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INFINITELY MANY HIGH ENERGY SOLUTIONS FOR A FOURTH-ORDER EQUATIONS OF KIRCHHOFF TYPE IN $\mathbb{R}^N$

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In this paper we study the following fourth-order elliptic equations of Kirchhoff type

$$\Delta^2 u - (a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $a, b > 0$ are constants, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$. Under some appropriate assumptions on $V(x)$ and $f(x,u)$, new results on the existence of infinitely many high energy solutions for the above equation are obtained via Symmetric Mountain Pass Theorem.

Key words : Fourth-order equations of Kirchhoff type; infinitely many high energy solutions; symmetric mountain pass theorem.

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1. Introduction and Main Results

Consider the following fourth-order elliptic equations of Kirchhoff type

$$\Delta^2 u - (a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $a, b > 0$ are constants, $V(x)$ and $f(x,u)$ are continuous functions and satisfy

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0$. Moreover, there exists a constant $d > 0$ such that

$$\lim_{|y|\to+\infty} \text{meas}\{x \in \mathbb{R}^N : |x-y| \leq d, V(x) \leq M\} = 0, \quad \forall M > 0,$$

where $\text{meas}(.)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

(f_1) there exist $c_1, c_2 > 0$ and $p \in (4, 2_*)$ such that

$$|f(x,t)| \leq c_1|t| + c_2|t|^{p-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where 2_* is the critical Sobolev exponent given by

$$2_* = \begin{cases} \frac{2N}{N-4}, & N > 4, \\ +\infty, & N \leq 4. \end{cases}$$

(f_2) $\lim\limits_{|t|\to+\infty} \frac{F(x,t)}{t^4} = +\infty$, a.e. $x \in \mathbb{R}^N$, where $F(x,t) = \int_0^t f(x,s)ds$.

(f_3) there exist two constants $L > 0$ and $\rho > 0$ such that

$$f(x,t)t - 4F(x,t) \geq -\rho|t|^2, \quad \text{a.e. } x \in \mathbb{R}^N \quad \text{and} \quad \forall |t| \geq L.$$

(f_4) there exists $\theta \geq 1$ such that

$$\theta\mathcal{F}(x,u) \geq \mathcal{F}(x,\tau u), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \quad \text{and} \quad \tau \in [0,1], \tag{1.2}$$

where $\mathcal{F}(x,t) = f(x,t)t - 4F(x,t)$.

(f_5) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Problem (1.1) is a nonlocal problem because the term $\int_{\mathbb{R}^N} |\nabla u|^2 dx$ is involving in the equation, this provokes some analysis difficulties when using variational methods to study the existence and multiplicity of solutions of equation (1.1). Therefore, the study of equation (1.1) has a particularly interesting. Let $V(x) = 0$, we consider the problem in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and set $u = \Delta u = 0$ on $\partial\Omega$, then problem (1.1) is reduced to the following fourth order Kirchhoff-type equation

$$\begin{cases} \Delta^2 u - \left(a + b\int_\Omega |\nabla u|^2 dx\right) \Delta u = f(x,u), & \text{in} \quad \Omega, \\ u = \Delta u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Equation (1.3) is related to the stationary analogue of the following Kirchhoff equation

$$\Delta^2 u + u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), \quad x \in \Omega. \tag{1.4}$$

When $N \leq 2$, the equation (1.4) can explain some phenomena appeared in physics, engineering and other sciences, for more physical background we refer the readers to [2-4]. Moreover, many authors have studied the Kirchhoff type problem on bounded domain like (1.3) under various growth conditions on f by using variational methods, readers can see [11-14, 20, 24, 25, 27].

Recently, Kirchhoff type problems setting on the unbounded domain or the whole space $\mathbb{R}^N$ have also attracted widespread attention. In order to obtain the existence and multiplicity of solutions for Kirchhoff type problems in $\mathbb{R}^N$, many solvability conditions on the nonlinearity f have been given, some interesting results can be founded in [8-10, 15-19, 22, 23] and the references therein. Wang and An [22] studied the following Kirchhoff-type problem

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + cu = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $c > 0$ is a constant and $N > 4$. By using variational methods and truncation, the authors proved the existence of positive solutions for (1.5). In [16], Xu and Chen obtained infinitely many negative nontrivial solutions for problem (1.1) in $\mathbb{R}^3$ by assuming that the nonlinear term $f(x,u)$ is sublinear at infinity with respect to u. Xu and Chen [18] established the existence of infinitely many large solutions for (1.1) in $\mathbb{R}^3$ by using the Symmetric Mountain Pass Theorem when the nonlinearity $f(x,u)$ is superlinear at infinity with respect to u. Khoutir and Chen [10] combined the variational methods and the Nehari manifold techniques to obtain the existence of least energy sign-changing solutions for equation (1.1) with a pure power nonlinearity (i.e., $f(x,u) = |u|^{p-2}u$, with $4 < p < 2_*$).

Motivated by the results mentioned above, in the present paper, we shall study the existence of infinitely many nontrivial solutions via variational methods for equations (1.1).

The main results of this paper are the following.

***Theorem* 1.1** — *Assume that (V), (f_1), (f_2), (f_3) and (f_5) hold. Then problem (1.1) possesses an unbounded sequence of nontrivial solutions $\{u_k\}$ such that when $k \to \infty$,*

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(a|\Delta u_k|^2 + |\nabla u_k|^2 + V(x)u_k^2\right) dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_k|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u_k)dx \to \infty.$$

***Theorem* 1.2** — *Assume that (V), (f_1), (f_2), (f_4) and (f_5) hold. Then problem (1.1) possesses an unbounded sequence of nontrivial solutions $\{u_k\}$ such that when $k \to \infty$,*

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(a|\Delta u_k|^2 + |\nabla u_k|^2 + V(x)u_k^2\right) dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_k|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u_k)dx \to \infty.$$

Remark 1.1 : Since problem (1.1) is defined on the whole space $\mathbb{R}^N$, it is well known that the main difficulty is the lack of compactness of the Sobolev embedding. To overcome this difficulty, we always assume that the potential $V(x)$ satisfies the condition (V), which was introduced by Bartsch *et al*. [7].

Remark 1.2 : (i) In [18], the authors assumed that the nonlinearity *f* satisfies the variant Ambrosetti-Rabinowitz condition ((AR) for short), that is, there exists $\mu > 4$ and $L > 0$ such that

$$\mu F(x,t) \leq tf(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, |t| > L.$$

Obviously, the (AR) condition implies (f_2) and (f_3). Thus, our conditions are weaker than the (AR) condition.

Condition (f_4), which is weaker than the variant Nehari-type monotonicity condition (i.e., $\frac{f(x,t)}{|t|^3}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$) is originally due to Jeanjean [26]. Therefore, our assumption (f_4) is weaker than the variant Nehari-type monotonicity condition used in [23].

2. Variational Framework and Technical Lemmas

Let $1 \leq p < \infty$ and

$$L^p(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \mid u \quad \text{is measureable and } \int_{\mathbb{R}^N} |u|^p dx < \infty \right\},$$

with the norm

$$||u||_{L^p} := |u|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}.$$

Let

$$H := H^2(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \ : \ \nabla u, \Delta u \in L^2(\mathbb{R}^N) \right\},$$

with the inner product and the norm

$$\langle u, v \rangle_H = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \nabla v + uv) dx, \quad ||u||_H = \langle u, u \rangle_H^{\frac{1}{2}}.$$

We define the working space by

$$E = \left\{ u \in H \ : \ \int_{\mathbb{R}^N} V(x) u^2 dx < +\infty \right\},$$

with the inner product and norm

$$\langle u, v \rangle = \int_{\mathbb{R}^N} (\Delta u \Delta v + a \nabla u \nabla v + V(x) uv) dx, \quad ||u|| = \langle u, u \rangle^{\frac{1}{2}},$$

where $||.||$ is equivalent to the norm $||.||_H$. Since the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for all $p \in [2, 2_*]$, then there exists $\eta_p > 0$ such that

$$|u|_p \leq \eta_p ||u||, \quad \forall u \in E. \tag{2.1}$$

Furthermore, motivated by [7, Lemma 3.1], we can prove the following compactness lemma.

Lemma 2.1 — Under the assumption (V), the continuous embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $2 \leq p < 2_*$.

Recall that $u \in E$ is called a weak solution of (1.1) if

$$\langle u, v\rangle + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\int_{\mathbb{R}^N} \nabla u \nabla v dx = \int_{\mathbb{R}^N} f(x,u) v dx, \quad \forall v \in E.$$

We define the energy functional $I : E \to \mathbb{R}$ associated with problem (1.1) by

$$I(u) = \frac{1}{2}||u||^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)dx, \tag{2.2}$$

where $F(x,t) = \int_0^t f(x,s)ds$. By a standard argument, it is easy to verify that $I \in C^1(E, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \langle u, v\rangle + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\int_{\mathbb{R}^N} \nabla u \nabla v dx - \int_{\mathbb{R}^N} f(x,u) v dx, \tag{2.3}$$

for all $u, v \in E$. Furthermore, the critical points of I are weak solutions of (1.1).

Definition 2.1 — Let $J \in C^1(E, \mathbb{R})$. We say that J satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ ((P.S)$_c$ for short), if any sequence $\{u_n\} \subset E$ along with $J(u_n) \to c$ and $J'(u_n) \to 0$ possesses a convergent subsequence; J satisfies the (P.S)-condition if J satisfies (P.S)$_c$ for all $c \in \mathbb{R}$.

In order to prove the main results of this paper, we shall make use of the following symmetric mountain pass theorem.

Proposition 2.1 — [5]. Let E be an infinite dimensional Banach space and let $J \in C^1(E, \mathbb{R})$ be even, satisfy the (P.S)-condition and $J(0) = 0$. If $E = Y \oplus Z$, where Y is finite dimensional, and J satisfies

(I) there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho \cap Y} \geq \alpha$;

(II) for each finite dimensional subspace $\widetilde{E} \subset E$, there exists $R = R(\widetilde{E}) > 0$ such that $J \leq 0$ on $\widetilde{E} \setminus B_R$.

Then J possesses an unbounded sequence of critical value.

Lemma 2.2 — Suppose that (V) and $(f_1)-(f_3)$ hold. Then any (PS)-sequence $\{u_n\} \subset E$ is bounded.

PROOF : Let $\{u_n\} \subset E$ be such that

$$I(u_n) \to c, \quad I'(u_n) \to 0. \tag{2.4}$$

Arguing by contradiction, suppose that $\{u_n\}$ is unbounded in E, i.e., $\|u_n\| \to +\infty$ as $n \to \infty$. By (f_1) we have

$$|F(x,u)| \le \frac{c_1}{2}|u|^2 + \frac{c_2}{p}|u|^p, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.5}$$

Recall that $\mathcal{F}(x,t) = f(x,t)t - 4F(x,t)$, then by (f_1) and (2.5), there exists $c_3 > 0$ such that

$$|\mathcal{F}(x,u)| \le c_3|u|^2, \quad \forall x \in \mathbb{R}^N,\ \forall |u| \le L,$$

where L is given by (f_3). This together with (f_3) implies that there is a constant $\beta > 0$ such that

$$\mathcal{F}(x,u) \ge -\beta|u|^2, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.6}$$

Hence, combined (2.4) with (2.6), for all $n \in \mathbb{N}$, one has

$$\begin{aligned} 1 + c &\ge I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}||u_n||^2 + \frac{1}{4}\int_{\mathbb{R}^N} [f(x,u_n)u_n - 4F(x,u_n)]\,dx \\ &= \frac{1}{4}||u_n||^2 + \frac{1}{4}\int_{\mathbb{R}^N} \mathcal{F}(x,u_n)dx \\ &\ge \frac{1}{4}||u_n||^2 - \frac{\beta}{4}\int_{\mathbb{R}^N} u_n^2 dx \\ &= \frac{1}{4}||u_n||^2 - \frac{\beta}{4}|u_n|_2^2, \end{aligned} \tag{2.7}$$

which implies that

$$\frac{|u_n|_2^2}{||u_n||^2} \ge \frac{1}{\beta} - \frac{4(c+1)}{\beta||u_n||^2}.$$

Therefore, for sufficiently large $n \in \mathbb{N}$ such that $\frac{4(c+1)}{||u_n||^2} \le \frac{1}{2}$, we get

$$\frac{|u_n|_2^2}{||u_n||^2} \ge \frac{1}{2\beta} > 0. \tag{2.8}$$

Let $v_n = \frac{u_n}{||u_n||}$, then $\|v_n\| = 1$ and $|v_n|_s \le \eta_s \|v_n\| = \eta_s$ for $s \in [2, 2_*]$. Moreover, by (2.8) we have

$$|v_n|_2^2 > 0. \tag{2.9}$$

Set $\Omega_n = \{x \in \mathbb{R}^N \ : \ |u_n(x)| \leq L\}$ and $A_n = \{x \in \mathbb{R}^N : v_n \neq 0\}$, then (2.9) implies that $\text{meas}(A_n) > 0$. Furthermore, under the assumption that $||u_n|| \to \infty$ as $n \to \infty$, we obtain

$$|u_n(x)| \to \infty \quad \text{as} \quad n \to \infty, \quad \forall x \in A_n. \tag{2.10}$$

Therefore, for n sufficiently large, we have $A_n \subset \mathbb{R}^N \setminus \Omega_n$. Consequently, it follows from (2.1), (2.2), (2.4), (2.5), (2.10), (f_2) and Fatou's Lemma that

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \frac{I(u_n)}{||u_n||^4} \\
&= \lim_{n\to+\infty} \left[\frac{1}{2\|u_n\|^2} + \frac{b}{4\|u_n\|^4} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^2 - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^4} dx \right] \\
&\leq \frac{b}{4} - \lim_{n\to+\infty} \left[\int_{\Omega_n} \frac{F(x,u_n)}{|u_n|^4} v_n^4 dx - \int_{\mathbb{R}^N\setminus\Omega_n} \frac{F(x,u_n)}{|u_n|^4} v_n^4 dx \right] \\
&\leq \frac{b}{4} + \lim_{n\to+\infty} \frac{1}{\|u_n\|^2} \left(\frac{c_1}{2} + \frac{c_2}{p} L^{p-2} \right) \eta_2^2 - \liminf_{n\to\infty} \int_{\mathbb{R}^N\setminus\Omega_n} \frac{F(x,u_n)}{|u_n|^4} v_n^4 dx \\
&\leq \frac{b}{4} - \int_{\mathbb{R}^N/\Omega_n} \liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^4} v_n^4 dx \\
&= \frac{b}{4} - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^4} [\chi_{A_n}(x)] v_n^4 dx \\
&= -\infty.
\end{aligned} \tag{2.11}$$

Obviously, this is a contradiction. Hence $\{u_n\} \subset E$ is bounded.

Lemma 2.3 — Suppose that (V), (f_1), (f_2) and (f_4) hold. Then any (PS)-sequence $\{u_n\} \subset E$ is bounded.

PROOF : To prove the boundedness of $\{u_n\}$, arguing by contradiction, assume that $\|u_n\| \to \infty$ as $\to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $\|v_n\|_p \leq \eta_p \|v_n\| = \eta_p$ for $2 \leq p < 2_*$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^p(\mathbb{R}^N)$, for $2 \leq p < 2_*$ and $v_n \to v$ a.e. in $\mathbb{R}^N$. So, two cases need to be discussed here: $v = 0$ or $v \neq 0$. For the case that $v \neq 0$, by a similar argument as (2.11) we can conclude a contradiction. Next, for the case that $v = 0$, motivated by [26], we choose a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n).$$

For any fixed $M > 0$, letting $w_n = \sqrt{4M} v_n$, we have

$$\begin{aligned}
&w_n \to 0 \quad \text{in} \quad L^p(\mathbb{R}^N), 2 \leq p < 2_*, \\
&w_n \to 0, \quad \text{a.e.} \quad x \in \mathbb{R}^N.
\end{aligned} \tag{2.12}$$

Then, by (2.5), (2.12) and Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,w_n)dx \le \lim_{n\to\infty}\left(\frac{c_1}{2}\int_{\mathbb{R}^N}|w_n|^2dx + \frac{c_2}{p}\int_{\mathbb{R}^N}|w_n|^p dx\right) = 0.$$

Therefore, for n large enough, we obtain

$$I(t_nu_n) \ge I(w_n) = 2M + 4bM^2\left(\int_{\mathbb{R}^N}|\nabla v_n|^2dx\right)^2 - \int_{\mathbb{R}^N}F(x,w_n)dx \ge 2M$$

which implies that $\liminf_{n\to\infty} I(t_nu_n) \ge 2M$. Since $M > 0$ is arbitrary, we conclude that

$$\lim_{n\to\infty} I(t_nu_n) = +\infty$$

On the other hand, since $I(tu_n)$ attains maximum at $t_n \in (0,1)$, one has

$$\langle I'(t_nu_n), t_nu_n\rangle = o(1).$$

for large $n \in \mathbb{N}$. Therefore, it follows from (f_4) that

$$\begin{aligned}
I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle &= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^N}(f(x,u_n)u_n - 4F(x,u_n))dx \\
&= \frac{1}{4}\|u_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^N}\mathcal{F}(x,u_n)dx \\
&\ge \frac{1}{4\theta}\|t_nu_n\|^2 + \frac{1}{4\theta}\int_{\mathbb{R}^N}\mathcal{F}(x,t_nu_n)dx \\
&= \frac{1}{\theta}\left[\frac{1}{4}\|t_nu_n\|^2 + \frac{1}{4}\int_{\mathbb{R}^N}(f(x,t_nu_n)t_nu_n - 4F(x,t_nu_n))dx\right] \\
&= \frac{1}{\theta}\left(I(t_nu_n) - \frac{1}{4}\langle I'(t_nu_n), t_nu_n\rangle\right) \to +\infty \quad \text{as} \quad n\to\infty.
\end{aligned}$$

This contradicts (2.4). Hence, $\{u_n\}$ is bounded in E. The proof is completed.

Lemma 2.4 — Suppose that (V) and (f_1) are satisfied. Then, any sequence $\{u_n\} \subset E$ defined by (2.4) has a convergent subsequence in E.

PROOF : Let $\{u_n\} \subset E$ be a bounded sequence. Then by Lemma 2.1, going if necessary to a subsequence, we may assume that

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in} \quad E, \\
&u_n \to u \quad \text{in} \quad L^p(\mathbb{R}^N), \quad 2 \le p < 2_*, \\
&u_n(x) \to u(x) \quad a.e.\ x \in \mathbb{R}^N.
\end{aligned} \tag{2.13}$$

By (f_1), the boundedness of $\{u_n\}$ and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(x,u_n)-f(x,u)||u_n-u|dx \\
&\leq \int_{\mathbb{R}^N} (|f(x,u_n)|+|f(x,u)|)|u_n-u|dx \\
&\leq c_1\int_{\mathbb{R}^N}(|u_n|+|u|)|u_n-u|dx + c_2\int_{\mathbb{R}^N}(|u_n|^{p-1}+|u|^{p-1})|u_n-u|dx \\
&\leq c_1\left[\left(\int_{\mathbb{R}^N}|u_n|^2dx\right)^{\frac{1}{2}}+\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{1}{2}}\right]\left(\int_{\mathbb{R}^N}|u_n-u|^2dx\right)^{\frac{1}{2}} \\
&+ c_2\left[\left(\int_{\mathbb{R}^N}|u_n|^pdx\right)^{\frac{p-1}{p}}+\left(\int_{\mathbb{R}^N}|u|^pdx\right)^{\frac{p-1}{p}}\right]\left(\int_{\mathbb{R}^N}|u_n-u|^pdx\right)^{\frac{1}{p}} \\
&\leq C_1|u_n-u|_2+C_2|u_n-u|_p \to 0 \quad \text{as} \quad n\to+\infty.
\end{aligned} \tag{2.14}$$

On the other hand, by (2.3), we obtain

$$\begin{aligned}
&\langle I'(u_n)-I'(u),u_n-u\rangle \\
&= \int_{\mathbb{R}^N}|\Delta(u_n-u)|^2dx + \left(a+b\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^N}|\nabla(u_n-u)|^2dx \\
&+ \int_{\mathbb{R}^N}V(x)|u_n-u|^2dx - b\left(\int_{\mathbb{R}^N}|\nabla u|^2dx-\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)dx \\
&- \int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx \\
&\geq ||u_n-u||^2 - b\left(\int_{\mathbb{R}^N}|\nabla u|^2dx-\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)dx \\
&- \int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx.
\end{aligned} \tag{2.15}$$

We then get

$$\begin{aligned}
||u_n-u||^2 \leq & \langle I'(u_n)-I'(u),u_n-u\rangle \\
&+ b\left(\int_{\mathbb{R}^N}|\nabla u|^2dx-\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)\int_{\mathbb{R}^N}\nabla u\nabla(u_n-u)dx \\
&+ \int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))(u_n-u)dx.
\end{aligned} \tag{2.16}$$

Define the functional $T_u : E\to\mathbb{R}$ by

$$T_u(v) = \int_{\mathbb{R}^N}\nabla u\nabla v dx, \quad \forall v\in E.$$

Obviously, T_u is a linear functional on E. Furthermore,

$$|T_u(v)| \leq \int_{\mathbb{R}^N} |\nabla u \nabla v| dx \leq ||u|| ||v||,$$

which implies that T_u is bounded on E, i.e., $T_u \in E^*$. Hence we get $\lim_{n\to\infty} T_{u_n}(u_n) = T_u(u)$, since $u_n \rightharpoonup u$ in E, that is, $\int_{\mathbb{R}^N} \nabla u.\nabla(u_n - u)dx \to 0$ as $n \to \infty$. Consequently, using the boundedness of $\{u_n\}$, we conclude that

$$b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \nabla u \nabla (u_n - u) dx \to 0, \tag{2.17}$$

as $n \to +\infty$. Moreover, it is clear that $\langle I'(u_n) - I'(u), u_n - u\rangle \to 0$. Hence, the combination of (2.15)-(2.17) implies that $||u_n - u|| \to 0$ in E. This completes the proof.

3. Proof of Main Results

In this section, we shall give the proof of Theorem 1.1 by using the Symmetric Mountain Pass Theorem 2.2.

Let $\{e_j\}$ is an orthonormal basis of E and define $X_j = \mathbb{R}e_j$,

$$Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \bigoplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{Z}. \tag{3.1}$$

Lemma 3.1 — Assume that (V) and (f_1) hold. Then, there exist constants $\rho, \alpha > 0$ and $m \in \mathbb{N}$ such that $I|_{\partial B_\rho \cap Z_m} \geq \alpha$.

PROOF : Set

$$\beta_k(s) = \sup_{u\in Z_k, ||u||=1} |u|_s, \quad \forall k \in \mathbb{N}, \quad 2 \leq s < 2_*. \tag{3.2}$$

Since E is compactly embedded into $L^s(\mathbb{R}^N)$ for $2 \leq s < 2_*$, there holds (see [5, Lemma 3.8])

$$\beta_k(s) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.3}$$

Combining (2.1), (2.2) with (2.6) we have

$$\begin{aligned} I(u) &= \frac{1}{2}||u||^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(x,u)dx \\ &\geq \frac{1}{2}||u||^2 - \frac{c_1}{2}|u|_2^2 - \frac{c_2}{p}|u|_p^p \\ &\geq \frac{1}{2}||u||^2 - \frac{c_1}{2}\beta_k^2(2)||u||^2 - \frac{c_2}{p}\beta_k^p(p)||u||^p. \end{aligned} \tag{3.4}$$

By (3.3), there exists a positive integer $m \geq 1$ such that

$$\beta_k^2(2) \leq \frac{1}{2c_1} \quad \text{and} \quad \beta_k^p(p) \leq \frac{p}{4c_2}, \quad \forall k \geq m.$$

Hence, there exists $\rho \in (0,1)$ such that

$$I(u) \geq \frac{1}{4}\rho^2(1-\rho^{p-2}) = \alpha > 0,$$

where $\|u\| = \rho$, since $p > 2$. The proof is completed.

Lemma 3.2 — Assume that (V) and $(f_1)-(f_3)$ hold. Then, for any finite dimensional subspace $\widetilde{E} \subset E$, there holds

$$I(u) \to -\infty \quad \text{as} \quad ||u|| \to +\infty, \quad u \in \widetilde{E}. \tag{3.5}$$

Proof : Arguing by contradiction, assume that for some sequence $\{u_n\} \subset \widetilde{E}$ with $\|u_n\| \to \infty$, there exists $M > 0$ such that $I(u_n) \geq -M$ for all $n \in \mathbb{N}$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E. Since $\widetilde{E}$ is finite dimensional, then $v_n \to v \in \widetilde{E}$ in E, $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^N$, and so $\|v\| = 1$. Hence, we can conclude a contradiction by a similar way as (2.11).

Corollary 3.1 — Assume that (V) and $(f_1)-(f_2)$ hold. Then, for any finite dimensional subspace $\widetilde{E} \subset E$, there exists $R = R(\widetilde{E}) > 0$ such that

$$I(u) \leq 0, \quad \forall u \in \widetilde{E} \setminus B_R.$$

PROOF OF THEOREM 1.1 : Clearly, $I \in C^1(E,\mathbb{R})$, $I(0) = 0$ and I is even in view of (f_5). Lemmas 2.2 and 2.4 imply that I satisfies the (P.S)-condition. On the other hand, Lemma 3.1 and Corollary 3.1 imply that I satisfies the conditions $(I)-(II)$ of Proposition 2.1. Hence, I has a sequence of nonrivial critical points $\{u_k\} \subset E$ such that

$$\lim_{k\to\infty} ||u_k|| = +\infty \quad \text{and} \quad \lim_{k\to\infty} I(u_k) = +\infty.$$

Thus, problem (1.1) possesses infinitely many nontrivial solutions.

PROOF OF THEOREM 1.2 : Similar to the proof of Theorem 1.1, we have $I \in C^1(E,\mathbb{R})$, $I(0) = 0$ and I is even in view of (f_5). Moreover, Lemmas 2.3, 2.4, 3.1 and Corollary 3.1 imply that I satisfies all the conditions of Proposition 2.1. Hence, I has a sequence of nonrivial critical points $\{u_k\} \subset E$ such that

$$\lim_{k\to\infty} ||u_k|| = +\infty \quad \text{and} \quad \lim_{k\to\infty} I(u_k) = +\infty.$$

Then, problem (1.1) possesses infinitely many nontrivial solutions.

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