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MULTIDIMENSIONAL LINEAR FUNCTIONAL CONNECTED WITH DOUBLE STRONG CESÀRO SUMMABILITY

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In 1964 Borwein presented functional characterization of the normed linear spaces w_p and W_p. These two spaces are clearly linked to Cesàro summability $[C, 1]$ in particular it should be noted that a sequence x in w_p if and only if x is Cesàro summable. The goal of this paper includes extension of these notions to double function space thus producing multidimensional analog of Borwein's results.

Key words : Discrete functional analysis; sequences spaces; double functions; Pringsheim; P-convergent.

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1. INTRODUCTION

Throughout this paper we will consider $1 \leq p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and observe that the normed linear spaces w_p and W_p are defined as follows.

Definition 1.1 — The space w_p is a collection of real sequences $\{x_k\}$ for which there exists a real number $l = l_x$ such that

$$\sum_{k=1}^{N} |x_k - l|^p = o(N)$$

with norm

$$||x|| = \sup_{N\geq 1}\left(\frac{1}{N}\sum_{k=1}^{N}|x_k|^p\right)^{\frac{1}{p}}.$$

Definition 1.2 — The space W_p is a collection of real valued function x, measurable in the Lebesgue sense on the interval $(1,\infty)$ for which there exists a real number $l = l_x$ such that

$$\int_1^T |x(t) - l|^p dt = o(T)$$

with norm

$$||x|| = \sup_{T\geq 1}\left(\frac{1}{T}\int_1^T |x(t)|^p dt\right)^{\frac{1}{p}}.$$

In addition to these definitions Borwein [2] also presented the following sequences:

Example 1.1 : Let $\alpha = \{\alpha_k\}$ be a real sequence and define the sequence $\{m_k(\alpha, p)\}$ as follows

$$m_k(\alpha,p) = \begin{cases} \sup_{2^k\leq n\leq 2^{k+1}} |n\alpha_n| & \text{if} \quad p = 1; \\ \left(\frac{1}{2^k}\sum_{n=2^k}^{2^{k+1}-1}|n\alpha_n|^q\right)^{\frac{1}{q}} & \text{if} \quad p > 1. \end{cases}$$

Example 1.2 : Let α be a real measurable function define on $(1,\infty)$ and define the sequence $\{M_k(\alpha,p)\}$ as follows

$$M_k(\alpha,p) = \begin{cases} ess.\sup_{2^k\leq t\leq 2^{k+1}} |t\alpha(t)| & \text{if} \quad p = 1; \\ \left(\frac{1}{2^k}\int_{2^k}^{2^{k+1}}|t\alpha(t)|^q dt\right)^{\frac{1}{q}} & \text{if} \quad p > 1. \end{cases}$$

Using these two definitions and examples Borwein [2] presented two functional characterization of the spaces w_p and W_p. The goal of the paper includes extended these notions to double function space and the presentation of multidimensional analog of Borwein's results.

2. Definitions, Notations, and Preliminary Results

Definition 2.1 — The space w_p'' is a factorable real double sequence space whose terms are of the form $x = \{x_{k,l}\}$ for which there is a number L_x such that

$$\sum_{k,l=1,1}^{M,N}|x_{k,l} - L_x|^p = o(MN),$$

with norm

$$||x|| = \sup_{M,N\geq 1,1}\left(\frac{1}{MN}\sum_{k,l=1,1}^{M,N}|x_{k,l}|^p\right)^{\frac{1}{p}}.$$

Definition 2.2 — The space $W_p^{''}$ consist of measurable real valued double functions $x(s,t)$ on the interval $(1,\infty)\times(1,\infty)$ for which there is a number L_x such that

$$\int_1^S \int_1^T |x(s,t)-L_x|^p = o(ST),$$

with norm

$$||x|| = \sup_{S,T\geq 1,1}\left(\frac{1}{ST}\int_1^S\int_1^T |x(s,t)|^p dsdt\right)^{\frac{1}{p}}.$$

Definition 2.3 — For a given double real sequence $\alpha=\{\alpha_{k,l}\}$ let us define the double sequence $\{\phi_{k,l}(\alpha,p)\}$ as follows

$$\phi_{k,l}(\alpha,p)=\begin{cases}\sup_{\{2^k\leq\zeta\leq 2^{k+1}\ \&\ 2^l\leq\eta\leq 2^{l+1}\}}|\zeta\eta\alpha_{\zeta,\eta}|, & \text{if} \quad p=1;\\ \left(\frac{1}{2^{k+l}}\sum_{\zeta=2^k,\eta=2^l}^{2^{k+1}-1,2^{l+1}-1}|\zeta\eta\alpha_{\zeta,\eta}|^q\right)^{\frac{1}{q}}, & \text{if} \quad p>1.\end{cases}$$

Definition 2.4 — For a given double real measurable function α in $(1,\infty)\times(1,\infty)$ let us define a double sequence $\{\Phi_{k,l}(\alpha,p)\}$ by the following:

$$\Phi_{k,l}(\alpha,p)=\begin{cases}ess.\sup_{\{2^k\leq s\leq 2^{k+1}\ \&\ 2^l\leq t\leq 2^{l+1}\}}|st\alpha(s,t)|, & \text{if} \quad p=1;\\ \left(\frac{1}{2^{k+l}}\int_{2^k}^{2^{k+1}}\int_{2^l}^{2^{l+1}}|st\alpha(s,t)|^q dsdt\right)^{\frac{1}{q}}, & \text{if} \quad p>1.\end{cases}$$

Henceforth, when we refer to f it shall be a two dimensional, factorable, additive, homogeneous functions.

***Theorem* 2.1** — *(1) If f is a function on $W_p^{''}$, then there is a real number a and a real-valued function α in $(1,\infty)\times(1,\infty)$ such that*

$$f(x)=aL_x+\int_1^\infty\int_1^\infty \alpha(s,t)x(s,t)dsdt$$

for every $x\in W_p^{''}$ and

$$\sum_{k,l=0,0}^{\infty,\infty}\Phi_{k,l}(\alpha,p)<\infty.$$

(2) In addition, if a is a real number and α is a real-valued measurable function on $(1,\infty)\times(1,\infty)$ such that $\sum_{k,l=0,0}^{\infty,\infty}\Phi_{k,l}(\alpha,p)<\infty$ then

$$f(x)=aL_x+\int_1^\infty\int_1^\infty \alpha(s,t)x(s,t)dsdt$$

defines a function f in $W_p^{''}$ with

$$||f|| \leq |a| + 4^{\frac{1}{p}} \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha, p)$$

and the double integrals

$$f(x) = aL_x + \int_1^\infty \int_1^\infty \alpha(s,t)x(s,t)dsdt$$

is absolutely P-convergent for every $x \in W_p''$.

PROOF : Let L_p'' be the space of real-valued measurable functions x on $(1,\infty)\times(1,\infty)$ for which

$$\int_1^\infty \int_1^\infty |x(s,t)|^p dsdt < \infty,$$

with norm

$$||x||_{L_p''} = \left(\int_1^\infty \int_1^\infty |x(s,t)|^p dsdt\right)^{\frac{1}{p}}.$$

It should be noted that if $x \in L_p''$ then $x \in W_p''$, with $L_x = 0$ and

$$||x|| = ||x||_{W_p''} \leq ||x||_{L_p''}.$$

Thus restriction to L_p'' of the given function f on W_p'' is additive and homogeneous on L_p''. The following is a slight extension of standard results in [1, pp 64-65], that there is a real valued function α measurable in $(1,\infty)\times(1,\infty)$ such that

$$f(x) = \int_1^\infty \int_1^\infty \alpha(s,t)x(s,t)dsdt \tag{2.1}$$

for all $x \in L_p''$ and either

$$ess. \sup_{1<s<\infty \,\&\, 1<t<\infty} |\alpha(s,t)| < \infty, \text{ if } p = 1$$

or

$$\int_1^\infty \int_1^\infty |\alpha(s,t)|^q dsdt < \infty, \text{ if } p > 1.$$

To show that α must be necessarily satisfy

$$\sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha, p) < \infty.$$

We consider the cases $p = 1$ and $p > 1$ separately. To that end, let us examine the case when $p = 1$ first. Let $\Phi_{k,l} = \Phi_{k,l}(\alpha, 1)$ there is a two dimensional measurable set $e_{k,l}$ of positive measure $|e_{k,l}|$ in the two dimensional interval $(2^k, 2^{k+1}) \times (2^l, 2^{l+1})$ such that

$$|st\alpha(s,t)| > \Phi_{k,l} - \frac{1}{2^{k+l}}$$

for all $(s,t)\in e_{k,l}$. Also let us consider the following

$$x(s,t)=\begin{cases}\frac{2^{k+l}}{|e_{k,l}|}\text{ sign }\alpha(s,t), & \text{if } (s,t)\in e_{k,l}\text{ for } k\le\Delta \;\&\; l\le\Delta';\\ 0, & \text{otherwise.}\end{cases}$$

Observe that since $x\in L_1''$ and

$$f(x)=\int_1^\infty\int_1^\infty \alpha(s,t)x(s,t)dsdt$$

we are granted the following:

$$\begin{aligned}||f||\cdot||x|| &\ge f(x)=\int_1^\infty\int_1^\infty\alpha(s,t)x(s,t)dsdt \qquad (2.2)\\ &= \sum_{k,l=0,0}^{\Delta,\Delta'}\iint_{e_{k,l}}\frac{2^{k+l}}{|e_{k,l}|}|\alpha(s,t)|dsdt\\ &\ge \frac{1}{4}\sum_{k,l=0,0}^{\Delta,\Delta'}\frac{1}{|e_{k,l}|}\iint_{e_{k,l}}|st\alpha(s,t)|dsdt\\ &\ge \frac{1}{4}\sum_{k,l=0,0}^{\Delta,\Delta'}\left(\Phi_{k,l}-\frac{1}{2^{k+l}}\right).\end{aligned}$$

Additionally for $2^\psi\le S\le 2^{\psi+1}\le 2^{\Delta+1}$ and $2^{\psi'}\le T\le 2^{\psi'+1}\le 2^{\Delta'+1}$ we are granted the following

$$\begin{aligned}\frac{1}{ST}\int_1^S\int_1^T|x(s,t)|dsdt &\le \frac{1}{2^{\psi+\psi'}}\int_1^{2^{\psi+1}}\int_1^{2^{\psi'+1}}|x(s,t)|dsdt\\ &= \frac{1}{2^{\psi+\psi'}}\sum_{k,l=0,0}^{\psi,\psi'}\iint_{e_{k,l}}|x(s,t)|dsdt\\ &\le \frac{1}{2^{\psi+\psi'}}\sum_{k,l=0,0}^{\psi,\psi'}2^{k+l}<4.\end{aligned}$$

Also note that for $S>2^{\Delta+1}$ and $T>2^{\Delta'+1}$ we have

$$\frac{1}{ST}\int_1^S\int_1^T|x(s,t)|dsdt\le\frac{1}{2^{\Delta+\Delta'+2}}\int_1^{2^{\Delta+1}}\int_1^{2^{\Delta'+1}}|x(s,t)|dsdt<1.$$

Therefore with $||x||<4$ the following holds by (2.2)

$$4||f(x)||+\frac{1}{4}\sum_{k,l=0,0}^{\infty,\infty}\frac{1}{2^{k+l}}=4||f(x)||+2\ge\frac{1}{4}\sum_{k,l=0,0}^{\infty,\infty}\Phi_{k,l}.$$

Hence

$$\sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha,p) < \infty$$

when $p = 1$. Now let us consider the second case, that is when $p > 1$ and let $\Phi_{k,l} = \Phi_{k,l}(\alpha,p)$ and define x as follows

$$x(s,t) = \begin{cases} \frac{(st)^q}{2^{k+l}} \left|\frac{\alpha(s,t)}{\Phi_{k,l}}\right|^{\frac{q}{p}} \text{ sign } \alpha(s,t), & \text{if } \quad 2^k \leq s < 2^{k+1} \leq 2^{\Delta+1} \\ & \qquad \& \\ & 2^l \leq t < 2^{l+1} \leq 2^{\Delta'+1}; \\ & \qquad \Phi_{k,l} \neq 0; \\ 0, & \qquad \text{otherwise.} \end{cases}$$

Then $x \in L_p''$ grants us the following by (2.1)

$$\begin{aligned} f(x) &= \int_1^{2^{\Delta+1}} \int_1^{2^{\Delta'+1}} |\alpha(s,t)x(s,t)| dsdt \qquad (2.3) \\ &= \sum_{k,l=0,0}^{\Delta,\Delta'} \int_{2^k}^{2^{k+1}} \int_{2^l}^{2^{l+1}} |x(s,t)\alpha(s,t)| dsdt \\ &= \sum_{k,l=0,0}^{\Delta,\Delta'} \Phi_{k,l} \leq ||f|| \cdot ||x||. \end{aligned}$$

Further, for $2^\psi \leq S \leq 2^{\psi+1} \leq 2^{\Delta+1}$ and $2^{\psi'} \leq T \leq 2^{\psi'+1} \leq 2^{\Delta'+1}$ we are granted the following

$$\begin{aligned} \frac{1}{ST}\int_1^S \int_1^T |x(s,t)|^p dsdt &\leq \frac{1}{2^{\psi+\psi'}} \int_1^{2^{\psi+1}} \int_1^{2^{\psi'+1}} |x(s,t)|^p dsdt \\ &= \frac{1}{2^{\psi+\psi'}} \sum_{k,l=0,0}^{\psi,\psi'} \int_{2^k}^{2^{k+1}} \int_{2^l}^{2^{l+1}} |x(s,t)|^p dsdt \\ &\leq 2^{2p-\psi-\psi'} \sum_{k,l=0,0}^{\psi,\psi'} 2^{k+l} < 2^{2p+2}. \end{aligned}$$

In addition for $S \geq 2^{\Delta+1}$ and $T \geq 2^{\Delta'+1}$ we have the following

$$\frac{1}{ST}\int_1^S \int_1^T |x(s,t)|^p dsdt \leq \frac{1}{2^{\Delta+\Delta'+2}} \int_1^{2^{\Delta+1}} \int_1^{2^{\Delta'+1}} |x(s,t)|^p dsdt < 4^p.$$

Thus $||x|| < 4^{1+\frac{1}{p}}$, and so by (2.3),

$$\sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l} \leq 4^{1+\frac{1}{p}}||f||.$$

Thus $\sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha, p) < \infty$ is established for the case when is $p > 1$. Suppose that $p \geq 1$, $\Phi_{k,l} = \Phi_{k,l}(\alpha, p)$ and $x \in W_p^{''}$. Then by the multidimensional Hölder's inequality [6] we are granted the following

$$\begin{aligned}
\int_1^\infty \int_1^\infty |\alpha(s,t)x(s,t)|dsdt &= \sum_{k,l=0,0}^{\infty,\infty} \int_{2^k}^{2^{k+1}} \int_{2^l}^{2^{l+1}} |\alpha(s,t)x(s,t)|dsdt \\
&\leq \sum_{k,l=0,0}^{\infty,\infty} \left[\left(\frac{1}{2^{k+l}} \int_{2^k}^{2^{k+1}} \int_{2^l}^{2^{l+1}} |stx(s,t)|^q dsdt \right)^{\frac{1}{q}} \right. \qquad (2.4) \\
&\times \left. \left(2^{p(1-\frac{1}{p})(k+l)} \int_{2^k}^{2^{k+1}} \int_{2^l}^{2^{l+1}} \left| \frac{x(s,t)}{st} \right|^p dsdt \right)^{\frac{1}{p}} \right] \\
&\leq \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l} \left(2^{p(1-\frac{1}{p})(k+l)} \int_{2^k}^{2^{k+1}} \int_{2^l}^{2^{l+1}} \left| \frac{x(s,t)}{st} \right|^p dsdt \right)^{\frac{1}{p}} \\
&\leq \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l} \left(2^{-(k+l)} \int_{2^k}^{2^{k+1}} \int_{2^l}^{2^{l+1}} |x(s,t)|^p \, dsdt \right)^{\frac{1}{p}} \\
&\leq 4^{\frac{1}{p}} ||x|| \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}.
\end{aligned}$$

Thus

$$\int_1^\infty \int_1^\infty |\alpha(s,t)x(s,t)|dsdt < \infty$$

whenever $x \in W_p^{''}$ and since the characteristic functions on $(1, \infty) \times (1, \infty)$ is in $W_p^{''}$ thus

$$\int_1^\infty \int_1^\infty |\alpha(s,t)|dsdt < \infty.$$

Thus suppose that $x \in W_p^{''}$, $L = L_x$, $y(s,t) = x(s,t) - L$ and define $y_{k,l}(s,t)$ as follows

$$y_{k,l}(s,t) = \begin{cases} y(s,t), & \text{if} \quad 1 \leq s \leq k \\ & \qquad 1 \leq t \leq l; \\ 0, & \text{if} \quad s > k \\ & \qquad t > l. \end{cases}$$

Therefore when $y \in W_p''$ and $y_{k,l} \in L_p''$ the following holds

$$||y_{k,l} - y|| = \sup_{S,T>k,l} \left(\frac{1}{ST} \int_k^S \int_l^T |x(s,t) - L|^p \right)^{\frac{1}{p}} = o(1)$$

in the Pringsheim sense. However

$$|f(y_{k,l} - y) = |f(y_{k,l}) - f(y)| \leq ||y_{k,l} - y|| \cdot ||f||.$$

Thus by (2.1) we obtain the following

$$\begin{aligned} f(y(s,t)) &= P - \lim_{k,l} f(y_{k,l}(s,t)) \\ &= P - \lim_{k,l} \int_1^k \int_1^l y(s,t)\alpha(s,t)dsdt \\ &= \int_1^\infty \int_1^\infty y(s,t)\alpha(s,t)dsdt - L \int_1^\infty \int_1^\infty \alpha(s,t)dsdt. \end{aligned} \tag{2.5}$$

Since the following double integrals are absolutely convergent $\int_1^\infty \int_1^\infty y(s,t)\alpha(s,t)dsdt$, $\int_1^\infty \int_1^\infty \alpha(s,t)dsdt$ we can let δ be the characteristic function on $(1,\infty) \times (1,\infty)$ and observe the following

$$f(x) = f(y + L\delta) = f(y) + Lf(\delta) = \int_1^\infty \int_1^\infty x(s,t)\alpha(s,t)dsdt + La,$$

where

$$a = f(\delta) - \int_1^\infty \int_1^\infty \alpha(s,t)dsdt.$$

This complete the proof of (1). Now let us now consider (2) If follows from (2.5) that if $x \in W_p''$, and $\Phi_{k,l} = \Phi_{k,l}(\alpha, p)$ then

$$\begin{aligned} |f(x)| &= \left| \int_1^\infty \int_1^\infty \alpha(s,t)x(s,t)dsdt + aL \right| \\ &\leq 4^{\frac{1}{p}} ||x|| \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha, p) + |aL| \end{aligned} \tag{2.6}$$

Thus by the multidimensional Minkowski's inequality [5, Theorem 202]

$$\begin{aligned} \left(\left(1 - \frac{1}{S}\right) \left(1 - \frac{1}{T}\right) \right)^{\frac{1}{p}} |L| &\leq \left(\frac{1}{ST} \int_1^T \int_1^S |x(s,t) - L|^p dsdt \right)^{\frac{1}{p}} \\ &+ \left(\frac{1}{ST} \int_1^T \int_1^S |x(s,t)|^p dsdt \right)^{\frac{1}{p}} \end{aligned}$$

and the first term on the right on side is a Pringshem null sequence. Thus $|L| \leq ||x||$ which yields by (2.6)

$$|f(x)| \leq ||x|| \left(|a| + 4^{\frac{1}{p}} \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha, p) \right)$$

for every $x \in W_p^{''}$. Thus note the additive and homogeneous two dimensional function f defined by

$$f(x) = aL_x + \int_1^\infty \int_1^\infty \alpha(s,t)x(s,t)dsdt$$

is continuous on $W_p^{''}$ and

$$||f|| \leq |a| + 4^{\frac{1}{p}} \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha, p).$$

Thus by (2.4)

$$\int_1^\infty \int_1^\infty \alpha(s,t)x(s,t)dsdt$$

is absolutely P-convergent. □

***Theorem* 2.2** — *(3) If f is a function on $w_p^{''}$, then there is a real number a and a real double sequence $\alpha = \{\alpha_{k,l}\}$such that*

$$f(x) = aL_x + \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l} x_{k,l} \tag{2.7}$$

for every $x \in w_p^{''}$ and

$$\sum_{k,l=0,0}^{\infty,\infty} \phi_{k,l}(\alpha, p) < \infty.$$

(4) In addition, if a is a real number and $\alpha = \{\alpha_{k,l}\}$ is a double real sequence such that $\sum_{k,l=0,0}^{\infty,\infty} \phi_{k,l}(\alpha, p) < \infty$ then $f(x) = aL_x + \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l} x_{k,l}$ defines a function f in $w_p^{''}$ with

$$||f|| \leq |a| + 4^{\frac{1}{p}} \sum_{k,l=0,0}^{\infty,\infty} \phi_{k,l}(\alpha, p)$$

and the double series

$$f(x) = aL_x + \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l} x_{k,l}$$

is absolutely P-convergent for every $x \in w_p^{''}$.

PROOF : For any real double sequence $x = \{x_{k,l}\}$ let us define this corresponding function

$$x^*(s,t) = x_{k,l} \text{ for } k < s \leq k+1 \text{ and } l < t \leq l+1; k,l = 1,2,3,\ldots$$

similar to the single dimensional case there is a one-to-one corresponding between $w_p^{''}$ and a linear subspace of ${W''}^{*}_{p}$ of $W_p^{''}$ such that

$$L_{x^*} = L \text{ and } ||x^*|| \leq ||x|| \leq 4^{\frac{1}{p}}||x^*||.$$

Thus given a function f in $w_p^{''}$, the function f^* is defined by

$$f^*(x^*) = f(x)$$

is additive and homogeneous on ${W''}^{*}_{p}$.

By Nachbin Theorem [3] there is a real number a and a real valued function α^*, integrable over $(1,\infty)\times(1,\infty)$, such that

$$\sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha^*, p) < \infty$$

and, for every $x \in w_p^{''}$,

$$\begin{aligned} f(x) = f^*(x^*) &= aL_{x^*} + \int_1^\infty \int_1^\infty \alpha^*(s,t)x^*(s,t)dsdt \\ &= aL_{x^*} + \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}x_{k,l} \end{aligned}$$

where

$$\alpha_{k,l} = \int_k^{k+1}\int_l^{l+1} \alpha^*(s,t)dsdt.$$

Note also that for $\alpha = \{\alpha_{k,l}\}$

$$\sum_{k,l=0,0}^{\infty,\infty} \phi_{k,l}(\alpha, p) \leq \sum_{k,l=0,0}^{\infty,\infty} \Phi_{k,l}(\alpha^*, p)$$

This complete the proof of part (3). Now let us consider part (4). Using the multidimensional Hölder's [6] and Minkowski's [5] inequalities we are granted that following as in (2).

$$\begin{aligned} f(s,t) &= aL + \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}x_{k,l} \\ &\leq |aL| + \sum_{k,l=1,1}^{\infty,\infty} |\alpha_{k,l}x_{k,l}| \\ &\leq |aL| + 4^{\frac{1}{p}}||x|| \sum_{k,l=0,0}^{\infty,\infty} \phi_{k,l} \\ &\leq ||x|| \left(|\alpha| + 4^{\frac{1}{p}} \sum_{k,l=0,0}^{\infty,\infty} \phi_{k,l} \right). \end{aligned}$$

The function f defined by (2.7) is therefore additive and homogeneous on W_p'',

$$||f|| \leq |a| + 4^{\frac{1}{p}} \sum_{k,l=0,0}^{\infty,\infty} \phi_{k,l}$$

and the series in (2.7) is absolutely P-convergent. This complete the proof of Theorem 2.2. □

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