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# EXACT CONTROLLABILITY OF MULTI-TERM TIME-FRACTIONAL DIFFERENTIAL SYSTEM WITH SEQUENCING TECHNIQUES

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In this paper, an abstract multi-term time-fractional differential system is considered and the existence, uniqueness and exact controllability results are investigated. In this theory, we use the notion of bounded integral contractor introduced by Altman to come up with a new set of sufficient conditions for the exact controllability by constructing a sequencing technique. Moreover, in this technique, we are not required to define induced inverse operator and Lipschitz continuity of nonlinear functions. Finally, an application is given to illustrate the obtained results.

**Key words** : Exact controllability; multi-term time-fractional differential system; $(\beta, \gamma_j)$-resolvent family; bounded integral contractor type operator.

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## 1. Introduction

In the last few decades, the theory of fractional differential systems has attracted the interest of many researchers towards itself due to numerous applications in widespread areas of science and engineering such as in models of medicine, electrical engineering, biochemistry and for more applications, see [3, 7, 10, 23]. The fractional differential equations of arbitrary order are capable to describe the dynamical behavior of a real life phenomenon more precisely. In addition, due to the memory and hereditary properties, in many areas of science like as identification systems, signal processing and control theory, fractional differential operators seem more appropriate in modeling than the classical integer operators. For recent development in this field, one can see the cited papers [2, 4, 11, 13, 14, 18-20], the monographs [10, 15, 25, 28] and references therein.

On the other hand, the theory of controllability is well developed for the control problems described as abstract differential equations using different approaches in finite and infinite dimensional spaces. The notion of controllability plays a crucial role in many control problems in both deterministic and stochastic control theories, such as stabilization of unstable systems by feedback etc. Introduced by Kalman [12], the theory of controllability started at the beginning of the sixties. For more details, see the papers [2, 4, 13, 16, 17, 20-22, 24] and references therein. Triggiani [29] studied the exact controllability by assuming that the controllability operator have an induced inverse on a quotient space. However, if the controllability operator or semigroup associated with the control system is compact, then the controllability operator is also compact and hence the induced inverse does not exists [29]. Thus, the existence of induced inverse in study of exact controllability is a strong condition.

Recently, multi-term time-fractional differential systems are generating a great interest among the mathematicians and engineers. For instance, in the papers [14, 19, 30] a two-term time-fractional differential system is studied in the abstract context, which include a concrete example of fractional diffusion-wave problems. Moreover, for multi-term time-fractional diffusion systems in [11, 18] authors studied analytic and numerical solutions, and in [27] Pardo and Lizama studied the existence of mild solutions. However, in the best of our knowledge, no work has been reported for the exact controllability of multi-term time-fractional differential systems in the literature.

Motivated by the above facts, in this paper, we study the existence, uniqueness and exact controllability results for the following abstract multi-term time-fractional differential system with the notion of bounded integral contractor by constructing a sequencing technique.

$$\begin{aligned} {}^cD^{1+\beta}y(t) \quad + \quad & \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} y(t) = Ay(t) + Bu(t) + F(t, y(t)), \qquad t \in (0, T], \\ & y(0) = \varphi \in \mathbb{X}, \qquad y'(0) = \chi \in \mathbb{X}, \end{aligned} \tag{1.1}$$

where $\alpha_j \geq 0$ and $\gamma_j$ are positive real numbers for all $j = 1, 2, ..., n$ such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. The state $y(\cdot)$ takes values in the Banach space $\mathbb{X}$ equipped with norm $\|\cdot\|$ and the control function $u(\cdot)$ belongs to the space $L^2([0, T], \mathbb{U})$, a Banach space of admissible control functions with $\mathbb{U}$ as a Banach space and $B : L^2([0, T], \mathbb{U}) \to L^2([0, T], \mathbb{X})$ is a bounded linear operator. We assume that $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator which generates a strongly continuous cosine family. ${}^cD^{\eta}$ stands for the Caputo fractional derivative of order $\eta > 0$.

In the sequencing technique, existence of induced inverse of controllability operator and Lipschitz continuity of nonlinear functions are not required. The concept of bounded integral contractor introduced by Altman [1]. Later, this concept was used to establish the existence and uniqueness of solutions for nonlinear evolution equations [26]. In [2, 5, 6], the authors have obtained some controllability results using the concept of bounded integral contractor.

The work of this paper is organized as follows. Section $2$ is devoted to recall some basic concepts of fractional calculus and formulation of the mild solution. In Section $3$ and $4$, the existence, uniqueness and exact controllability results are obtained by utilization of a sequencing technique. In Section $5$, an example is provided to show the feasibility of the theory discussed in this paper.

## 2. Preliminaries

We denote the space of bounded and linear operators, from $\mathbb{X}$ to $\mathbb{U}$, by $\mathcal{L}(\mathbb{X}, \mathbb{U})$ endowed with uniform operator topology and the notation abbreviated by $\mathcal{L}(\mathbb{X})$ when $\mathbb{X} = \mathbb{U}$. Moreover, we denote by $\mathcal{C}([0,T], \mathbb{X})$ the Banach space of all continuous functions form $[0,T]$ to $\mathbb{X}$ with the norm $\|y\|_{\mathcal{C}} = \sup_{t\in[0,T]}\{\|y(t)\| : y \in \mathbb{X}\}$. Let $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real numbers and natural numbers, respectively. For a linear operator $A$ on $\mathbb{X}$, $\mathcal{R}(A), \mathcal{D}(A)$ and $\rho(A)$ represent the range, domain and resolvent of $A$, respectively.

To give an appropriate representation of mild solution in terms of certain family of bounded and linear operators, we define following family of operators.

*Definition* 2.1 — [27]. Let $A$ be a closed linear operator on a Banach space $\mathbb{X}$ with the domain $\mathcal{D}(A)$ and let $\beta > 0, \gamma_j, \alpha_j,\ j = 1, 2, .., n$ be the real positive numbers. Then $A$ is called the generator of a $(\beta, \gamma_j)$-resolvent family if there exists $\omega \geq 0$ and a strongly continuous function $\mathcal{S}_{\beta,\gamma_j} : \mathbb{R}^+ \to \mathcal{L}(\mathbb{X})$ such that $\{\lambda^{\beta+1} + \sum_{j=1}^{n} \alpha_j \lambda^{\gamma_j} : \mathrm{Re}\,\lambda > \omega\} \subset \rho(A)$ and

$$\lambda^{\beta}\left(\lambda^{\beta+1} + \sum_{j=1}^{n} \alpha_j \lambda^{\gamma_j} - A\right)^{-1} y = \int_0^\infty e^{-\lambda t}\mathcal{S}_{\beta,\gamma_j}(t) y dt, \quad \mathrm{Re}\,\lambda > \omega, y \in \mathbb{X}. \tag{2.1}$$

The following result guarantees the existence of $(\beta, \gamma_j)$-resolvent family under some suitable conditions.

***Theorem* 2.2** — *[27]. Let* $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ *and* $\alpha_j \geq 0,\ j = 1, 2, ..., n$ *be given and let* $A$ *be a generator of a bounded and strongly continuous cosine family* $\{C(t)\}_{t\in\mathbb{R}}$*. Then,* $A$ *generates a bounded* $(\beta, \gamma_j)$*-resolvent family* $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$.

Now, we recall some definitions and basic results on fractional calculus (for more details see [27, 28]). Define $g_\eta(t)$ for $\eta > 0$ by

$$g_\eta(t) = \begin{cases} \frac{1}{\Gamma(\eta)} t^{\eta-1}, & t > 0; \\ 0, & t \leq 0, \end{cases}$$

where $\Gamma$ denotes the gamma function. The function $g_\eta$ satisfies the properties $(g_a * g_b)(t) = g_{a+b}(t)$, for $a, b > 0$ and $\widehat{g_\eta}(\lambda) = \frac{1}{\lambda^\eta}$ for $\eta > 0$ and Re $\lambda > 0$, where $\widehat{(\cdot)}$ and $(\cdot * \cdot)(\cdot)$ denote the Laplace transformation and convolution, respectively.

*Definition* 2.3 — The Riemann-Liouville fractional integral of a function $f \in L^1_{loc}([0,\infty), \mathbb{X})$ of order $\eta > 0$ with lower limit zero is defined as follows

$$I^\eta f(t) = (g_\eta * f)(t) = \int_0^t g_\eta(t-s) f(s) ds, \quad t > 0,$$

and $I^0 f(t) := f(t)$, provided that right side integral is point-wise defined in $[0, \infty)$.

This fractional integral satisfies the properties $I^\eta \circ I^b = I^{\eta+b}$ for $b > 0$ and $\widehat{I^\eta f}(\lambda) = \frac{1}{\lambda^\eta} \widehat{f}(\lambda)$ for Re $\lambda > 0$.

*Definition* 2.4 — Let $\eta > 0$ be given and denote $m = \lceil \eta \rceil$. The Caputo fractional derivative of order $\eta > 0$ of a function $f \in \mathcal{C}^m([0,\infty), \mathbb{X})$ with lower limit zero is given by

$${}^cD^\eta f(t) = I^{m-\eta} D^m f(t) = \int_0^t g_{m-\eta}(t-s) D^m f(s) ds,$$

and ${}^cD^0 f(t) := f(t)$, where $D^m = \frac{d^m}{dt^m}$. In addition, we have ${}^cD^\eta f(t) = (g_{m-\eta} * D^m f)(t)$ and the Laplace transformation of Caputo fractional derivative is given by

$$\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda) - \sum_{i=0}^{m-1} f^{(i)}(0) \lambda^{\eta-1-i}, \quad \lambda > 0. \tag{2.2}$$

and

$$(I^\eta \circ {}^cD^\eta) f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0) g_{i+1}(t), \quad t > 0. \tag{2.3}$$

*Remark* 2.5 : If $f^{(i)}(0) = 0$, for $i = 1, 2, 3, \ldots m-1$, then $(I^\eta \circ {}^cD^\eta) f(t) = f(t)$ and $\widehat{{}^cD^\eta f}(t) = \lambda^\eta \widehat{f}(\lambda)$.

Now, consider the initial value problem

$${}^cD^{1+\beta} y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} y(t) = Ay(t) + f(t), \quad t \in (0, T], \tag{2.4}$$

$$y(0) = p, \quad y'(0) = q, \tag{2.5}$$

where $A$ is a generator of a bounded and strongly continuous cosine family, $p, q \in \mathbb{X}$, $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$, $\alpha_j \geq 0,\ j = 1, 2, ..., n$, and $f$ is Hölder continuous.

With the aim to construct mild solution representation for the problem (2.4)-(2.5) in the term of the family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$, we apply the Laplace transformation on the both sides of (2.4), then we obtain

$$\lambda^{1+\beta}\widehat{y}(\lambda) - \sum_{i=0}^{\lceil 1+\beta\rceil-1} y^{(i)}(0)\lambda^{\beta-i} + \sum_{j=1}^{n}\alpha_j\Big[\lambda^{\gamma_j}\widehat{y}(\lambda) - \sum_{i=0}^{\lceil\gamma_j\rceil-1} y^{(i)}(0)\lambda^{\gamma_j-1-i}\Big] = A\widehat{y}(\lambda) + \widehat{f}(\lambda).$$

Using initial data given by (2.5), we have

$$\lambda^{1+\beta}\widehat{y}(\lambda) - \lambda^{\beta}p - \lambda^{\beta-1}q + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j}\widehat{y}(\lambda) - \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j-1}p = A\widehat{y}(\lambda) + \widehat{f}(\lambda).$$

This is equivalent to

$$\Big(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\Big)\widehat{y}(\lambda) = \lambda^{\beta}p + \lambda^{\beta-1}q + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j-1}p + \widehat{f}(\lambda).$$

Now, by Theorem 2.2 assuming the existence of $(\beta,\gamma_j)$- resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$, we have

$$\begin{aligned}\widehat{y}(\lambda) =& \lambda^{\beta}\Big(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\Big)^{-1}p + \lambda^{\beta-1}\Big(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\Big)^{-1}q\\ &+ \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j-1}\Big(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\Big)^{-1}p + \Big(\lambda^{1+\beta} + \sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} - A\Big)^{-1}\widehat{f}(\lambda).\end{aligned}$$

By inverse Laplace transformation, we obtain

$$\begin{aligned}y(t) =& \mathcal{S}_{\beta,\gamma_j}(t)p + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)q + \sum_{j=1}^{n}\alpha_j\int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)p\,ds\\ &+ \int_0^t (g_\beta * \mathcal{S}_{\beta,\gamma_j})(t-s)f(s)ds.\end{aligned}$$

The above representation, allows us to define the mild solution for the system (1.1).

*Definition* 2.6 — Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0,\ j = 1, 2, ..., n$ be given and let $A$ be a generator of a bounded $(\beta,\gamma_j)$-resolvent family $\{\mathcal{S}_{\beta,\gamma_j}(t)\}_{t\geq 0}$. Then a function $y \in \mathcal{C}([0,T],\mathbb{X})$

is called a mild solution of the system (1.1) if $y(0)=\varphi$ and $y'(0)=\chi$ and satisfies the equation

$$y(t) = \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n}\alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Big[Bu(s)+F(s,y(s))\Big]ds, \tag{2.6}$$

where $\mathcal{T}_{\beta,\gamma_j}(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\mathcal{S}_{\beta,\gamma_j}(s)ds$.

We denote by $y_t(\varphi,\chi,u)$ the state value of system (1.1) corresponding to the control $u$ at the time $t$, In particular, at time $T$ the state $y_T(\varphi,\chi,u)$ of the system (1.1) is known at the terminal state with control $u$. The set $\mathfrak{R}_T(F,\varphi,\chi) := \{y_T(\varphi,\chi,u) : u(\cdot) \in L^2([0,T],\mathbb{U})\}$ is called the reachable set of the system (1.1) corresponding to the nonlinear function $F$, and we denote by $\mathfrak{R}_T(0,\varphi,\chi)$ the reachable set corresponding to linear system.

*Definition* 2.7 — The system (1.1) is said to be exact controllable on the time interval $[0,T]$ if and only if $\mathfrak{R}_T(F,\varphi,\chi) = \mathbb{X}$.

In other words, we say that the system (1.1) is exact controllable on the time interval $[0,T]$ if and only if for any final state $y_T \in \mathbb{X}$, there exists a control function $u \in L^2([0,T],\mathbb{U})$ such that the mild solution $y(t)$ of the system (1.1) satisfies $y(0)=\varphi$ and $y(T)=y_T$.

*Definition* 2.8 — [6]. Let $\Pi : [0,T]\times\mathbb{X} \to \mathcal{L}(\mathbb{X})$ be a bounded operator and there exists a constant $\delta > 0$ such that

$$\left\| F\left(t, y(t)+z(t) + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s,y(s))z(s)ds\right) - F(t,y(t)) - \Pi(t,y(t))z(t)\right\| \le \delta\|z(t)\|, \tag{2.7}$$

for all $y,z \in \mathcal{C}([0,T],\mathbb{X})$ and $t \in [0,T]$. Then, we say that $F(t,y(t))$ has a bounded integral contractor $\Pi$ with respect to $\mathcal{T}_{\beta,\gamma_j}(t)$.

*Remark* 2.9 : The Lipschitz condition follows by (2.7) in case $\Pi \equiv 0$ i.e.

$$\|F(t,y(t)+z(t)) - F(t,y(t))\| \le \delta\|z(t)\|. \tag{2.8}$$

As we are aware that unique mild solutions may be obtained for the given system (1.1) by using Lipschitz continuity of nonlinear function $F$, but the condition (2.7) does not ensure the uniqueness of the mild solution of the system (1.1). The uniqueness of mild solution for the given system is acquired by regularity of integral contractor operator [6].

*Definition* 2.10 — A bounded integral contractor $\Pi$ is said to be regular if the integral equation

$$x(t) = z(t) + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s, y(s))z(s)ds \tag{2.9}$$

admits a solution $z \in \mathcal{C}([0,T], \mathbb{X})$ for every $x, y \in \mathcal{C}([0,T], \mathbb{X})$.

*Remark* 2.11 : We observe that if $F(t, y(t))$ is uniformly Lipschitz continuous for all $t$, then it has a regular integral contractor $\Pi \equiv 0$. For other existence conditions of a bounded integral contractor for $F(t, y(t))$, see Altman [1] and Govindan [8].

*Lemma* 2.12 — [9]. (Generalized Gronwall's inequality). Let $a \geq 0$, $\beta > 0$, $c(t)$ and $u(t)$ be the nonnegative locally integrable functions on $0 \leq t < T < +\infty$, such that

$$u(t) \leq c(t) + a\int_0^t (t-s)^{\beta-1}u(s)ds,$$

then

$$u(t) \leq c(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(a\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}c(s)\right]ds, \quad 0 < t \leq T.$$

## 3. EXISTENCE AND UNIQUENESS RESULTS

In this section, the existence and uniqueness results for the system (1.1) are established under the following assumption:

$(A_1)$: The nonlinear function $F(t, y(t))$ has a regular integral contractor.

For brevity of notations, we denote $S_0 = \sup_{t\in[0,T]} \|\mathcal{S}_{\beta,\gamma_j}(t)\|_{\mathcal{L}}$ and $\sup_{s\in[0,T],\ y\in\mathcal{C}([0,T],\mathbb{X})} \|\Pi(s, y(s))\|_{\mathcal{L}} = \sigma$. Moreover, we have $\|\mathcal{T}_{\beta,\gamma_j}(t)y\|_{\mathcal{L}} = \frac{S_0 t^{\beta}}{\Gamma(1+\beta)}\|y\|$.

***Theorem* 3.1** — *Let* $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ *and* $\alpha_j \geq 0$, $j = 1, 2, ..., n$ *be given and the assumption* $(A_1)$ *holds. Then, the multi-term time-fractional differential system (1.1) has a unique mild solution.*

PROOF : Existence of mild solution : To apply the iterative technique, we consider the sequences $\{y_n\}$ and $\{z_n\}$ in $\mathcal{C}([0,T], \mathbb{X})$. For $n = 0, 1, 2, 3, ...,$ we define

$$\begin{aligned} y_0(t) =& \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)Bu(s)ds, \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
z_n(t) =& y_n(t) - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s,y_n(s))ds - y_0(t), \qquad (3.2)\\
y_{n+1}(t) =& y_n(t) - \left[z_n(t) + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s,y_n(s))z_n(s)ds\right]\\
=& \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s,y_n(s))ds - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s,y_n(s))z_n(s)ds + y_0(t). \qquad (3.3)
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
z_{n+1}(t) =& y_{n+1}(t) - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s,y_{n+1}(s))ds - y_0(t),\\
=& \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s,y_n(s))ds - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s,y_n(s))z_n(s)ds\\
&- \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\left[F\left(s, y_n(t) - z_n(t) - \int_0^s \mathcal{T}_{\beta,\gamma_j}(s-\xi)\Pi(\xi,y_n(\xi))z_n(\xi)d\xi\right)\right]ds.
\end{aligned}
$$

Assuming as $z_n = -z$ and $y_n = y$ in the above expression and using the Definition 2.7, we obtain

$$
\begin{aligned}
\|z_{n+1}(t)\| \leq& \frac{\delta S_0 T^\beta}{\Gamma(1+\beta)}\int_0^t \|z_n(s)\|ds\\
\leq& \left(\frac{\delta S_0 T^\beta}{\Gamma(1+\beta)}\right)\int_0^t \left(\frac{\delta S_0 T^\beta}{\Gamma(1+\beta)}\right)\left(\int_0^s \|z_{n-1}(s_1)\|ds_1\right)ds\\
\leq& \left(\frac{\delta S_0 T^\beta}{\Gamma(1+\beta)}\right)^2\left(\int_0^t\int_0^s \|z_{n-1}(s_1)\|ds_1 ds\right)\\
\leq& \left(\frac{\delta S_0 T^\beta}{\Gamma(1+\beta)}\right)^3\left(\int_0^t\int_0^s\int_0^{s_1} \|z_{n-2}(s_2)\|ds_2 ds_1 ds\right)\\
\leq& \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
\leq& \left(\frac{\delta S_0 T^\beta}{\Gamma(1+\beta)}\right)^{n+1}\left(\int_0^t\int_0^s\int_0^{s_1}\cdots\int_0^{s_{n-1}} \|z_0(s_n)\|ds_n\cdots ds_2 ds_1 ds\right).
\end{aligned}
$$

Now, using the following integral relation

$$
\int_a^t\int_a^{s_1}\cdots\int_a^{s_{n-1}} f(s_n)ds_n\cdots ds_2 ds_1 = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}f(s)ds, \ a\in\mathfrak{R},
$$

we have

$$
\begin{aligned}
\|z_{n+1}(t)\| \leq& \left(\frac{\delta S_0 T^\beta}{\Gamma(1+\beta)}\right)^{n+1}\frac{1}{n!}\int_0^t (t-s)^n\|z_0(s)\|ds\\
\leq& \left(\frac{\delta S_0 T^{\beta+1}}{\Gamma(1+\beta)}\right)^{n+1}\frac{1}{(n+1)!}\|z_0\|_{\mathcal{C}}.
\end{aligned}
$$

This implies that $\{z_n(t)\}$ converges to zero as $n \to \infty$ in $\mathbb{X}$.

Further, we show that the sequence $\{y_n\}$ converges to the a solution of the system (1.1). From the constructed sequences, we have

$$y_{n+1}(t) - y_n(t) = -z_n(t) - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s, y_n(s))z_n(s)ds.$$

Taking norm on both sides, we have

$$\begin{aligned}\|y_{n+1}(t) - y_n(t)\| \leq & \|z_n(t)\| + \left\| \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s, y_n(s))z_n(s)ds \right\| \\ \leq & \left(\frac{\delta S_0 T^{\beta+1}}{\Gamma(1+\beta)}\right)^n \frac{1}{(n)!}\|z_0\|_{\mathcal{C}} + \left(\frac{\delta S_0 T^{\beta+1}}{\Gamma(1+\beta)}\right)^n \frac{1}{(n)!}\frac{\sigma S_0 T^{1+\beta}}{\Gamma(2+\beta)}\|z_0\|_{\mathcal{C}} \\ \leq & \left(\frac{\delta S_0 T^{\beta+1}}{\Gamma(1+\beta)}\right)^n \frac{1}{(n)!}\left[1 + \frac{\sigma S_0 T^{1+\beta}}{\Gamma(2+\beta)}\right]\|z_0\|_{\mathcal{C}}.\end{aligned}$$

Now, for a fixed positive integer $m$ such that $m < n$, following the above procedure, we conclude

$$\begin{aligned}\|y_n(t) - y_m(t)\| \leq & \|y_n(t) - y_{n-1}(t)\| + \|y_{n-1}(t) - y_{n-2}(t)\| + \cdots + \|y_{m+1}(t) - y_m(t)\| \\ \leq & \left[1 + \frac{\sigma S_0 T^{1+\beta}}{\Gamma(2+\beta)}\right]\|z_0\|_{\mathcal{C}} \sum_{k=m}^{n-1} \frac{1}{(k)!}\left(\frac{\delta S_0 T^{\beta+1}}{\Gamma(1+\beta)}\right)^k.\end{aligned}$$

It is straightforward by right hand side of above inequality that the sequence $\{y_n\}$ is a Cauchy sequence in $\mathcal{C}([0,T],\mathbb{X})$. Therefore, being a Cauchy sequence $\{y_n\}$ will converges uniformly to $y^*$ (say) in $\mathcal{C}([0,T],\mathbb{X})$. Hence, we obtain from (3.2) that

$$\begin{aligned}\lim_{n\to\infty} z_n(t) = & \lim_{n\to\infty} y_n(t) - \lim_{n\to\infty} \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y_n(s))ds - y_0(t) \\ 0 = & y^*(t) - \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)F(s, y^*(s))ds - y_0(t).\end{aligned}$$

Putting the value of $y_0(t)$, we have

$$\begin{aligned}y^*(t) = & \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^n \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)}\mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ & + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Big[Bu(s) + F(s, y^*(s))\Big]ds.\end{aligned}$$

Thus, $y^*$ is a mild solution of the system (1.1).

*Uniqueness of mild solution*

We establish the uniqueness of a mild solution with the regularity of the integral contractor type operator. Let us assume that $y_1$ and $y_2$ be two mild solutions of the system (1.1) for a given control $u \in L^2([0,T],\mathbb{U})$. By using the regularity condition (2.9) with $y = y_1$ and $x = y_2 - y_1$, there exists a $z \in L^2([0,T],\mathbb{X})$ such that

$$z(t) + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s,y_1(s))z(s)ds = y_2(t) - y_1(t). \tag{3.4}$$

Hence

$$y_2(t) = y_1(t) + z(t) + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s,y_1(s))z(s)ds.$$

Since $y_1$ and $y_2$ are two mild solutions of the system (1.1) for a given control $u \in L^2([0,T],\mathbb{U})$, so by (2.6), we have

$$\begin{aligned} y_2(t) - y_1(t) &= \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)[F(s,y_2(s)) - F(s,y_1(s))]ds, \\ &= \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Big[F\Big(s, y_1(s) + z(s) + \int_0^s \mathcal{T}_{\beta,\gamma_j}(s-\xi)\Pi(\xi,y_1(\xi))z(\xi)d\xi\Big) \\ &\quad - F(s,y_1(s)) - \Pi(s,y_1(s))z(s)\Big]ds + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s,y_1(s))z(s)ds. \end{aligned} \tag{3.5}$$

Now, from (3.4) and (3.5), we have

$$\begin{aligned} z(t) = &\int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Big[F\Big(s, y_1(s) + z(s) + \int_0^s \mathcal{T}_{\beta,\gamma_j}(s-\xi)\Pi(\xi,y_1(\xi))z(\xi)d\xi\Big) \\ &- F(s,y_1(s)) - \Pi(s,y_1(s))z(s)\Big]ds. \end{aligned}$$

Using the definition of integral contractor with $y = y_1$ after taking norm on both sides, we have

$$\begin{aligned} \|z(t)\| = &\int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|_{\mathcal{L}}\Big\|F\Big(s, y_1(s) + z(s) + \int_0^s \mathcal{T}_{\beta,\gamma_j}(s-\xi)\Pi(\xi,y_1(\xi))z(\xi)d\xi\Big) \\ &- F(s,y_1(s)) - \Pi(s,y_1(s))z(s)\Big\|ds \\ \leq &\delta \int_0^t \|\mathcal{T}_{\beta,\gamma_j}(t-s)\|_{\mathcal{L}}\|z(s)\|ds \\ \leq &\frac{\delta S_0}{\Gamma(1+\beta)} \int_0^t (t-s)^{(\beta+1)-1}\|z(s)\|ds. \end{aligned}$$

Now, applying Generalized Gronwall's inequality (Lemma 2.12), we obtain that $z = 0$. Therefore from (3.4), we have $y_1 = y_2$ which shows the uniqueness of mild solution of the system (1.1). This completes the proof of the theorem.

## 4. Controllability Results

In this section, we will establish the exact controllability results for the system (1.1). So, in order to obtain the results, we consider the following assumptions

$(A_2)$ The linear system

$$\begin{aligned} {}^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} y(t) &= Ay(t) + Bv(t), \quad t \in (0,T], \\ y(0) = \varphi, \qquad y'(0) &= \chi, \end{aligned} \tag{4.1}$$

corresponding to the system (1.1) is exact controllable with control $v$.

$(A_3)$ $\mathcal{R}(F) \subseteq \mathcal{R}(B)$.

***Theorem* 4.1** — *Let $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$ and $\alpha_j \geq 0$, $j = 1, 2, ..., n$ be given and the assumptions $(A_1) - (A_3)$ hold. Then, the multi-term time-fractional differential systems (1.1) is exactly controllable.*

PROOF : *Consider the multi-term time-fractional linear differential system*

$$\begin{aligned} {}^cD^{1+\beta}z(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} z(t) &= Az(t) + Bv(t), \quad t \in (0,T], \\ z(0) = \mu = \varphi, \qquad z'(0) &= \nu = \chi. \end{aligned} \tag{4.2}$$

The mild solution of (4.2) is defined by

$$\begin{aligned} z(t) =& \mathcal{S}_{\beta,\gamma_j}(t)\mu + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\nu + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\mu ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)Bv(s)ds. \end{aligned} \tag{4.3}$$

Let $z(t)$ be the mild solution of (4.3) system with control $v$. Now, consider the perturbed semilinear system

$$\begin{aligned} {}^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j {}^cD^{\gamma_j} y(t) &= Ay(t) + F(t, y(t)) + Bv(t) \\ &\quad - F\bigg(t, z(t) + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s, y(s))(z-y)(s)ds\bigg), \\ y(0) = \varphi = \mu, \qquad y'(0) = \chi = \nu. \end{aligned} \tag{4.4}$$

On comparing the system (1.1) and the system (4.4), we have

$$Bu(t) = Bv(t) - F\bigg(t, z(s) + \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s, y(s))(z-y)(s)ds\bigg). \tag{4.5}$$

The above relation hold due to the assumptions $(A_3)$.

Mild solution of the system (4.4) is given by

$$\begin{aligned} y(t) =& \mathcal{S}_{\beta,\gamma_j}(t)\varphi + (g_1 * \mathcal{S}_{\beta,\gamma_j})(t)\chi + \sum_{j=1}^{n} \alpha_j \int_0^t \frac{(t-s)^{\beta-\gamma_j}}{\Gamma(1+\beta-\gamma_j)} \mathcal{S}_{\beta,\gamma_j}(s)\varphi ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Big[Bv(s) + F(s, y(s))\Big]ds \\ &- \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\bigg[F\bigg(s, z(s) + \int_0^s \mathcal{T}_{\beta,\gamma_j}(s-\xi)\Pi(\xi, y(\xi))(z-y)(\xi)d\xi\bigg)\bigg]ds. \end{aligned} \tag{4.6}$$

From (4.3) and (4.6) we have

$$\begin{aligned} z(t) - y(t) =& \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\bigg[F\bigg(s, z(s) + \int_0^s \mathcal{T}_{\beta,\gamma_j}(s-\xi)\Pi(\xi, y(\xi))(z-y)(\xi)d\xi\bigg) \\ &- F(s, y(s)) - \Pi(s, y(s))(z-y)(s)\bigg]ds \\ &+ \int_0^t \mathcal{T}_{\beta,\gamma_j}(t-s)\Pi(s, y(s))(z-y)(s)ds. \end{aligned}$$

Taking norm on both sides and using the definition of integral contractor, we have

$$\begin{aligned} \|z(t) - y(t)\| \leq& \frac{\delta S_0}{\Gamma(1+\beta)} \int_0^t (t-s)^{\beta} \|z(s) - y(s)\|_{\mathcal{C}} ds \\ &+ \frac{\sigma S_0}{\Gamma(1+\beta)} \int_0^t (t-s)^{\beta} \|z(s) - y(s)\| ds \\ \leq& \frac{(\sigma+\delta) S_0}{\Gamma(1+\beta)} \int_0^t (t-s)^{(\beta+1)-1} \|z(s) - y(s)\| ds. \end{aligned}$$

Now, using the Generalized Gronwall's inequality (Lemma 2.12), we conclude that $\|z - y\|_{\mathcal{C}} = 0$ i.e. $z(t) = y(t)$ for all $t \in [0, T]$. Therefore, the equation implies that

$$Bu(t) = Bv(t) - F(t, y(t)). \tag{4.7}$$

This is known as feedback control.

Since every mild solution of the linear system (4.1) with control $v$ is also a mild solution of the perturbed semilinear system (4.4) with $u$ which implies that $\Re_T(0, \varphi, \chi) \subseteq \Re_T(F, \varphi, \chi)$. Exact controllability of linear system (4.1) (by $(A_2)$) follows that $\Re_T(0, \varphi, \chi)$ is equal to the whole space $X$. Hence $\Re_T(F, \varphi, \chi)$ is also equal to whole space $\mathbb{X}$. Therefore, the multi-term time-fractional semilinear differential system (1.1) is exact controllable. □

## 5. EXAMPLE

The fractional order diffusion wave equations have great applications in varies fields of science and engineering. These equations represent propagation of mechanical waves through viscoelastic media, charge transport in amorphous semiconductors [7, 10, 23], and may be used in thermodynamics and the flow of fluid through fissured rocks [3].

We provide a concrete example to illustrate the applicability of the established results. Let $\beta, \gamma_j > 0$, $j = 1, 2, 3, \ldots, n$ be given such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$. We consider the space $\mathbb{X} = L^2([0, \pi])$. We consider the following system

$$^{c}D^{1+\beta} z(t,x) + \sum_{j=1}^{n} \alpha_j {}^{c}D^{\gamma_j} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + \mu(t,x) + f(t, z(t,x)), \tag{5.1}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in [0,T], \tag{5.2}$$

$$z(0,x) = a_0, \ 0 \leq x \leq \pi, \qquad \frac{\partial u(t,x)}{\partial t}|_{t=0} = b_0, \tag{5.3}$$

where $t \in [0, T]$, $a_0, b_0 \in \mathbb{X}$. The arbitrary nonlinear function $f$ satisfies regular integral contractor conditions. The function $\mu(t, x) : [0, T] \times [0, \pi] \to \Re$ is continuous in $t$. Define a operator $A : \mathcal{D}(A) \subset \mathbb{X} \to \mathbb{X}$ by

$$Aw = w'', \quad w \in \mathcal{D}(A),$$

where $\mathcal{D}(A) := \{w \in \mathbb{X} : w, w'$ are absolutely continuous, $w'' \in \mathbb{X}, w(0) = w(\pi) = 0\}$.

Then the operator $A$ has spectral representation given by

$$Aw = \sum_{n=1}^{\infty} -n^2 (w, w_n) w_n,$$

where $w_n(t) = (\sqrt{2/\pi}) \sin nt$, $n = 1, 2, \ldots$, is the orthogonal set of eigenfunctions corresponding to the eigenvalues $\lambda_n = -n^2$ of $A$. Then $A$ will be a generator of cosine family such that

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, w_n)w_n,$$

Thus $A$ generates a strongly continuous cosine family. Then, for $\beta, \gamma_j > 0$, $j = 1, 2, 3, \ldots, n$ such that $0 < \beta \leq \gamma_n \leq \cdots \leq \gamma_1 \leq 1$, by Theorem 2.2, we conclude that $A$ generates a bounded $(\beta, \gamma_j)-$ resolvent family

$$\mathcal{S}_{\beta,\gamma_j}(t)y = \int_0^{\infty} \frac{1}{t^{\frac{(1+\beta)}{2}}} \Phi_{\frac{(1+\beta)}{2}}(st^{-\frac{(1+\beta)}{2}})C(s)yds, \quad y \in \mathbb{X},\ t \in [0, T],$$

where

$$\Phi_{\frac{(1+\beta)}{2}}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-(\beta(n+1)) - n)}, z \in \mathcal{C},$$

is the Wright functions. Put $y(t)(x) = z(t, x)$, $\mu(t)(x) = \mu(t, x)$ where $\mu(t, x) : [0, T] \times [0, \pi] \to [0, \pi]$ is continuous function then $f(t, z(t, x)) = f(t, y(t))$ and $Ay(t) = \frac{\partial^2}{\partial x^2} z(t, x)$. Let the control operator $Bu : [0, T] \to \mathbb{X}$ defined by

$$(Bu)(t)(x) = \mu(t, x), \quad x \in [0, \pi].$$

Then the system (5.1)-(5.3) has the abstract form of the system

$$^cD^{1+\beta}y(t) + \sum_{j=1}^{n} \alpha_j{}^cD^{\gamma_j}y(t) = Ay(t) + Bu(t) + f(t, y(t)), \quad t \in (0, T], \tag{5.4}$$

$$y(0) = a_0, \qquad y'(0) = b_0. \tag{5.5}$$

If we take function $f(t, y(t)) := \frac{e^t}{1+e^t} \sin(y(t))$, then $f$ has a regular integral contractor $\Pi \equiv 0$ and $\mathcal{R}(F) \subseteq \mathcal{R}(B)$. Therefore, by Theorem 3.1 the system (5.1)-(5.3) has unique mild solution. Further, if the corresponding linear system of the system (5.1)-(5.3) is exact controllable, then by Theorem 4.1 the system (5.1)-(5.3) is exact controllable on $[0, T]$.

## 6. Conclusion

In this paper, a sequencing technique has been developed to obtain existence, uniqueness and exact controllability results for the multi-term time-fractional differential systems (1.1). In this approach, we assume that a compatible bounded regular integral contractor type operator exists for the system which is a weaker condition than the Lipschitz condition. First, we obtained unique mild solution by generalizing the semigroup as $(\beta, \gamma_j)$- resolvent family. Further, exact controllability results are established for the system with the exact controllability of corresponding linear system. By adapting the ideas developed in this paper, one may establish exact controllability results with impulsive effects which are very effective in study of a phenomenon with discontinuous jumps.

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