\in\mathbb{N}}$ is a Lie higher derivation. It is easy to verify that every higher derivation is a Lie higher derivation. But, the converse statement is in general not true. Similar to the local behaviours of Lie derivations, one can study the local actions of Lie higher derivations on $\mathcal{A}$. Assume that $D = \{d_n\}_{n\in\mathbb{N}}$ is a higher derivation on $\mathcal{A}$. Let us construct a sequence of $\mathcal{R}$-linear mappings
$$\delta_n = d_n + \tau_n, \tag{1}$$
where $\tau_n (n \in \mathbb{N})$ is an $\mathcal{R}$-linear mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ vanishing on all commutators $[x, y]$ of $\mathcal{A}$ with $xy = 0$ (resp. $xy = p$, where p is a fixed nontrivial idempotent in $\mathcal{A}$). Or we can write it as
$$\delta_n = i_n + \tau_n,$$
where $\{i_n\}_{n\in\mathbb{N}}$ is an inner higher derivation on $\mathcal{A}$. It is not difficult to see that $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ is a family of $\mathcal{R}$-linear mappings such that
$$\delta_n([x, y]) = \sum_{i+j=n} [\delta_i(x), \delta_j(y)] \tag{2}$$
for any $x, y \in \mathcal{A}$ with $xy = 0$ (resp. $xy = p$), but not a higher derivation if $\tau_n \neq 0$ for some $n \in \mathbb{N}$. Conversely, if a family $\Delta = \{\delta_n\}_{n\in\mathbb{N}}$ of $\mathcal{R}$-linear mappings on $\mathcal{A}$ satisfies the condition (2), is it has the decomposition form (1) ? This is the main motivation of the current work.

The major objective of this work is to investigate a family $\Delta=\{\delta_n\}_{n\in\mathbb{N}}$ of $\mathcal{R}$-linear mappings satisfying the condition (2) on a triangular algebra $\mathcal{T}=\left[\begin{smallmatrix}\mathcal{A} & \mathcal{M}\\ 0 & \mathcal{B}\end{smallmatrix}\right]$. The framework of our paper is as follows. In the second section we characterize Lie higher derivations determined by acting on zero products (Theorem 2.2). More precisely, they are a family $\Delta=\{\delta_n\}_{n\in\mathbb{N}}$ of $\mathcal{R}$-linear mappings satisfying the condition

$$\delta_n([X,Y])=\sum_{i+j=n}[\delta_i(X),\delta_j(Y)]$$

for any $X,Y\in\mathcal{T}$ with $XY=0$. In the third section we characterize Lie higher derivations determined by acting on idempotent products (Theorem 3.1). More precisely, they are a family $\Delta=\{\delta_n\}_{n\in\mathbb{N}}$ of $\mathcal{R}$-linear mappings satisfying the condition

$$\delta_n([X,Y])=\sum_{i+j=n}[\delta_i(X),\delta_j(Y)]$$

for any $X,Y\in\mathcal{T}$ with $XY=P$, where P is a fixed nontrivial idempotent of $\mathcal{T}$. Then we immediately apply the obtained results to the background of nest algebras and describe Lie higher derivations by local actions on these algebras.

2. Lie Higher Derivations Determined by Acting on Zero Products

Let us begin with the definition of triangular algebras over a commutative ring $\mathcal{R}$. Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ and $\mathcal{B}$ be two $\mathcal{R}$-algebras with units I_1, I_2, respectively. Let ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ be a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Then one can define

$$\mathcal{T}=\begin{bmatrix}\mathcal{A} & \mathcal{M}\\ O & \mathcal{B}\end{bmatrix}=\left\{\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\,\middle|\, a\in\mathcal{A}, m\in\mathcal{M}, b\in\mathcal{B}\right\}$$

to be an associative algebra under matrix-like addition and matrix-like multiplication. Then $\mathcal{T}$ is an algebra with the unit $I=\left[\begin{smallmatrix}I_1 & 0\\ 0 & I_2\end{smallmatrix}\right]$ and has a nontrivial idempotent element $P=\left[\begin{smallmatrix}I_1 & 0\\ 0 & 0\end{smallmatrix}\right]$, which is called the *standard idempotent*.

By [10, Proposition 3] we know that the center $\mathcal{Z}(\mathcal{T})$ of $\mathcal{T}$ is

$$\mathcal{Z}(\mathcal{T})=\left\{\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\,\middle|\, a\in\mathcal{A}, b\in\mathcal{B},\ am=mb \text{ for all } m\in\mathcal{M}\right\}.$$

Let us define two natural projections $\pi_{\mathcal{A}}:\mathcal{T}\longrightarrow\mathcal{A}$ and $\pi_{\mathcal{B}}:\mathcal{T}\longrightarrow\mathcal{B}$ by

$$\pi_{\mathcal{A}}\left(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\right)=a \ \text{ and }\ \pi_{\mathcal{B}}\left(\begin{bmatrix}a & m\\ 0 & b\end{bmatrix}\right)=b,$$

respectively. Then $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) \subseteq \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) \subseteq \mathcal{Z}(\mathcal{B})$, and there exists a unique algebraic homomorphism $\eta : \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) \longrightarrow \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T}))$ such that $\eta(b)m = mb$ for all $m \in \mathcal{M}, b \in \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T}))$.

The next lemma will be frequently invoked in the rest part of this paper.

Lemma 2.1 — Let $\mathcal{T} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{smallmatrix}\right]$ be a triangular algebra. If $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$, then there is unique algebraic isomorphism $\eta : \mathcal{Z}(\mathcal{B}) \longrightarrow \mathcal{Z}(\mathcal{A})$ such that $\left[\begin{smallmatrix} \eta(b) & 0 \\ 0 & b \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{T})$ for any $b \in \mathcal{Z}(\mathcal{B})$.

For reader's convenience, we give some conventional notations. $P_1 = P = \left[\begin{smallmatrix} I_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $P_2 = I - P = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I_2 \end{smallmatrix}\right]$. Let us set $\mathcal{T}_{11} = \{ \left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] \mid a \in \mathcal{A} \}$, $\mathcal{T}_{12} = \{ \left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] \mid m \in \mathcal{M} \}$, and $\mathcal{T}_{22} = \{ \left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right] \mid b \in \mathcal{B} \}$. Thus we can write $\mathcal{T} = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{22}$. In what follows, when we write a_{ij}, it implies that $a_{ij} \in \mathcal{T}_{ij}$ and the corresponding element in $\mathcal{A}, \mathcal{M}$ or $\mathcal{B}$. Note that $a_{ij}a_{kl} = 0$ whenever $j \neq k$.

***Theorem* 2.2** — *Let $\mathcal{T} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{smallmatrix}\right]$ be a triangular algebra satisfying the condition $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$. Suppose that $\Delta = \{\delta_n\}_{n=0}^{\infty}$ is a family of $\mathcal{R}$-linear mappings from $\mathcal{T}$ into itself such that*

$$\delta_n([X, Y]) = \sum_{i+j=n} (\delta_i(X)\delta_j(Y) - \delta_j(Y)\delta_i(X))$$

for all $X, Y \in \mathcal{T}$ with $XY = 0$. Then there exist a higher derivation $D = \{d_n\}_{n=0}^{\infty}$ and an $\mathcal{R}$-linear mapping $\tau_n : \mathcal{T} \longrightarrow \mathcal{Z}(\mathcal{T})$ vanishing on commutators $[X, Y]$ with $XY = 0$ such that

$$\delta_n(X) = d_n(X) + \tau_n(X)$$

for all $X \in \mathcal{T}$.

PROOF : The proof will be realized via a series of claims.

Claim 1 : $P_1\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 \in \mathcal{Z}(\mathcal{T})$ and $\delta_n(X_{12}) \in \mathcal{T}_{12}$.

Note that $X_{12}P_1 = 0$ for all $X_{12} \in \mathcal{T}_{12}$. Thus we have

$$\delta_n(X_{12}) = \delta_n([P_1, X_{12}]) = \sum_{i+j=n} (\delta_i(P_1)\delta_j(X_{12}) - \delta_j(X_{12})\delta_i(P_1)).$$

Let's check its correctness by complete induction on n. When $n = 1$, due to [21, Theorem 2.1], it is clear that

$$P_1\delta_1(P_1)P_1 + P_2\delta_1(P_1)P_2 \in \mathcal{Z}(\mathcal{T}) \text{ and } \delta_1(X_{12}) \in \mathcal{T}_{12}.$$

Suppose that

$$P_1\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2 \in \mathcal{Z}(\mathcal{T}) \text{ and } \delta_j(X_{12}) \in \mathcal{T}_{12}$$

for all $1 \leq i, j < n$. We therefore get

$$\begin{aligned}\delta_i(P_1)\delta_j(X_{12}) &= (P_1\delta_i(P_1)P_1 + P_1\delta_i(P_1)P_2 + P_2\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2)\delta_j(X_{12}) \\ &= P_1\delta_i(P_1)P_1\delta_j(X_{12}) \text{ this is due to } P_2\delta_i(P_1)P_1 = 0 \\ &= \delta_j(X_{12})P_2\delta_i(P_1)P_2 \text{ since } P_1\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2 \in \mathcal{Z}(\mathcal{T}) \\ &= \delta_j(X_{12})(P_1\delta_i(P_1)P_1 + P_1\delta_i(P_1)P_2 + P_2\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2) \\ &\quad \text{since } \delta_j(X_{12}) \in \mathcal{T}_{12} \text{ for all } 1 \leq j < n \\ &= \delta_j(X_{12})\delta_i(P_1).\end{aligned}$$

Thus we arrive at

$$\delta_n(X_{12}) = \delta_n(P_1)X_{12} - \delta_n(X_{12})P_1 + P_1\delta_n(X_{12}) - X_{12}\delta_n(P_1). \qquad (*)$$

Multiplying $(*)$ from left by P_1 and from the right by P_2 gives

$$P_1\delta_n(P_1)X_{12} = X_{12}\delta_n(P_1)P_2.$$

This implies that

$$P_1\delta_n(P_1)P_1X_{12} = X_{12}P_2\delta_n(P_1)P_2.$$

By the definition of center $\mathcal{Z}(\mathcal{T})$ we conclude that

$$P_1\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 \in \mathcal{Z}(\mathcal{T}).$$

Multiplying $(*)$ from left and right by $P_k(k = 1, 2)$ and using the facts $X_{12}\delta_n(P_1)P_1 = 0$ and $P_2\delta_n(P_1)X_{12} = 0$ yields

$$P_1\delta_n(X_{12})P_1 = P_2\delta_n(X_{12})P_2 = 0.$$

Thus we obtain

$$\begin{aligned}\delta_n(X_{12}) &= P_1\delta_n(X_{12})P_1 + P_1\delta_n(X_{12})P_2 + P_2\delta_n(X_{12})P_1 + P_2\delta_n(X_{12})P_2 \\ &= P_1\delta_n(X_{12})P_2 \text{ note that } P_2\delta_n(X_{12})P_1 = 0.\end{aligned}$$

So $\delta_n(X_{12}) = P_1\delta_n(X_{12})P_2 \in \mathcal{T}_{12}$.

On the other hand, note that $P_2X_{12} = 0$ for all $X_{12} \in \mathcal{T}_{12}$. Similarly, one can shows that

Claim 2 : $P_1\delta_n(P_2)P_1 + P_2\delta_n(P_2)P_2 \in \mathcal{Z}(\mathcal{T})$.

Claim 3 : $\delta_n(I) = P_1\delta_n(I)P_1 + P_2\delta_n(I)P_2 \in \mathcal{Z}(\mathcal{T})$.

Since $P_1(I-P_1)=0$, we have

$$0=\delta_n([P_1,I-P_1])=\sum_{i+j=n}(\delta_i(P_1)\delta_j(I)-\delta_j(I)\delta_i(P_1)).$$

We also need to check its correctness by complete induction on n. In the case of $n=1$, due to [21, Theorem 2.1], we get

$$\delta_1(I)=P_1\delta_1(I)P_1+P_2\delta_1(I)P_2\in\mathcal{Z}(\mathcal{T}).$$

Suppose that

$$\delta_k(I)=P_1\delta_k(I)P_1+P_2\delta_k(I)P_2\in\mathcal{Z}(\mathcal{T})$$

for all $1\leq k<n$. Thus we obtain

$$0=P_1\delta_n(I)-\delta_n(I)P_1$$

Multiplying the above equality on the right by P_2 gives

$$P_1\delta_n(I)P_2=0.$$

So

$$\begin{aligned}\delta_n(I)&=P_1\delta_n(I)P_1+P_1\delta_n(I)P_2+P_2\delta_n(I)P_1+P_2\delta_n(I)P_2\\&=P_1\delta_n(I)P_1+P_2\delta_n(I)P_2\\&=P_1\delta_n(P_1)P_1+P_1\delta_n(P_2)P_1+P_2\delta_n(P_1)P_2+P_2\delta_n(P_2)P_2.\end{aligned}$$

In view of Claim 1 and Claim 2, we immediately see that

$$\delta_n(I)=P_1\delta_n(I)P_1+P_2\delta_n(I)P_2\in\mathcal{Z}(\mathcal{T}).$$

For each $n\in\mathbb{N}$, let us define a linear mapping

$$\begin{aligned}f_n:\mathcal{T}&\longrightarrow\mathcal{T}\\X&\longrightarrow\delta_n(X)+[P_1\delta_n(P_1)P_2,X].\end{aligned}$$

One easily verify that

$$f_n([X,Y])=\sum_{i+j=n}(f_i(X)f_j(Y)-f_j(Y)f_i(X)),\quad\forall n\in\mathbb{N}$$

for all $X, Y \in \mathcal{T}$ with $XY = 0$. By Claim 1 we can calculate that

$$\begin{aligned} f_n(P_1) &= \delta_n(P_1) - P_1\delta_n(P_1)P_2 \\ &= P_1\delta_n(P_1)P_1 + P_1\delta_n(P_1)P_2 + P_2\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 - P_1\delta_n(P_1)P_2 \\ &= P_1\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 \in \mathcal{Z}(\mathcal{T}). \end{aligned}$$

By Claim 3 we further get $f_n(I) = \delta_n(I) \in \mathcal{Z}(\mathcal{T})$. Consequently $f_n(P_2) = f_n(I) - f_n(P_1) \in \mathcal{Z}(\mathcal{T})$.

Claim 4 : $f_n(X_{12}) \in \mathcal{T}_{12}$ for all $X_{12} \in \mathcal{T}_{12}$.

For any $X_{12} \in \mathcal{T}_{12}$, clearly $f_n(X_{12}) = \delta_n(X_{12}) + [P_1\delta_n(P_1)P_2, X_{12}] = \delta_n(X_{12}) \in \mathcal{T}_{12}$.

Claim 5 : $f_n(\mathcal{T}_{11}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{22}$ and $f_n(\mathcal{T}_{22}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{22}$.

For any $X_{11} \in \mathcal{T}_{11}$, since $X_{11}P_2 = 0$ and $f_n(P_2) \in \mathcal{Z}(\mathcal{T})$, we have

$$\begin{aligned} 0 &= f_n([X_{11}, P_2]) \\ &= \sum_{i+j=n} (f_i(X_{11})f_j(P_2) - f_j(P_2)f_i(X_{11})) \\ &= f_n(X_{11})P_2 - P_2 f_n(X_{11}). \end{aligned}$$

We therefore obtain $P_1 f_n(X_{11})P_2 = 0$. So

$$\begin{aligned} f_n(X_{11}) &= P_1 f_n(X_{11})P_1 + P_1 f_n(X_{11})P_2 + P_2 f_n(X_{11})P_1 + P_2 f_n(X_{11})P_2 \\ &= P_1 f_n(X_{11})P_1 + P_2 f_n(X_{11})P_2 \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}. \end{aligned}$$

On the other hand, $X_{22}P_1 = 0$ for all $X_{22} \in \mathcal{T}_{22}$ and $f_n(P_1) \in \mathcal{Z}(\mathcal{T})$. By an analogous manner one can show that $f_n(X_{22}) \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}$.

Claim 6 : There exists an $\mathcal{R}$-linear mapping $\tau_n^{(1)} : \mathcal{T}_{11} \longrightarrow \mathcal{Z}(\mathcal{T})$ such that

$$f_n(X_{11}) - \tau_n^{(1)}(X_{11}) \in \mathcal{T}_{11}$$

for all $X_{11} \in \mathcal{T}_{11}$. Similarly, there exists an $\mathcal{R}$-linear mapping $\tau_n^{(2)} : \mathcal{T}_{22} \longrightarrow \mathcal{Z}(\mathcal{T})$ such that

$$f_n(X_{22}) - \tau_n^{(2)}(X_{22}) \in \mathcal{T}_{22}$$

for all $X_{22} \in \mathcal{T}_{22}$.

Now, by Claim 5 we know that $P_1 f_n(X_{11}) P_2 = 0$ for all $X_{11} \in \mathcal{T}_{11}$ and that $P_1 f_n(X_{22}) P_2 = 0$ for all $X_{22} \in \mathcal{T}_{22}$. For any $X_{22} \in \mathcal{T}_{22}$, since $X_{11}X_{22} = 0$, we get

$$\begin{aligned}
0 &= f_n([X_{11}, X_{22}]) \\
&= \sum_{i+j=n} (f_i(X_{11}) f_j(X_{22}) - f_j(X_{22}) f_i(X_{11})) \\
&= \sum_{i+j=n} (P_1 f_i(X_{11}) P_1 + P_2 f_i(X_{11}) P_2)(P_1 f_j(X_{22}) P_1 + P_2 f_j(X_{22}) P_2) \\
&\quad - (P_1 f_j(X_{22}) P_1 + P_2 f_j(X_{22}) P_2)(P_1 f_i(X_{11}) P_1 + P_2 f_i(X_{11}) P_2) \\
&= \sum_{i+j=n} (P_1 f_i(X_{11}) P_1 f_j(X_{22}) P_1 + P_2 f_i(X_{11}) P_2 f_j(X_{22}) P_2 \\
&\quad - P_1 f_j(X_{22}) P_1 f_i(X_{11}) P_1 - P_2 f_j(X_{22}) P_2 f_i(X_{11}) P_2) \\
&= \sum_{i+j=n} (P_1 f_i(X_{11}) P_1 f_j(X_{22}) P_1 - P_1 f_j(X_{22}) P_1 f_i(X_{11}) P_1) \\
&\quad + \sum_{i+j=n} (P_2 f_i(X_{11}) P_2 f_j(X_{22}) P_2 - P_2 f_j(X_{22}) P_2 f_i(X_{11}) P_2)
\end{aligned}$$

Next we will use induction on n. Let us first see the case of $n = 1$. By the proof of [21, Theorem 2.1] we know that

$$P_1 f_1(X_{22}) P_1 \in \mathcal{Z}(\mathcal{A}) \ \text{ and } \ P_2 f_1(X_{11}) P_2 \in \mathcal{Z}(\mathcal{B})$$

Suppose that

$$P_1 f_k(X_{22}) P_1 \in \mathcal{Z}(\mathcal{A}) \ \text{ and } \ P_2 f_k(X_{11}) P_2 \in \mathcal{Z}(\mathcal{B})$$

whenever $1 \leq k < n$. By the above equality we see that

$$0 = X_{11} P_1 f_n(X_{22}) P_1 - P_1 f_n(X_{22}) P_1 X_{11} + P_2 f_n(X_{11}) P_2 X_{22} - X_{22} P_2 f_n(X_{11}) P_2.$$

Multiplying the above equality on two sides by $P_k (k = 1, 2)$ gives

$$X_{11} P_1 f_n(X_{22}) P_1 = P_1 f_n(X_{22}) P_1 X_{11}, \quad \forall X_{11} \in \mathcal{T}_{11}$$

$$P_2 f_n(X_{11}) P_2 X_{22} = X_{22} P_2 f_n(X_{11}) P_2, \quad \forall X_{22} \in \mathcal{T}_{22}.$$

In view of the fact $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$, we conclude that

$$P_1 f_n(X_{22}) P_1 \in \mathcal{Z}(\mathcal{A}) = P_1 \mathcal{Z}(\mathcal{T}) P_1, \quad \forall X_{22} \in \mathcal{T}_{22}$$

$$P_2 f_n(X_{11}) P_2 \in \mathcal{Z}(\mathcal{B}) = P_2 \mathcal{Z}(\mathcal{T}) P_2, \quad \forall X_{11} \in \mathcal{T}_{11}.$$

Let us define $\tau_n^{(1)} : \mathcal{T}_{11} \longrightarrow \mathcal{Z}(\mathcal{T})$ by $\tau_n^{(1)}(X_{11}) = \eta(P_2 f_n(X_{11})P_2) \oplus P_2 f_n(X_{11})P_2$, where η is the mapping defined in Lemma 2.1. Thus we arrive at

$$\begin{aligned} f_n(X_{11}) - \tau_n^{(1)}(X_{11}) &= P_1 f_n(X_{11})P_1 + P_2 f_n(X_{11})P_2 - \eta(P_2 f_n(X_{11})P_2) - P_2 f_n(X_{11})P_2 \\ &= P_1 f_n(X_{11})P_1 - \eta(P_2 f_n(X_{11})P_2) \in \mathcal{T}_{11}. \end{aligned}$$

Since f_n is $\mathcal{R}$-linear, one can verify that $\tau_n^{(1)}$ is also $\mathcal{R}$-linear. Similarly, we can prove that there exists an $\mathcal{R}$-linear mapping $\tau_n^{(2)} : \mathcal{T}_{22} \longrightarrow \mathcal{Z}(\mathcal{T})$ such that

$$f_n(X_{22}) - \tau_n^{(2)}(X_{22}) \in \mathcal{T}_{22}$$

for all $X_{22} \in \mathcal{T}_{22}$.

Now for any $X = X_{11} + X_{12} + X_{22} \in \mathcal{T}$, we establish two $\mathcal{R}$-linear mappings $\tau_n : \mathcal{T} \longrightarrow \mathcal{Z}(\mathcal{T})$ and $d_n : \mathcal{T} \longrightarrow \mathcal{T}$ by

$$\tau_n(X) = \tau_n^{(1)}(X_{11}) + \tau_n^{(2)}(X_{22}) \text{ and } d_n(X) = f_n(X) - \tau_n(X).$$

Then $d_n(\mathcal{T}_{ij}) \subseteq \mathcal{T}_{ij}$ for all $1 \leq i \leq j \leq 2$ and $d_n(X_{12}) = f_n(X_{12})$.

Claim 7 : $\{d_n\}_{n \in \mathbb{N}}$ is a higher derivation.

We divide the proof into the following several steps.

Step 1 : For any $X_{11} \in \mathcal{T}_{11}, X_{12} \in \mathcal{T}_{12}$, since $X_{12}X_{11} = 0, \tau_n^{(1)}(X_{11}) \in \mathcal{Z}(\mathcal{T})$ and $d_n(\mathcal{T}_{ij}) \subseteq \mathcal{T}_{ij}(1 \leq i \leq j \leq 2)$, we get

$$\begin{aligned} d_n(X_{11}X_{12}) &= f_n(X_{11}X_{12}) \\ &= f_n([X_{11}, X_{12}]) \\ &= \sum_{i+j=n} (f_i(X_{11})f_j(X_{12}) - f_j(X_{12})f_i(X_{11})) \\ &= \sum_{i+j=n} ((d_i(X_{11}) + \tau_i(X_{11}))d_j(X_{12}) - d_j(X_{12})(d_i(X_{11}) + \tau_i(X_{11}))) \\ &= \sum_{i+j=n} ((d_i(X_{11}) + \tau_i^{(1)}(X_{11}))d_j(X_{12}) - d_j(X_{12})(d_i(X_{11}) + \tau_i^{(1)}(X_{11}))) \\ &= \sum_{i+j=n} d_i(X_{11})d_j(X_{12}) \end{aligned}$$

Similarly, one can prove that $d_n(X_{12}X_{22}) = \sum_{i+j=n} d_i(X_{12})d_j(X_{22})$.

Step 2 : Let us take arbitrary elements $X_{11}, Y_{11} \in \mathcal{T}_{11}$, for any $X_{12} \in \mathcal{T}_{12}$, by virtue of Step 1, we have

$$d_n(X_{11}Y_{11}X_{12}) = \sum_{k+j=n} d_k(X_{11}Y_{11})d_j(X_{12}).$$

On the other hand, by Step 1 again, we also get

$$\begin{aligned} d_n(X_{11}Y_{11}X_{12}) &= \sum_{h+l=n} d_h(X_{11})d_l(Y_{11}X_{12}) \\ &= \sum_{i+j=l, h+l=n} d_h(X_{11})d_i(Y_{11})d_j(X_{12}). \end{aligned}$$

Comparing the above two equalities gives

$$\sum_{h+i=k, k+j=n} (d_k(X_{11}Y_{11}) - d_h(X_{11})d_i(Y_{11}))d_j(X_{12}) = 0.$$

By the proof of [21, Theorem 2.1] we know that for any $X_{11}, Y_{11} \in \mathcal{T}_{11}$,

$$d_1(X_{11}Y_{11}) = d_1(X_{11})Y_{11} + X_{11}d_1(Y_{11})$$

whenever $n = 1$. Suppose that for any $X_{11}, Y_{11} \in \mathcal{T}_{11}$,

$$d_k(X_{11}Y_{11}) = \sum_{h+i=k} d_h(X_{11})d_i(Y_{11})$$

whenever $1 \leq k < n$. We therefore arrive at

$$(d_n(X_{11}Y_{11}) - \sum_{i+j=n} d_i(X_{11})d_j(Y_{11}))X_{12} = 0.$$

Since $\mathcal{T}_{12}$ is faithful as a left $\mathcal{T}_{11}$-module, we immediately see

$$d_n(X_{11}Y_{11}) = \sum_{i+j=n} d_i(X_{11})d_j(Y_{11})$$

for all $X_{11}, Y_{11} \in \mathcal{T}_{11}$. Using an analogous manner one can show that

$$d_n(X_{22}Y_{22}) = \sum_{i+j=n} d_i(X_{22})d_j(Y_{22})$$

for all $X_{22}, Y_{22} \in \mathcal{T}_{22}$.

Step 3 : For arbitrary elements $X, Y \in \mathcal{T}$, let us write them $X = X_{11} + X_{12} + X_{22}, Y = Y_{11} + Y_{12} + Y_{22}$. Taking into account the Step 1 and Step 2 we calculate that

$$\begin{aligned} d_n(XY) &= d_n(X_{11}Y_{11} + X_{11}Y_{12} + X_{12}Y_{22} + X_{22}Y_{22}) \\ &= \sum_{i+j=n} (d_i(X_{11})d_j(Y_{11}) + d_i(X_{11})d_j(Y_{12}) \\ &\quad + d_i(X_{12})d_j(Y_{22}) + d_i(X_{22})d_j(Y_{22})) \\ &= \sum_{i+j=n} (d_i(X_{11})d_j(Y_{11}) + d_i(X_{11})d_j(Y_{12}) + d_i(X_{11})d_j(Y_{22}) \\ &\quad + d_i(X_{12})d_j(Y_{11}) + d_i(X_{12})d_j(Y_{12}) + d_i(X_{12})d_j(Y_{22}) \\ &\quad + d_i(X_{22})d_j(Y_{11}) + d_i(X_{22})d_j(Y_{12}) + d_i(X_{22})d_j(Y_{22})) \\ &= \sum_{i+j=n} (d_i(X_{11})d_j(Y) + d_i(X_{12})d_j(Y) + d_i(X_{22})d_j(Y)) \\ &= \sum_{i+j=n} d_i(X)d_j(Y). \end{aligned}$$

This shows that $\{d_n\}_{n=0}^{\infty}$ is a higher derivation of $\mathcal{T}$.

Now we come to the last step of the proof of this theorem.

Claim 8 : τ_n vanishes on all the commutators $[X, Y]$ with $XY = 0$ for all $X, Y \in \mathcal{T}$.

For any $X, Y \in \mathcal{T}$ with $XY = 0$, by the definition of $d_n = f_n - \tau_n$ we come up with conclusion

$$\begin{aligned} \tau_n([X, Y]) &= f_n([X, Y]) - d_n([X, Y]) \\ &= \sum_{i+j=n} (f_i(X)f_j(Y) - f_j(Y)f_i(X)) - d_n([X, Y]) \\ &= \sum_{i+j=n} ((d_i(X) + \tau_i(X))(d_j(Y) + \tau_j(Y))) \\ &\quad - \sum_{i+j=n} ((d_j(Y) + \tau_j(Y))(d_i(X) + \tau_i(X))) - d_n([X, Y]) \\ &= \sum_{i+j=n} (d_i(X)d_j(Y) - d_j(Y)d_i(X)) - d_n([X, Y]) \\ &= d_n(XY) - d_n(YX) - d_n(XY - YX) \\ &= 0. \square \end{aligned}$$

As a direct application of Theorem 2.2, we now consider the nest algebra case.

Let $\mathcal{X}$ be a Banach space over the real or complex field $\mathbb{F}$, $\mathcal{B}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. A nest $\mathcal{N}$ on $\mathcal{X}$ is a complete totally ordered subspace lattice, i.e., a chain of closed (under norm topology) subspaces of X which is closed under the formation of arbitrary closed linear span (denoted by $\bigvee$) and intersection (denoted by $\bigwedge$), and which contains $\{0\}$ and $\mathcal{X}$. The nest algebra associated with the nest $\mathcal{N}$, denoted by $\mathrm{Alg}\mathcal{N}$, is the weakly closed operator algebra consisting of all operators that leave $\mathcal{N}$ invariant, i.e.,

$$\mathrm{Alg}\mathcal{N} = \{\ T \in \mathcal{B}(\mathcal{X}) \mid TN \subseteq N \text{ for all } N \in \mathcal{N}\ \}.$$

It should be remarked that arbitrary finite dimensional nest algebra is isomorphic to a real of complex block upper triangular matrix algebra. When the nest $\mathcal{N} \neq \{0, \mathcal{X}\}$, we say that the nest $\mathcal{N}$ is non-trivial. If $\mathcal{N}$ is trivial, then $\mathrm{Alg}\mathcal{N} = \mathcal{B}(\mathcal{X})$, which is a prime algebra over the real or complex field $\mathbb{F}$. In this paper we only consider the nontrivial nest algebras.

Let $\mathcal{N}$ be a nest on a complex Banach space X such that there exists a $N \in \mathcal{N}$ complemented in $\mathcal{X}$ and Alg$\mathcal{N}$ be the associated nest algebra. Then Alg$\mathcal{N}$ is a triangular algebra over $\mathbb{C}$. Indeed, Since $N \in \mathcal{N}$ is complemented in $\mathcal{X}$, there is a bounded idempotent operator P with range N. It is easy to check that $P \in$ Alg$\mathcal{N}$. Let us denote $M = (I-P)(\mathcal{X})$, and let $\mathcal{A} = P\text{Alg}\mathcal{N}|_N, \mathcal{M} = P\text{Alg}\mathcal{N}|_M$ and $\mathcal{B} = (I-P)\text{Alg}\mathcal{N}|_M$. Then ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ is faithful as left $\mathcal{A}$-module and right $\mathcal{B}$-module. Therefore we can say that Alg$\mathcal{N} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{smallmatrix}\right]$.

In particular, if $\mathcal{X}$ is a Hilbert space, then every nontrivial nest algebra is a triangular algebra. Indeed, if $N \in \mathcal{N}\backslash\{0, \mathcal{X}\}$ and E is the orthogonal projection onto N, then $\mathcal{N}_1 = E(\mathcal{N})$ and $\mathcal{N}_2 = (1-E)(\mathcal{N})$ are nests of N and $N^{\perp}$, respectively. Moreover, $\mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E, \mathcal{T}(\mathcal{N}_2) = (1-E)\mathcal{T}(\mathcal{N})(1-E)$ are nest algebras and

$$\mathcal{T}(\mathcal{N}) = \begin{bmatrix} \mathcal{T}(\mathcal{N}_1) & E\mathcal{T}(\mathcal{N})(1-E) \\ O & \mathcal{T}(\mathcal{N}_2) \end{bmatrix}$$

or

$$\mathcal{T}'(\mathcal{N}) = \begin{bmatrix} \mathcal{T}(\mathcal{N}_1) & O \\ (1-E)\mathcal{T}(\mathcal{N})E & \mathcal{T}(\mathcal{N}_2) \end{bmatrix}.$$

However, it is not always the case for a nest $\mathcal{N}$ on a general Banach space $\mathcal{X}$, since $N \in \mathcal{N}$ may be not complemented. We refer the reader to [12] for the theory of nest algebras.

Corollary 2.3 — Let $\mathcal{N}$ be an arbitrary nest on a complex Banach space $\mathcal{X}$ such that there is $N \in \mathcal{N}$ complemented in X and be $\mathrm{Alg}\mathcal{N}$ the nest algebra associated with $\mathcal{N}$. Let $\Delta = \{\delta_n\}_{n=0}^{\infty}$ be a family $\mathbb{F}$-linear mappings from $\mathrm{Alg}\mathcal{N}$ into itself such that

$$\delta_n([X,Y]) = \sum_{i+j=n} (\delta_i(X)\delta_j(Y) - \delta_j(Y)\delta_i(X))$$

for all $X, Y \in \mathrm{Alg}\mathcal{N}$ with $XY = 0$. Then there exist a higher derivation $D = \{d_n\}_{n=0}^{\infty}$ and an $\mathcal{R}$-linear mapping $\tau_n : \mathrm{Alg}\mathcal{N} \longrightarrow \mathbb{F}I$ vanishing on commutators $[X, Y]$ with $XY = 0$ such that

$$\delta_n(X) = d_n(X) + \tau_n(X)$$

for all $X \in \mathrm{Alg}\mathcal{N}$.

In particular, we have

Corollary 2.4 — Let $\mathcal{N}$ be an arbitrary nest on a Hilbert space $\mathbf{H}$ with $\dim \mathbf{H} \geq 2$ and $\mathrm{Alg}\mathcal{N}$ be the nest algebra associated with $\mathcal{N}$. Let $\Delta = \{\delta_n\}_{n=0}^{\infty}$ be a family $\mathbb{F}$-linear mappings from $\mathrm{Alg}\mathcal{N}$ into itself such that

$$\delta_n([X, Y]) = \sum_{i+j=n} (\delta_i(X)\delta_j(Y) - \delta_j(Y)\delta_i(X))$$

for all $X, Y \in \mathrm{Alg}\mathcal{N}$ with $XY = 0$. Then there exist a higher derivation $D = \{d_n\}_{n=0}^{\infty}$ and an $\mathcal{R}$-linear mapping $\tau_n : \mathrm{Alg}\mathcal{N} \longrightarrow \mathbb{F}I$ vanishing on commutators $[X, Y]$ with $XY = 0$ such that

$$\delta_n(X) = d_n(X) + \tau_n(X)$$

for all $X \in \mathrm{Alg}\mathcal{N}$.

3. Lie Higher Derivations by Acting on Idempotent Products

This section is aimed to studying Lie higher derivations at idempotents on triangular algebras. More precisely, we will give the higher version corresponding to [21, Theorem 2.2]

***Theorem* 3.1** — *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over commutative ring $\mathcal{R}$ with unit I_1 and I_2, respectively. Let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. The triangular algebra $\mathcal{T} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{smallmatrix}\right]$ satisfies the following condition*

- $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ *and* $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$.
- *For every $a \in \mathcal{A}$, there exists some integer n such that $nI_1 - a$ is invertible.*

Suppose that $\Delta = \{\delta_n\}_{n=0}^{\infty}$ is a family of $\mathcal{R}$-linear mappings from $\mathcal{T}$ into itself such that

$$\delta_n([X, Y]) = \sum_{i+j=n} (\delta_i(X)\delta_j(Y) - \delta_j(Y)\delta_i(X))$$

for all $X, Y \in \mathcal{T}$ with $XY = P_1$. Then there exist a higher derivation $D = \{d_n\}_{n=0}^{\infty}$ and an $\mathcal{R}$-linear mapping $\tau_n : \mathcal{T} \longrightarrow \mathcal{Z}(\mathcal{T})$ vanishing on commutators $[X, Y]$ with $XY = P_1$ such that

$$\delta_n(X) = d_n(X) + \tau_n(X)$$

for all $X \in \mathcal{T}$.

PROOF : The proof will be realized via a series of claims.

Claim 1 : $P_1\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 \in \mathcal{Z}(\mathcal{T})$ and $\delta_n(X_{12}) \in \mathcal{T}_{12}$.

Since $(P_1 + X_{12})P_1 = P_1$ for all $X_{12} \in \mathcal{T}_{12}$. Thus we have

$$\delta_n(X_{12}) = \delta_n([P_1, P_1 + X_{12}]) = \sum_{i+j=n} (\delta_i(P_1)\delta_j(X_{12}) - \delta_j(X_{12})\delta_i(P_1)).$$

Let's check its correctness by complete induction on n. When $n = 1$, due to [21, Theorem 2.2], it is clear that

$$P_1\delta_1(P_1)P_1 + P_2\delta_1(P_1)P_2 \in \mathcal{Z}(\mathcal{T}) \text{ and } \delta_1(X_{12}) = P_1\delta_1(X_{12})P_2 \in \mathcal{T}_{12}.$$

Suppose that

$$P_1\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2 \in \mathcal{Z}(\mathcal{T}) \text{ and } \delta_j(X_{12}) = P_1\delta_j(X_{12})P_2 \in \mathcal{T}_{12}$$

for all $1 \le i, j < n$. We therefore get

$$\begin{aligned}
\delta_i(P_1)\delta_j(X_{12}) &= (P_1\delta_i(P_1)P_1 + P_1\delta_i(P_1)P_2 + P_2\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2)\delta_j(X_{12}) \\
&= P_1\delta_i(P_1)P_1\delta_j(X_{12}) \ \text{ this is due to } P_2\delta_i(P_1)P_1 = 0 \\
&= \delta_j(X_{12})P_2\delta_i(P_1)P_2 \ \text{ since } P_1\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2 \in \mathcal{Z}(\mathcal{T}) \\
&= \delta_j(X_{12})(P_1\delta_i(P_1)P_1 + P_1\delta_i(P_1)P_2 + P_2\delta_i(P_1)P_1 + P_2\delta_i(P_1)P_2) \\
&\quad \text{since } \delta_j(X_{12}) \in \mathcal{T}_{12} \text{ for all } 1 \le j < n \\
&= \delta_j(X_{12})\delta_i(P_1).
\end{aligned}$$

Thus we arrive at

$$\delta_n(X_{12}) = \delta_n(P_1)X_{12} - \delta_n(X_{12})P_1 + P_1\delta_n(X_{12}) - X_{12}\delta_n(P_1).$$

Multiplying the above equality from left by P_1 and from the right by P_2 gives

$$P_1\delta_n(P_1)X_{12} = X_{12}\delta_n(P_1)P_2.$$

This implies that

$$P_1\delta_n(P_1)P_1X_{12} = X_{12}P_2\delta_n(P_1)P_2.$$

By the definition of center $\mathcal{Z}(\mathcal{T})$ we conclude that

$$P_1\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 \in \mathcal{Z}(\mathcal{T}).$$

Multiplying the above equality from left and right by $P_k(k = 1, 2)$ and using the facts $X_{12}\delta_n(P_1)P_1 = 0$ and $P_2\delta_n(P_1)X_{12} = 0$ yields

$$P_1\delta_n(X_{12})P_1 = P_2\delta_n(X_{12})P_2 = 0.$$

Thus we obtain

$$\begin{aligned}\delta_n(X_{12}) &= P_1\delta_n(X_{12})P_1 + P_1\delta_n(X_{12})P_2 + P_2\delta_n(X_{12})P_1 + P_2\delta_n(X_{12})P_2\\ &= P_1\delta_n(X_{12})P_2 \text{ not that } P_2\delta_n(X_{12})P_1 = 0.\end{aligned}$$

So $\delta_n(X_{12}) = P_1\delta_n(X_{12})P_2 \in \mathcal{T}_{12}$.

For each $n \in \mathbb{N}$, we now define an $\mathcal{R}$-linear mapping

$$\begin{aligned}f_n : \mathcal{T} &\longrightarrow \mathcal{T}\\ X &\longrightarrow \delta_n(X) + [P_1\delta_n(P_1)P_2, X].\end{aligned}$$

By Claim 1 we can calculate that

$$\begin{aligned}f_n(P_1) &= \delta_n(P_1) - P_1\delta_n(P_1)P_2\\ &= P_1\delta_n(P_1)P_1 + P_1\delta_n(P_1)P_2 + P_2\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 - P_1\delta_n(P_1)P_2\\ &= P_1\delta_n(P_1)P_1 + P_2\delta_n(P_1)P_2 \in \mathcal{Z}(\mathcal{T}).\end{aligned}$$

One easily verify that

$$f_n([X,Y]) = \sum_{i+j=n} (f_i(X)f_j(Y) - f_j(Y)f_i(X)), \quad \forall n \in \mathbb{N}$$

fro all $X, Y \in \mathcal{T}$ with $XY = P$. Moreover, for any $X_{12} \in \mathcal{T}_{12}$, we have $f_n(X_{12}) = \delta_n(X_{12}) + [P_1\delta_n(P_1)P_2, X_{12}] = \delta_n(X_{12}) \in \mathcal{T}_{12}$.

Claim 2 : $f_n(I) = P_1f_n(I)P_1 + P_2f_n(I)P_2 \in \mathcal{Z}(\mathcal{T})$ and $f_n(P_2) \in \mathcal{Z}(\mathcal{T})$.

Since $IP_1 = P_1 = P$, we get

$$\begin{aligned}0 &= f_n([I, P_1])\\ &= \sum_{i+j=n} (f_i(I)f_j(P_1) - f_j(P_1)f_i(I))\\ &= f_n(I)P_1 - P_1f_n(I).\end{aligned}$$

Multiplying the above equality from the right side by P_2 leads to $P_1 f_n(I)P_2 = 0$. This implies that

$$f_n(I) = P_1 f_n(I)P_1 + P_2 f_n(I)P_2.$$

Note that $(P_1 - X_{12})(I + X_{12}) = P_1$ for all $X_{12} \in \mathcal{T}_{12}$. We therefore arrive at

$$\begin{aligned}
f_n(X_{12}) &= f_n([P_1 - X_{12}, I + X_{12}]) \\
&= \sum_{i+j=n} (f_i(P_1 - X_{12})f_j(I + X_{12}) - f_j(I + X_{12})f_i(P_1 - X_{12})) \\
&= \sum_{i+j=n} (f_i(P_1)f_j(I) - f_j(I)f_i(P_1)) \\
&\quad + \sum_{i+j=n} (f_i(P_1)f_j(X_{12}) - f_j(X_{12})f_i(P_1)) \\
&\quad + \sum_{i+j=n} (f_i(I)f_j(X_{12}) - f_j(X_{12})f_i(I)) \\
&= P_1 f_n(X_{12}) + \sum_{i+j=n} (f_i(I)f_j(X_{12}) - f_j(X_{12})f_i(I)).
\end{aligned}$$

Thus we conclude

$$\sum_{i+j=n} (f_i(I)f_j(X_{12}) - f_j(X_{12})f_i(I)) = 0.$$

Let us check the correctness of this claim by induction. By the proof of [21, Theorem 2.2] we know that

$$f_1(I) \in \mathcal{Z}(\mathcal{T})$$

whenever $n = 1$. Suppose that

$$f_k(I) \in \mathcal{Z}(\mathcal{T})$$

whenever $1 \leq k < n$. Consequently, we obtain

$$f_n(I)X_{12} = X_{12} f_n(I).$$

Multiplying the above equality from the left side by P_1 and from the right side by P_2 gives

$$P_1 f_n(I)P_1 X_{12} = X_{12} P_2 f_n(I)P_2.$$

This shows that

$$f_k(I) \in \mathcal{Z}(\mathcal{T})$$

whenever $k = n$. Thus we assert that for every $n \geq 1$,

$$f_n(I) = P_1 f_n(I) P_1 + P_2 f_n(I) P_2 \in \mathcal{Z}(\mathcal{T}).$$

This further leads to

$$f_n(P_2) = f_n(I) - f_n(P_1) \in \mathcal{Z}(\mathcal{T}).$$

Claim 3 : $f_n(\mathcal{T}_{11}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{22}$.

The proof of this claim is divided into two cases.

Let us choose an arbitrary element $X_{11} \in \mathcal{T}_{11}$. If X_{11} is invertible, i.e. there exists an element X_{11}^{-1} such that $X_{11} X_{11}^{-1} = X_{11}^{-1} X_{11} = P_1 = P$. It follows from $X_{11} X_{11}^{-1} = P$ and $(X_{11}^{-1} + P_2) X_{11} = P$ that

$$\begin{aligned} 0 &= f_n([X_{11}, X_{11}^{-1}]) \\ &= \sum_{i+j=n} (f_i(X_{11}) f_j(X_{11}^{-1}) - f_j(X_{11}^{-1}) f_i(X_{11})) \end{aligned}$$

and that

$$\begin{aligned} 0 &= f_n([X_{11}^{-1} + P_2, X_{11}]) \\ &= \sum_{i+j=n} (f_i(X_{11}^{-1} + P_2) f_j(X_{11}) - f_j(X_{11}) f_i(X_{11}^{-1} + P_2)) \\ &= \sum_{i+j=n} (f_i(P_2) f_j(X_{11}) - f_j(X_{11}) f_i(P_2)) \\ &= P_2 f_n(X_{11}) - f_n(X_{11}) P_2. \end{aligned}$$

Multiplying the above equality from the left side by P_1 gives

$$P_1 f_n(X_{11}) P_2 = 0.$$

We immediately claim that $f_n(X_{11}) = P_1 f_n(X_{11}) P_1 + P_2 f_n(X_{11}) P_2 \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}$.

If X_{11} is not invertible in $\mathcal{T}_{11}$, by the hypothesis of this theorem, there exists some integer k such that $kP_1 - X_{11}$ is invertible in $\mathcal{T}_{11}$. In view of the preceding case, we can say that $f_n(kP_1 - X_{11}) \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}$. Thus we assert that

$$f_n(X_{11}) = k f_n(P_1) - f_n(kP_1 - X_{11}) \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}.$$

Claim 4 : $f_n(\mathcal{T}_{22}) \subseteq \mathcal{T}_{11} \oplus \mathcal{T}_{22}$.

For an arbitrary element $X_{22} \in \mathcal{T}_{22}$, using the fact $(P_1 + X_{22})P_1 = P_1 = P$ we get

$$\begin{aligned} 0 &= f_n([P_1 + X_{22}, P_1]) \\ &= \sum_{i+j=n} (f_i(P_1 + X_{22})f_j(P_1) - f_j(P_1)f_i(P_1 + X_{22})) \\ &= \sum_{i+j=n} (f_i(X_{22})f_j(P_1) - f_j(P_1)f_i(X_{22})) \\ &= f_n(X_{22})P_1 - P_1 f_n(X_{22}). \end{aligned}$$

Likewise, we see that $P_1 f_n(X_{22})P_2 = 0$. This yields that $f_n(\mathcal{T}_{22}) = P_1 f_n(X_{22})P_1 + P_2 f_n(X_{22})P_2 \in \mathcal{T}_{11} \oplus \mathcal{T}_{22}$.

Claim 5 : There exists an $\mathcal{R}$-linear mapping $\tau_n^{(1)} : \mathcal{T}_{11} \longrightarrow \mathcal{Z}(\mathcal{T})$ such that

$$f_n(X_{11}) - \tau_n^{(1)}(X_{11}) \in \mathcal{T}_{11}$$

for all $X_{11} \in \mathcal{T}_{11}$. Similarly, there exists an $\mathcal{R}$-linear mapping $\tau_n^{(2)} : \mathcal{T}_{22} \longrightarrow \mathcal{Z}(\mathcal{T})$ such that

$$f_n(X_{22}) - \tau_n^{(2)}(X_{22}) \in \mathcal{T}_{22}$$

for all $X_{22} \in \mathcal{T}_{22}$.

Let us choose arbitrary elements $X_{11} \in \mathcal{T}_{11}, X_{22} \in \mathcal{T}_{22}$. By Claim 3 and Claim 4 we can write $f_n(X_{11}) = P_1 f_n(X_{11})P_1 + P_2 f_n(X_{11})P_2$ and $f_n(\mathcal{T}_{22}) = P_1 f_n(X_{22})P_1 + P_2 f_n(X_{22})P_2$.

Let us consider the first case that X_{11} is invertible in $\mathcal{T}_{11}$ with inverse element X_{11}^{-1}. Note that $X_{11}^{-1}X_{11} = P_1 = P$ and that $(X_{11}^{-1} + X_{22})X_{11} = P_1 = P$. We therefore calculate that

$$0 = f_n([X_{11}^{-1}, X_{11}]) = \sum_{i+j=n} (f_i(X_{11}^{-1})f_j(X_{11}) - f_j(X_{11})f_i(X_{11}^{-1}))$$

and that

$$\begin{aligned} 0 &= f_n([X_{11}^{-1} + X_{22}, X_{11}]) \\ &= \sum_{i+j=n} (f_i(X_{11}^{-1} + X_{22})f_j(X_{11}) - f_j(X_{11})f_i(X_{11}^{-1} + X_{22})) \\ &= \sum_{i+j=n} (f_i(X_{22})f_j(X_{11}) - f_j(X_{11})f_i(X_{22})) \\ &= \sum_{i+j=n} (P_1 f_i(X_{22})P_1 + P_2 f_i(X_{22})P_2)(P_1 f_j(X_{11})P_1 + P_2 f_j(X_{11})P_2) \end{aligned}$$

$$\begin{aligned}
&\quad -(P_1f_j(X_{11})P_1+P_2f_j(X_{11})P_2)(P_1f_i(X_{22})P_1+P_2f_i(X_{22})P_2)\\
&= \sum_{i+j=n}(P_1f_i(X_{22})P_1f_j(X_{11})P_1+P_2f_i(X_{22})P_2f_j(X_{11})P_2\\
&\quad -P_1f_j(X_{11})P_1f_i(X_{22})P_1-P_2f_j(X_{11})P_2f_i(X_{22})P_2)\\
&= \sum_{i+j=n}(P_1f_i(X_{22})P_1f_j(X_{11})P_1-P_1f_j(X_{11})P_1f_i(X_{22})P_1)\\
&\quad +\sum_{i+j=n}(P_2f_i(X_{22})P_2f_j(X_{11})P_2-P_2f_j(X_{11})P_2f_i(X_{22})P_2).
\end{aligned}$$

Next we will use induction on n. Let us first see the case of $n = 1$. By the proof of [21, Theorem 2.2] we know that

$$P_1f_1(X_{22})P_1 \in \mathcal{Z}(\mathcal{A}) \text{ and } P_2f_1(X_{11})P_2 \in \mathcal{Z}(\mathcal{B})$$

Suppose that

$$P_1f_k(X_{22})P_1 \in \mathcal{Z}(\mathcal{A}) \text{ and } P_2f_k(X_{11})P_2 \in \mathcal{Z}(\mathcal{B})$$

whenever $1 \le k < n$. By the above equality we have

$$0 = P_1f_n(X_{22})P_1X_{11} - X_{11}P_1f_n(X_{22})P_1 + X_{22}P_2f_n(X_{11})P_2 - P_2f_n(X_{11})P_2X_{22}.$$

Multiplying the above equality on two-sides by $P_l(l = 1, 2)$ gives

$$P_1f_n(X_{22})P_1X_{11} = X_{11}P_1f_n(X_{22})P_1, \quad \forall X_{11} \in \mathcal{T}_{11}$$

$$X_{22}P_2f_n(X_{11})P_2 = P_2f_n(X_{11})P_2X_{22}, \quad \forall X_{22} \in \mathcal{T}_{22}.$$

Taking into account the hypothesis $\pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(\mathcal{B})$, we conclude that

$$P_1f_n(X_{22})P_1 \in \mathcal{Z}(\mathcal{A}) = P_1\mathcal{Z}(\mathcal{T})P_1, \quad \forall X_{22} \in \mathcal{T}_{22}$$

$$P_2f_n(X_{11})P_2 \in \mathcal{Z}(\mathcal{B}) = P_2\mathcal{Z}(\mathcal{T})P_2, \quad \forall X_{11} \in \mathcal{T}_{11},$$

whenever $k = n$.

If X_{11} is not invertible in $\mathcal{T}_{11}$, by the hypothesis, there exists some integer r such that $rP_1 - X_{11}$ is invertible in $\mathcal{T}_{11}$. By virtue of the discussion of the preceding case it follows that

$$\begin{aligned}
0 &= \sum_{i+j=n}(f_i(X_{22})f_j(rP_1-X_{11})-f_j(rP_1-X_{11})f_i(X_{22}))\\
&= \sum_{i+j=n}(f_j(X_{11})f_i(X_{22})-f_i(X_{22})f_j(X_{11}))
\end{aligned}$$

$$\begin{aligned} &= \sum_{i+j=n}(P_1f_j(X_{11})P_1+P_2f_j(X_{11})P_2)(P_1f_i(X_{22})P_1+P_2f_i(X_{22})P_2)\\ &\quad -(P_1f_i(X_{22})P_1+P_2f_i(X_{22})P_2)(P_1f_j(X_{11})P_1+P_2f_j(X_{11})P_2)\\ &= \sum_{i+j=n}(P_1f_j(X_{11})P_1f_i(X_{22})P_1+P_2f_j(X_{11})P_2f_i(X_{22})P_2\\ &\quad -P_1f_i(X_{22})P_1f_j(X_{11})P_1-P_2f_i(X_{22})P_2f_j(X_{11})P_2)\\ &= \sum_{i+j=n}(P_1f_j(X_{11})P_1f_i(X_{22})P_1-P_1f_i(X_{22})P_1f_j(X_{11})P_1)\\ &\quad +\sum_{i+j=n}(P_2f_j(X_{11})P_2f_i(X_{22})P_2-P_2f_i(X_{22})P_2f_j(X_{11})P_2). \end{aligned}$$

The remainder proofs are similar to the preceding case. Likewise, we obtain

$$P_1f_n(X_{22})P_1 \in \mathcal{Z}(\mathcal{A}) = P_1\mathcal{Z}(\mathcal{T})P_1, \quad \forall X_{22}\in\mathcal{T}_{22}$$

$$P_2f_n(X_{11})P_2 \in \mathcal{Z}(\mathcal{B}) = P_2\mathcal{Z}(\mathcal{T})P_2, \quad \forall X_{11}\in\mathcal{T}_{11}.$$

Let us define $\tau_n^{(1)}:\mathcal{T}_{11}\longrightarrow\mathcal{Z}(\mathcal{T})$ by $\tau_n^{(1)}(X_{11})=\eta(P_2f_n(X_{11})P_2)\oplus P_2f_n(X_{11})P_2$, where η is the mapping defined in Lemma 2.1. Thus we arrive at

$$\begin{aligned} f_n(X_{11})-\tau_n^{(1)}(X_{11}) &= P_1f_n(X_{11})P_1+P_2f_n(X_{11})P_2-\eta(P_2f_n(X_{11})P_2)-P_2f_n(X_{11})P_2\\ &= P_1f_n(X_{11})P_1-\eta(P_2f_n(X_{11})P_2)\in\mathcal{T}_{11}. \end{aligned}$$

Since f_n is $\mathcal{R}$-linear, one can verify that $\tau_n^{(1)}$ is also $\mathcal{R}$-linear. Similarly, we can show that there exists an $\mathcal{R}$-linear mapping $\tau_n^{(2)}:\mathcal{T}_{22}\longrightarrow\mathcal{Z}(\mathcal{T})$ such that

$$f_n(X_{22})-\tau_n^{(2)}(X_{22})\in\mathcal{T}_{22}$$

for all $X_{22}\in\mathcal{T}_{22}$.

Now for any $X=X_{11}+X_{12}+X_{22}\in\mathcal{T}$, we establish two $\mathcal{R}$-linear mappings $\tau_n:\mathcal{T}\longrightarrow\mathcal{Z}(\mathcal{T})$ and $d_n:\mathcal{T}\longrightarrow\mathcal{T}$ by

$$\tau_n(X)=\tau_n^{(1)}(X_{11})+\tau_n^{(2)}(X_{22}) \text{ and } d_n(X)=f_n(X)-\tau_n(X).$$

Then $d_n(\mathcal{T}_{ij})\subseteq\mathcal{T}_{ij}$ for all $1\le i\le j\le 2$ and $d_n(X_{12})=f_n(X_{12})$.

Claim 6 : $\{d_n\}_{n\in\mathbb{N}}$ is a higher derivation.

The proof of this claim can be reached via several steps.

Step 1 : For any $X_{11} \in \mathcal{T}_{11}, X_{12} \in \mathcal{T}_{12}$, if X_{11} is invertible in $\mathcal{T}_{11}$ with invertible element X_{11}^{-1}, then by the fact $(X_{11}^{-1} + X_{11}^{-1}X_{12})X_{11} = P_1 = P$, we have

$$\begin{aligned} d_n(X_{12}) = f_n(X_{12}) &= f_n([X_{11}, X_{11}^{-1} + X_{11}^{-1}X_{12}]) \\ &= \sum_{i+j=n} (f_i(X_{11})f_j(X_{11}^{-1} + X_{11}^{-1}X_{12}) \\ &\quad - f_j(X_{11}^{-1} + X_{11}^{-1}X_{12})f_i(X_{11})) \\ &= \sum_{i+j=n} (f_i(X_{11})f_j(X_{11}^{-1}X_{12}) \\ &\quad - f_j(X_{11}^{-1}X_{12})f_i(X_{11})) \\ &= \sum_{i+j=n} (d_i(X_{11}) + \tau_i^{(1)}(X_{11}))d_j(X_{11}^{-1}X_{12}) \\ &\quad - \sum_{i+j=n} d_j(X_{11}^{-1}X_{12})(d_i(X_{11}) + \tau_i^{(1)}(X_{11})) \\ &= \sum_{i+j=n} d_i(X_{11})d_j(X_{11}^{-1}X_{12}). \end{aligned}$$

Substituting $X_{11}X_{12}$ for X_{12} in the above equality gives

$$d_n(X_{11}X_{12}) = \sum_{i+j=n} d_i(X_{11})d_j(X_{12}).$$

For any general $X_{11} \in \mathcal{T}_{11}$, letting $rP_1 - X_{11}$ be invertible in $\mathcal{T}_{11}$, then

$$\begin{aligned} d_n((rP_1 - X_{11})X_{12}) &= \sum_{i+j=n} d_i(rP_1 - X_{11})d_j(X_{12}) \\ &= \sum_{i+j=n} (d_i(rP_1)d_j(X_{12}) - d_i(X_{11})d_j(X_{12})). \end{aligned}$$

Since $d_n(rP_1X_{12}) = \sum_{i+j=n} d_i(rP_1)d_j(X_{12})$, we arrive at

$$d_n(X_{11}X_{12}) = \sum_{i+j=n} d_i(X_{11})d_j(X_{12}).$$

We therefore assert that $d_n(X_{11}X_{12}) = \sum_{i+j=n} d_i(X_{11})d_j(X_{12})$ regardless of the invertibility of X_{11}.

Step 2 : Let us choose arbitrary elements $X_{12} \in \mathcal{T}_{12}$ and $X_{22} \in \mathcal{T}_{22}$. Note that $(P_1 + X_{12})(P_1 +$

$X_{22} - X_{12}X_{22}) = P_1 = P$. Thus we calculate

$$\begin{aligned}
d_n(X_{12}) &= f_n(X_{12}) \\
&= f_n([P_1 + X_{22} - X_{12}X_{22}, P_1 + X_{12}]) \\
&= \sum_{i+j=n} (f_i(P_1 + X_{22} - X_{12}X_{22})f_j(P_1 + X_{12}) \\
&\quad - f_j(P_1 + X_{12})f_i(P_1 + X_{22} - X_{12}X_{22})) \\
&= \sum_{i+j=n} ((f_i(P_1) + f_i(X_{22}) - f_i(X_{12}X_{22}))(f_j(P_1) + f_j(X_{12})) \\
&\quad - (f_j(P_1) + f_j(X_{12}))(f_i(P_1) + f_i(X_{22}) - f_i(X_{12}X_{22}))) \\
&= \sum_{i+j=n} (f_i(P_1)f_j(X_{12}) - f_j(X_{12})f_i(P_1)) \\
&\quad + \sum_{i+j=n} (f_i(X_{22})f_j(P_1) - f_j(P_1)f_i(X_{22})) \\
&\quad + \sum_{i+j=n} (f_i(X_{22})f_j(X_{12}) - f_j(X_{12})f_i(X_{22})) \\
&= P_1f_n(X_{12}) - f_n(X_{12})P_1 + f_n(X_{22})P_1 - P_1f_n(X_{22}) + P_1f_n(X_{12}X_{22}) \\
&\quad + \sum_{i+j=n} ((d_i(X_{22}) + \tau_i^{(2)}(X_{22}))d_j(X_{12}) - d_j(X_{12})(d_i(X_{22}) + \tau_i^{(2)}(X_{22}))) \\
&= P_1d_n(X_{12}) - d_n(X_{12})P_1 + (d_n(X_{22}) + \tau_n^{(2)}(X_{22}))P_1 \\
&\quad - P_1(d_n(X_{22}) + \tau_n^{(2)}(X_{22})) + P_1d_n(X_{12}X_{22}) - \sum_{i+j=n} d_i(X_{12})d_j(X_{22}) \\
&= d_n(X_{12}) + d_n(X_{12}X_{22}) - \sum_{i+j=n} d_i(X_{12})d_j(X_{22}).
\end{aligned}$$

This shows that

$$d_n(X_{12}X_{22}) = \sum_{i+j=n} d_i(X_{12})d_j(X_{22}).$$

Step 3 : For arbitrary elements $X_{11}, Y_{11} \in \mathcal{T}_{11}, X_{12} \in \mathcal{T}_{12}$, by **Step 1** it follows that

$$d_n(X_{11}Y_{11}X_{12}) = \sum_{l+k=n} d_l(X_{11}Y_{11})d_k(X_{12}).$$

On the other hand, by **Step 1** again we have

$$\begin{aligned}
d_n(X_{11}Y_{11}X_{12}) &= \sum_{i+j=n} d_i(X_{11})d_j(Y_{11}X_{12}) \\
&= \sum_{h+k=j, i+j=n} d_i(X_{11})d_h(Y_{11})d_k(X_{12}).
\end{aligned}$$

Comparing the above two equalities yields

$$\sum_{i+h=l,l+k=n}(d_l(X_{11}Y_{11})-d_i(X_{11})d_h(Y_{11}))d_k(X_{12})=0.$$

Let us check the coreectness of this step by induction n. By the proof of [21, Theorem 2.2] we know that

$$d_1(X_{11}Y_{11})=d_1(X_{11})Y_{11}+X_{11}d_1(Y_{11})$$

whenever $n=1$. Suppose that

$$d_l(X_{11}Y_{11})=\sum_{i+h=l}d_i(X_{11})d_h(Y_{11})$$

whenever $1\leq l<n$. We therefore arrive at

$$(d_n(X_{11}Y_{11})-\sum_{i+j=n}d_i(X_{11})d_j(Y_{11}))X_{12}=0.$$

Since $\mathcal{T}_{12}$ as a left $\mathcal{T}_{11}$-module is faithful, we get

$$d_n(X_{11}Y_{11})=\sum_{i+j=n}d_i(X_{11})d_j(Y_{11}).$$

By an analogous manner one can show that

$$d_n(X_{22}Y_{22})=\sum_{i+j=n}d_i(X_{22})d_j(Y_{22}).$$

Step 4 : For arbitrary elements $X,Y\in\mathcal{T}$, let us write them $X=X_{11}+X_{12}+X_{22},Y=Y_{11}+Y_{12}+Y_{22}$. Taking into account **Step** 1, **Step** 2 and **Step** 3, we calculate that

$$\begin{aligned}
d_n(XY)&=d_n(X_{11}Y_{11}+X_{11}Y_{12}+X_{12}Y_{22}+X_{22}Y_{22})\\
&=\sum_{i+j=n}(d_i(X_{11})d_j(Y_{11})+d_i(X_{11})d_j(Y_{12})\\
&\quad+d_i(X_{12})d_j(Y_{22})+d_i(X_{22})d_j(Y_{22}))\\
&=\sum_{i+j=n}(d_i(X_{11})d_j(Y_{11})+d_i(X_{11})d_j(Y_{12})+d_i(X_{11})d_j(Y_{22})\\
&\quad+d_i(X_{12})d_j(Y_{11})+d_i(X_{12})d_j(Y_{12})+d_i(X_{12})d_j(Y_{22})\\
&\quad+d_i(X_{22})d_j(Y_{11})+d_i(X_{22})d_j(Y_{12})+d_i(X_{22})d_j(Y_{22}))\\
&=\sum_{i+j=n}(d_i(X_{11})d_j(Y)+d_i(X_{12})d_j(Y)+d_i(X_{22})d_j(Y))\\
&=\sum_{i+j=n}d_i(X)d_j(Y).
\end{aligned}$$

This shows that $\{d_n\}_{n\in\mathbb{N}}$ is a higher derivation of $\mathcal{T}$.

Now we come to the last step of the proof of this theorem.

Claim 7 : τ_n vanishes on all the commutators $[X,Y]$ with $XY=P$ for all $X,Y\in\mathcal{T}$.

For any $X,Y\in\mathcal{T}$ with $XY=P$, by the definition of $d_n=f_n-\tau_n$ we have

$$\begin{aligned}\tau_n([X,Y]) &= f_n([X,Y])-d_n([X,Y])\\ &= \sum_{i+j=n}(f_i(X)f_j(Y)-f_j(Y)f_i(X))-d_n([X,Y])\\ &= \sum_{i+j=n}((d_i(X)+\tau_i(X))(d_j(Y)+\tau_j(Y)))\\ &\quad-\sum_{i+j=n}((d_j(Y)+\tau_j(Y))(d_i(X)+\tau_i(X)))-d_n([X,Y])\\ &= \sum_{i+j=n}(d_i(X)d_j(Y)-d_j(Y)d_i(X))-d_n([X,Y])\\ &= d_n(XY)-d_n(YX)-d_n(XY-YX)\\ &= 0.\Box\end{aligned}$$

Corollary 3.2 — Let $\mathcal{N}$ be an arbitrary nest on a complex Banach space $\mathcal{X}$ such that there is $N\in\mathcal{N}$ complemented in X and be $\mathrm{Alg}\mathcal{N}$ the nest algebra associated with $\mathcal{N}$. Let $\Delta=\{\delta_n\}_{n=0}^{\infty}$ be a family $\mathbb{F}$-linear mappings from $\mathrm{Alg}\mathcal{N}$ into itself such that

$$\delta_n([X,Y])=\sum_{i+j=n}(\delta_i(X)\delta_j(Y)-\delta_j(Y)\delta_i(X))$$

for all $X,Y\in\mathrm{Alg}\mathcal{N}$ with $XY=P$. Then there exist a higher derivation $D=\{d_n\}_{n=0}^{\infty}$ and an $\mathcal{R}$-linear mapping $\tau_n:\mathrm{Alg}\mathcal{N}\longrightarrow\mathbb{F}I$ vanishing on commutators $[X,Y]$ with $XY=P$ such that

$$\delta_n(X)=d_n(X)+\tau_n(X)$$

for all $X\in\mathrm{Alg}\mathcal{N}$.

In particular, we have

Corollary 3.3 — Let $\mathcal{N}$ be an arbitrary nest on a Hilbert space $\mathbf{H}$ with $\dim\mathbf{H}\geq 2$ and $\mathrm{Alg}\mathcal{N}$ be the nest algebra associated with $\mathcal{N}$. Let $\Delta=\{\delta_n\}_{n=0}^{\infty}$ be a family $\mathbb{F}$-linear mappings from $\mathrm{Alg}\mathcal{N}$ into itself such that

$$\delta_n([X,Y])=\sum_{i+j=n}(\delta_i(X)\delta_j(Y)-\delta_j(Y)\delta_i(X))$$

for all $X, Y \in \mathrm{Alg}\mathcal{N}$ with $XY = P$. Then there exist a higher derivation $D = \{d_n\}_{n=0}^{\infty}$ and an $\mathcal{R}$-linear mapping $\tau_n : \mathrm{Alg}\mathcal{N} \longrightarrow \mathbb{F}I$ vanishing on commutators $[X, Y]$ with $XY = P$ such that

$$\delta_n(X) = d_n(X) + \tau_n(X)$$

for all $X \in \mathrm{Alg}\mathcal{N}$.

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