Indian J. Pure Appl. Math., **51**(1): 155-170, March 2020

DOI: 10.1007/s13226-020-0391-y

THE METHOD FOR SOLVING FIXED POINT PROBLEM OF G-NONEXPANSIVE MAPPING IN HILBERT SPACES ENDOWED WITH GRAPHS AND NUMERICAL EXAMPLE

Wongvisarut Khuangsatung* and Atid Kangtunyakarn**

*Department of Mathematics and Computer Science,Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi, Pathumthani, 12110, Thailand

**Department of Mathematics, Faculty of Science, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

e-mails: wongvisarut_k@rmatt.ac.th; k@rmutt.ac.th; beawrock@hotmail.com

(*Received* 4 *January* 2018; *after final revision* 23 *October* 2018; *accepted* 10 *January* 2019)

The main aim of this paper is to study a strong convergence theorem of viscosity approximation method for G-nonexpansive mapping defined on a Hilbert space endowed with a directed graph. By using our main result, we give a numerical expample to approximate the value of π.

Key words : G-nonexpansive mappings; viscosity approximation; edge-preserving.

2010 Mathematics Subject Classification: 47H09, 47H10, 05C69

1. Introduction

The fixed point theory plays an important role in nonlinear functional analysis and is a very useful tool in various fields. In particular, fixed point theorem has been applied in many branches of sciences. For a recent trend of fixed point problem, one of the most interesting problems is the combination of fixed point theory and graph theory. In the past few years, many researchers have studied fixed point theorems in a metric space endowed with a graphs; see [1-4] and references cited therein.

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be contraction if there is $0 < k < 1$ such that

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X.$$

The set of all fixed points of a mapping T is denoted by $F(T)$, i.e., $x \in F(T)$ if and only if $x = Tx$.

Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is a set of vertices of graph and $E(G)$ be a set of its edges, assume that G has no parallel edges, we denote G^{-1} as the directed graph obtained from G by reversing the direction of edges. That is,

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

If x and y are vertices in G, then a path in G from x to y of length $n \in \mathbb{N} \cup \{0\}$ is a sequence $\{x_i\}_{i=1}^{n}$ of $n+1$ vertices such that $x_0 = x$, $x_n = y$, $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$. A graph G is connected if there is a (directed) path between any two vertices of G.

For studying contractive-type mappings, the Banach contraction mapping principle, which was firstly introduced by Banach [5] in 1922, has been an important source for solving existence problems in fixed point theory. Some of the contractive-type mapping were studied in many directions, see [6, 7]. In 2008, Jachymski [8] combined the concept of fixed point theory and graph theory in a metric space to generalized Banach contraction mapping principle in a metric space endowed with a directed graph. He also introduced a contractive-type mapping with a directed graph as follows.

Definition 1.1 — [8]. Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $E(G)$ contains all loops, *i.e.*, $\triangle = \{(x, x) : x \in X\} \subseteq E(G)$.

We say that a mapping $f : X \to X$ is a G-contraction if f preserves edges of G, *i.e.*,

$$x, y \in X, (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G)$$

and there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$,

$$(x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

In the past few years, many authors have studied a concept of G-contraction in order to improve and extend the above definition, see for instance [9-12] and references cited therein. Let C be a nonempty convex subset of a Banach space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $T : C \to C$, then T is said to be G-nonexpansive if the following conditions hold:

(1) T is edge-preserving, *i.e.*, for any $x, y \in C$ such that $(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)$;

(2) $\|Tx - Ty\| \leq \|x - y\|$, whenever $(x, y) \in E(G)$ for any $x, y \in C$.

This mapping was introduced by Tiammee *et al*. [13] in 2015.

We know that Halpern iteration process is an important tool in fixed point problem and it can generate a strongly convergent sequence provided that the underlying space is smooth enough. So, in order to prove a strong convergence of the Halpern iteration process in a Hilbert space endowed with a directed graph , Tiammee *et al*. [13] introduced Property G and proved strong convergence of the Halpern iteration process for finding the set of fixed point of G-nonexpansive mappings in a Hilbert space endowed with a directed graph as the following theorem.

***Theorem* 1.2** — *Let C be a nonempty closed convex subset of a Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive mapping. Assume that there exists $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be a sequence satisfying*

$$x_0 \in C, x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, n \geq 0. \tag{1}$$

Let $\{x_n\}$ be a sequence defined by Halpern iteration, where $u = x_0$. If $\{x_n\}$ is dominated by Px_0 and $\{x_n\}$ dominates x_0, then $\{x_n\}$ converges strongly to Px_0, where P is the metric projection on $F(T)$.

One of the most interesting iteration processes is the viscosity approximation method introduced by Moudafi [15]. In 2004, Xu [14] studied the such method for a nonexpansive mapping in a Hilbert space and introduced an iterative scheme for finding the set of fixed points of a nonexpansive mapping in a Hilbert space as follows:

$$x_0 \in C, x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 0, \tag{2}$$

where $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, $f : C \to C$ is a contraction, and $\{\alpha_n\} \subseteq (0, 1)$. Then, they proved a strong convergence theorem under suitable conditions of the parameters $\{\alpha_n\}$.

In this paper, motivated by [13] and [14], we prove a strong convergence theorem for finding the set of fixed point of G-nonexpansive mapping in a Hilbert space endowed with a directed graph. By using our main result, we give a numerical expample to approximate the value of π.

2. Preliminaries

In this paper, we denote " weak and strong convergence " by notations " $\rightharpoonup$" and " $\to$", respectively. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$, there exists the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

In a real Hilbert space H, it is well known that H satisfies *Opial's condition* [19], *i.e.*, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|,$$

holds for every $y \in H$ with $y \neq x$.

The following lemmas are needed to prove the main theorem.

Definition 2.1 — A sequence $\{x_n\}$ in a Hilbert space H is said to converge weakly to $x \in H$ if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$. In this case, we write $x_n \rightharpoonup x$.

***Theorem* 2.2** — *[16]. Let X be a Banach space. Then X is reflexive if and only if every closed convex bounded subset C of X is weakly compact,* i.e., *every bounded sequence in C has a weakly convergent subsequence.*

Lemma 2.3 — [17]. Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \forall n \geq 0,$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) : $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) : $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 — [18]. Given $x \in H$ and $y \in C$. Then, $P_C x = y$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0, \forall z \in C.$$

Lemma 2.5 — Let H be a real Hilbert space. Then

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle,$$

for all $x, y \in H$.

Property G : [13]. Let C be a nonempty subset of a normed space X and let $G = (V(G), E(G))$, where $V(G) = C$, be a directed graph. Then C is said to have Property G if every sequence $\{x_n\}$ in C converging weakly to $x \in C$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

The following basic definitions of domination in graphs [20, 21] are needed to prove the main theorem.

Let $G = (V(G), E(G))$ be a directed graph. A set $X \subseteq V(G)$ is called a dominating set if every $z \in V(G) \setminus X$ there exists $x \in X$ such that $(x, z) \in E(G)$ and we say that x dominates z or z is dominated by x. Let $z \in V$, a set $X \subseteq V$ is dominated by z if $(z, x) \in E(G)$ for any $x \in X$ and we say that X dominates z if $(x, z) \in E(G)$ for all $x \in X$. In this paper, we always assume that $E(G)$ contains all loops.

***Theorem* 2.6** — *[13]. Let X be a normed space and $G = (V(G), E(G))$ a directed graph with $V(G) = X$. Suppose $T : X \to X$ is a G-nonexpansive mapping. If X has a Property G, then T is continuous.*

***Theorem* 2.7** — *[13]. Let X be a Hilbert space and C be a subset of X having Property G. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G)$ is convex. Suppose $T : C \to C$ is a G-nonexpansive mapping and $F(T) \times F(T) \subseteq E(G)$. Then $F(T)$ is closed and convex.*

Definition 2.8 — [13]. Let $G = (V(G), E(G))$ be a directed graph. A graph G is called transitive if for any $x, y, z \in V(G)$ such that (x, y) and (y, z) are in $E(G)$, then $(x, z) \in E(G)$.

3. Main Result

In this section, we prove a strong convergence theorem of viscosity approximation methods for G-nonexpansive mapping in Hilbert spaces endowed with a directed graph.

The following Proposition is needed to prove the main theorem.

Proposition 3.1 — Let C be a convex subset of a vector space X and $G = (V(G), E(G))$ a directed graph such that $V(G) = C$ and $E(G)$ is convex. Let G be transitive, $T : C \to C$ be edge-preserving, and $f : C \to C$ be a G-contraction mapping. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, n \geq 0, \end{cases}$$

where $(f(x_0), Tx_0)$ and $(x_0, f(x_0))$ are in $E(G)$. If $\{x_n\}$ dominates x_0, then (x_n, x_{n+1}), (x_0, x_n), (x_0, Tx_n), and (x_n, Tx_n) are in $E(G)$ for all $n \in \mathbb{N}$.

PROOF : We prove by induction. By transitivity of G and since $(x_0, f(x_0))$ and $(f(x_0), Tx_0)$ are in $E(G)$, we have $(x_0, Tx_0) \in E(G)$. Since $E(G)$ is convex, $(x_0, f(x_0))$ and (x_0, Tx_0) are in $E(G)$, we obtain

$$(\alpha_0 x_0 + (1-\alpha_0)x_0, \alpha_0 f(x_0) + (1-\alpha_0)Tx_0) = (x_0, x_1) \in E(G).$$

Since T is edge-preserving, f is G-contraction mapping and $(x_0, x_1) \in E(G)$, then $(Tx_0, Tx_1) \in E(G)$ and $((f(x_0), f(x_1))) \in E(G)$, respectively. By transitivity of G and since (x_0, Tx_0) and (Tx_0, Tx_1) are in $E(G)$, we obtain $(x_0, Tx_1) \in E(G)$. By assumption, $(x_1, x_0) \in E(G)$. Then, by transitivity of G and $(x_0, Tx_1) \in E(G)$, we get $(x_1, Tx_1) \in E(G)$. By transitivity of G and since $(f(x_0), Tx_0)$ and (Tx_0, Tx_1) are in $E(G)$, we obtain $(f(x_0), Tx_1) \in E(G)$. Since $E(G)$ is convex, $(f(x_0), Tx_1)$ and $(f(x_0), f(x_1))$ are in $E(G)$, we obtain

$$(\alpha_1 f(x_0) + (1-\alpha_1)f(x_0), \alpha_1 f(x_1) + (1-\alpha_1)Tx_1) = (f(x_0), x_2) \in E(G).$$

By transitivity of G and since (x_1, x_0) and $(x_0, f(x_0))$ are in $E(G)$, we obtain $(x_1, f(x_0)) \in E(G)$. Again, by transitivity of G and since $(x_1, f(x_0))$ and $(f(x_0), x_2)$ are in $E(G)$, we obtain $(x_1, x_2) \in E(G)$.

Next, assume that (x_k, x_{k+1}), (x_0, x_k), (x_0, Tx_k), and (x_k, Tx_k) are in $E(G)$. Since T is edge-preserving and $(x_k, x_{k+1}) \in E(G)$, then $(Tx_k, Tx_{k+1}) \in E(G)$. By transitivity of G, and (x_0, Tx_k), (Tx_k, Tx_{k+1}) are in $E(G)$, we have $(x_0, Tx_{k+1}) \in E(G)$. Since $\{x_n\}$ dominates x_0, we have $(x_{k+1}, x_0) \in E(G)$. By transitivity of G, and (x_{k+1}, x_0), (x_0, Tx_{k+1}) are in $E(G)$, we have $(x_{k+1}, Tx_{k+1}) \in E(G)$. By transitivity of G, and (x_0, x_k), (x_k, x_{k+1}) are in $E(G)$, we get $(x_0, x_{k+1}) \in E(G)$. Since T is edge-preserving, f is G-contraction mapping, and $(x_0, x_{k+1}) \in E(G)$, we have (Tx_0, Tx_{k+1}), $(f(x_0), f(x_{k+1}))$ are in $E(G)$, respectively. By transitivity of G and since $(f(x_0), Tx_0)$ and (Tx_0, Tx_{k+1}) are in $E(G)$, we obtain $(f(x_0), Tx_{k+1}) \in E(G)$. Since $E(G)$ is convex, $(f(x_0), Tx_{k+1})$ and $(f(x_0), f(x_{k+1}))$ are in $E(G)$, we obtain

$$\begin{aligned}(\alpha_{k+1} f(x_0) + (1-\alpha_{k+1})f(x_0), \alpha_{k+1} f(x_{k+1}) + (1-\alpha_{k+1})Tx_{k+1})& \\ = (f(x_0), x_{k+2}) \in E(G).&\end{aligned}$$

By transitivity of G and since (x_{k+1}, x_0) and $(x_0, f(x_0))$ are in $E(G)$, we obtain $(x_{k+1}, f(x_0)) \in E(G)$. Again, by transitivity of G and since $(x_{k+1}, f(x_0))$ and $(f(x_0), x_{k+2})$ are in $E(G)$, we obtain $(x_{k+1}, x_{k+2}) \in E(G)$.

So, by induction, we can conclude that (x_n, x_{n+1}), (x_0, x_n), and (x_n, Tx_n) are in $E(G)$ for any $n \in \mathbb{N}$. □

***Theorem* 3.2** — *Let C be a nonempty closed convex subset of a real Hilbert space H and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$, $E(G)$ is convex and G is transitive. Suppose C has Property G. Let $T : C \to C$ be a G-nonexpansive mapping. Let $f : C \to C$ be a G-contraction mapping with coefficient $\alpha \in (0, 1)$. Assume that there exists $x_0 \in C$ such that $(f(x_0), Tx_0)$ and $(x_0, f(x_0))$ are in $E(G)$. Suppose that $F(T) \neq \emptyset$ and $F(T) \times F(T) \subseteq E(G)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, n \geq 0, \end{cases} \tag{3}$$

where $\{\alpha_n\} \subseteq (0, 1)$ satisfies

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0\ , \text{(ii)} \sum_{n=0}^{\infty} \alpha_n = \infty\ , \text{(iii)} \sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

If $\{x_n\}$ dominates $P_{F(T)}f(x_0)$ and $\{x_n\}$ dominates x_0, then the sequence $\{x_n\}$ converge strongly to $x_0 = P_{F(T)}f(x_0)$.

PROOF : We divide the proof into five steps:

Step 1 : We show that the sequence $\{x_n\}$ is bounded. Let $x^* = P_{F(T)}f(x_0)$. From Proposition 3.1, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Since $x^* \in F(T)$ and $x^* = P_{F(T)}f(x_0)$ is dominated by $\{x_n\}$, we have $(x_n, x^*) \in E(G)$. From the definition of x_n, we get

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \alpha_n\|f(x_n) - x^*\| + (1 - \alpha_n)\|Tx_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n\|f(x_n) - f(x^*)\| + \alpha_n\|f(x^*) - x^*\| + (1 - \alpha_n)\|x_n - x^*\| \\ &\leq \alpha_n\alpha\|x_n - x^*\| + (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\| \\ &= (1 - \alpha_n(1 - \alpha))\|x_n - x^*\| + \alpha_n\|f(x^*) - x^*\|. \end{aligned}$$

By mathematical induction, we obtain that

$$\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1 - \alpha}\right\}, \forall n \in \mathbb{N}.$$

Therefore, $\{x_n\}$ is bounded and so are $\{Tx_n\}$ and $\{f(x_n)\}$.

Step 2 : We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From the definition of x_n and $(x_n,x_{n+1})\in E(G)$, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n f(x_n)+(1-\alpha_n)Tx_n-\alpha_{n-1}f(x_{n-1})\\
&\quad -\ (1-\alpha_{n-1})Tx_{n-1}\|\\
&= \|\alpha_n f(x_n)-\alpha_n f(x_{n-1})+\alpha_n f(x_{n-1})-\alpha_{n-1}f(x_{n-1})\\
&\quad +\ (1-\alpha_n)Tx_n-(1-\alpha_n)Tx_{n-1}+(1-\alpha_n)Tx_{n-1}\\
&\quad -\ (1-\alpha_{n-1})Tx_{n-1}\|\\
&= \|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1})\\
&\quad +\ (1-\alpha_n)(Tx_n-Tx_{n-1})+(\alpha_{n-1}-\alpha_n)Tx_{n-1}\|\\
&\leq \alpha_n\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\
&\quad +\ (1-\alpha_n)\|Tx_n-Tx_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Tx_{n-1}\|\\
&\leq \alpha_n\alpha\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\
&\quad +\ (1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Tx_{n-1}\|\\
&= (1-\alpha_n(1-\alpha))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|f(x_{n-1})\|\\
&\quad +\ |\alpha_n-\alpha_{n-1}|\|Tx_{n-1}\|. \qquad (4)
\end{aligned}$$

Applying Lemma 2.3, (4), and the conditions (i), (ii), (iii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (5)$$

Step 3 : We show that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. For each $n\in\mathbb{N}$, we have

$$\begin{aligned}
\|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\|\\
&\leq \|x_n-x_{n+1}\|+\alpha_n\|f(x_n)-Tx_n\|.
\end{aligned}$$

Because $\{Tx_n\}$ and $\{f(x_n)\}$ are bounded, from the condition (i), (ii), and (5), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \qquad (6)$$

Step 4 : We show that $\limsup_{n\to\infty}\langle f(z_0)-z_0,x_n-z_0\rangle\leq 0$ where $z_0=P_{F(T)}f(z_0)$. To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0)-z_0,x_n-z_0\rangle = \lim_{k\to\infty}\langle f(z_0)-z_0,x_{n_k}-z_0\rangle. \qquad (7)$$

Because all the $\{x_{n_k}\}$ lie in the weakly compact set C and C has Property G, we may assume without loss of generality that $\{x_{n_k}\} \rightharpoonup \omega$ for some $\omega \in C$ and $(x_{n_k}, \omega) \in E(G)$. Suppose $\omega \notin F(T)$, then $\omega \neq T\omega$. By G-nonexpansiveness of T, (6), and the Opial's condition, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \omega\| &< \liminf_{k\to\infty} \|x_{n_k} - T\omega\| \\
&\leq \liminf_{k\to\infty} \big(\|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - T\omega\|\big) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \omega\| .
\end{aligned}$$

This is a contradiction. Then $\omega \in F(T)$. Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\omega \in F(T)$. By (7) and Lemma 2.4, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle &= \lim_{k\to\infty} \langle f(z_0) - z_0, x_{n_k} - z_0\rangle \\
&= \langle f(z_0) - z_0, \omega - z_0\rangle \\
&\leq 0. \qquad (8)
\end{aligned}$$

Step 5 : Finally, we show that $\lim_{n\to\infty} x_n = z_0$, where $z_0 = P_{F(T)}f(z_0)$. By G-nonexpansiveness of T and $(z_0, x_n) \in E(G)$, and Lemma 2.5, we have

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\alpha_n(f(x_n) - z_0) + (1-\alpha_n)(Tx_n - z_0)\|^2 \\
&\leq \|(1-\alpha_n)(Tx_n - z_0)\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle \\
&= (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\|f(x_n) - f(z_0)\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + 2\alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\
&\leq (1-\alpha_n)^2\|x_n - z_0\|^2 + \alpha_n\alpha\|x_n - z_0\|^2 + \alpha_n\alpha\|x_{n+1} - z_0\|^2 \\
&\quad + 2\alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

It implies that

$$\|x_{n+1} - z_0\|^2 \leq \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}\|x_n - z_0\|^2 + \frac{2\alpha_n}{1-\alpha_n\alpha}\langle f(z_0) - z_0, x_{n+1} - z_0\rangle$$

$$= \left(1-\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\right)\|x_n-z_0\|^2+\frac{2\alpha_n(1-\alpha)}{1-\alpha_n\alpha}\left(\frac{\alpha_n}{2(1-\alpha)}\|x_n-z_0\|^2.\right.$$
$$\left.+\frac{1}{1-\alpha}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\right).$$

From the conditions (i), (ii), (8), and Lemma 2.3, we can conclude that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}f(z_0)$. This completes the proof. □

4. Numerical Results

The purpose of this section we give a numerical example to support our some result. The following example is given for supporting Theorem 3.2.

Example 4.1 : Let $H = \mathbb{R}$ and $C = [0,1]$ with the usual norm $\|x-y\| = |x-y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x,y) : x,y \in [0,\frac{3}{5}]$ such that $|x-y| \le \frac{1}{5}\}$. Let $f : C \to C$ be defined by $f(x) = \frac{x}{9}$, for all $x \in [0,1]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{10}x & \text{if } x \in [0,1), \\ \frac{8}{5} & \text{if } x = 1. \end{cases}$$

Solution : We observe that $F(T) = \{0\}$. Choose $x_0 = \frac{1}{5}$, then $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x,y) \in E(G)$. Then $x,y \in [0,\frac{3}{5}]$ and $|x-y| \le \frac{1}{5}$. It implies that

$$|Tx-Ty| \le \frac{1}{10}|x-y| \le |x-y| \le \frac{1}{5}.$$

Then, we have $(Tx,Ty) \in E(G)$ and $\|Tx-Ty\| \le \|x-y\|$. Thus T is G-nonexpansive. For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{2(n+1)}$. We rewrite (3) as follows:

$$x_{n+1} = \left(\frac{1}{2(n+1)}\right)\left(\frac{x_n}{9}\right)+\left(1-\frac{1}{2(n+1)}\right)\left(\frac{x_n}{10}\right). \tag{9}$$

Since $x_0 = \frac{1}{5} \in [0,\frac{1}{5}]$, from (9), we have

$$x_1 = \left(\frac{1}{2(1)}\right)\left(\frac{x_0}{9}\right)+\left(1-\frac{1}{2(1)}\right)\left(\frac{x_0}{10}\right).$$

By the convexity, we have $x_1 \in [0, \frac{1}{5}]$. Since $x_1 \in [0, \frac{1}{5}]$ and (9), we have

$$x_2 = \left(\frac{1}{2(2)}\right)\left(\frac{x_2}{9}\right) + \left(1 - \frac{1}{2(2)}\right)\left(\frac{x_2}{10}\right).$$

By the convexity, we have $x_2 \in [0, \frac{1}{5}]$. By continuing in this way, we have $x_n \in [0, \frac{1}{5}]$, for all $n \in \mathbb{N}$. It implies that $x_n \le \frac{1}{5}$ for all $n \in \mathbb{N}$. It follows that $(x_n, P_{F(T)}f(x_0)) = (x_n, 0) \in E(G)$. That is. $P_{F(T)}f(x_0)$ is dominated by $\{x_n\}$. It can be observed that parameters satisfy all the conditions of Theorem 3.2 and $C = [0, 1]$ satisfy Property G. Hence, the sequence $\{x_n\}$ converges strongly to 0.

Next, we show that T is not a nonexpansive mapping. Choose $x = 1$ and $y = \frac{3}{5}$, we have

$$\left|T(1) - T\left(\frac{3}{5}\right)\right| = \left|\frac{8}{5} - \frac{3}{50}\right| = \frac{77}{50} > \frac{2}{5} = \left|1 - \frac{3}{5}\right|.$$

Mathematicians know that the number π is an important mathematical constant. For the previous decades, many researcher have been trying to approximate the value of π; see [22, 23] and the references therein. By using our main result, we introduce the new method to approximate the value of π as shown in the following example.

Example 4.2 : Let $H = \mathbb{R}$ and $C = [3, 4]$ with the usual norm $\|x - y\| = |x - y|$ and let $G = (V(G), E(G))$ be such that $V(G) = C$, $E(G) = \{(x, y) : x, y \in [3, \frac{18}{5}]$ such that $|x - y| \le \frac{16}{5}\}$. Let $f : C \to C$ be defined by $f(x) = \frac{1}{5}x + \frac{4}{5}(\pi)$, for all $x \in [3, 4]$. Define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{3}x + \frac{2}{3}(\pi) & \text{if } x \in [3, 4), \\ \frac{56}{35} & \text{if } x = 4. \end{cases}$$

Solution : We observe that $F(T) = \{\pi\}$. Choose $x_0 = \frac{16}{5}$, then $(x_0, Tx_0) \in E(G)$. It is easy to see that $E(G)$ is convex. Let $(x, y) \in E(G)$. Then $x, y \in [3, \frac{18}{5}]$ and $|x - y| \le \frac{16}{5}$. It implies that

$$|Tx - Ty| = \left|\frac{1}{3}x + \frac{2}{3}(\pi) - \frac{1}{3}y - \frac{2}{3}(\pi)\right| \le \frac{1}{3}|x - y| \le |x - y| \le \frac{16}{5}.$$

Then, we have $(Tx, Ty) \in E(G)$ and $\|Tx - Ty\| \le \|x - y\|$. Thus T is G-nonexpansive. For every $n \in \mathbb{N}$, $\alpha_n = \frac{1}{2(n+1)}$. We rewrite (3) as follows:

$$x_{n+1} = \left(\frac{1}{2(n+1)}\right)\left(\frac{1}{5}x_n + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(n+1)}\right)\left(\frac{1}{3}x_n + \frac{2}{3}(\pi)\right). \tag{10}$$

Since $x_0 = \frac{16}{5} \in [3, \frac{16}{5}]$, from (10), we have

$$x_1 = \left(\frac{1}{2(1)}\right)\left(\frac{1}{5}x_0 + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(1)}\right)\left(\frac{1}{3}x_0 + \frac{2}{3}(\pi)\right).$$

By the convexity, we have $x_1 \in [3, \frac{16}{5}]$. Since $x_1 \in [3, \frac{16}{5}]$ and (10), we have

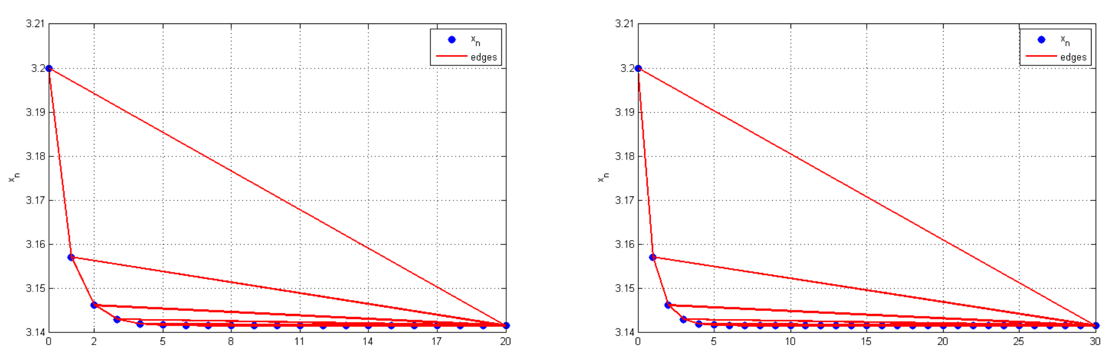
$$x_2 = \left(\frac{1}{2(2)}\right)\left(\frac{1}{5}x_1 + \frac{4}{5}(\pi)\right) + \left(1 - \frac{1}{2(2)}\right)\left(\frac{1}{3}x_1 + \frac{2}{3}(\pi)\right).$$

By the convexity, we have $x_2 \in (3, \frac{16}{5}]$. By continuing in this way, we have $x_n \in [3, \frac{16}{5}]$, for all $n \in \mathbb{N}$. It implies that $3 \leq x_n \leq \frac{16}{5}$ for all $n \in \mathbb{N}$. Then $|x_n - \pi| \leq \frac{16}{5}$ for all $n \in \mathbb{N}$. It follows that $(x_n, P_{F(T)}f(\pi)) = (x_n, \pi) \in E(G)$. That is. $P_{F(T)}f(\pi)$ is dominated by $\{x_n\}$. It can be observed that parameters satisfy all the conditions of Theorem 3.2 and $C = [3, 4]$ satisfy Property G. Since $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to π.

Next, we show that T is not a nonexpansive mapping. Choose $x = 4$ and $y = \frac{18}{5}$, we have

$$\begin{aligned}
\left|T(4) - T\left(\frac{18}{5}\right)\right| &= \left|\frac{56}{35} - \left(\frac{1}{3}\left(\frac{18}{5}\right) + \frac{2}{3}(\pi)\right)\right| \\
&\approx \left|\frac{56}{35} - \left(\frac{1}{3}\left(\frac{18}{5}\right) + \frac{2}{3}\left(\frac{22}{7}\right)\right)\right| \\
&= \frac{178}{105} \\
&> \frac{2}{5} \\
&= \left|4 - \frac{18}{5}\right|.
\end{aligned}$$

Using the algorithm (10) and choosing $x_0 = \frac{16}{5}$ with $n = 20$ and $n = 30$, we have the numerical result to approximate the value of π as shown in Table 1 and Figure 1.



(A) n=20 (B) n=30

Figure 1: The convergence of $\{x_n\}$ with initial values $x_0 = \frac{16}{5}$.

$n=20$		$n=30$	
n	x_n	n	x_n
0	3.200000000000000	0	3.200000000000000
1	3.157167945965848	1	3.157167945965848
2	3.146265241302610	2	3.146265241302610
3	3.143046347544892	3	3.143046347544892
4	3.142052990008907	4	3.142052990008907
⋮	⋮	⋮	⋮
10	3.141593185425522	15	3.141592659742157
⋮	⋮	⋮	⋮
16	3.141592654255843	26	3.141592653589803
17	3.141592653809197	27	3.141592653589797
18	3.141592653662115	28	3.141592653589794
19	3.141592653613647	29	3.141592653589793
20	3.141592653597665	30	3.141592653589793

Table 1: The values of the sequences $\{x_n\}$ with initial value $x_0 = \frac{16}{5}$.

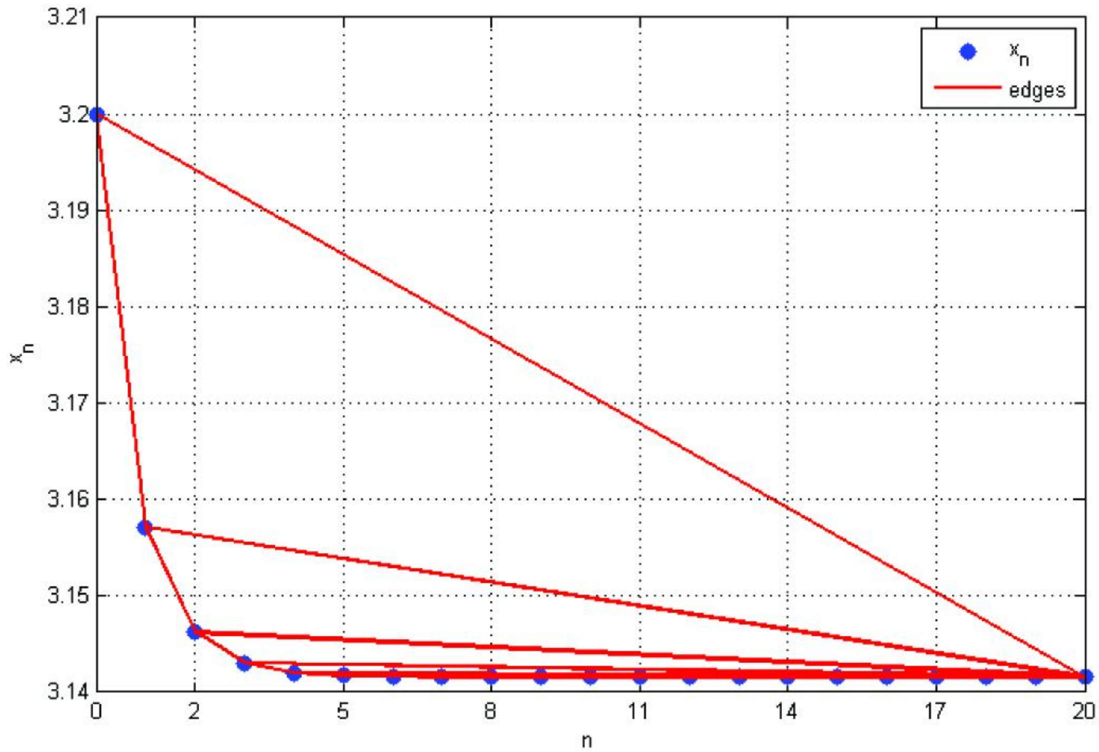
5. Conclusion

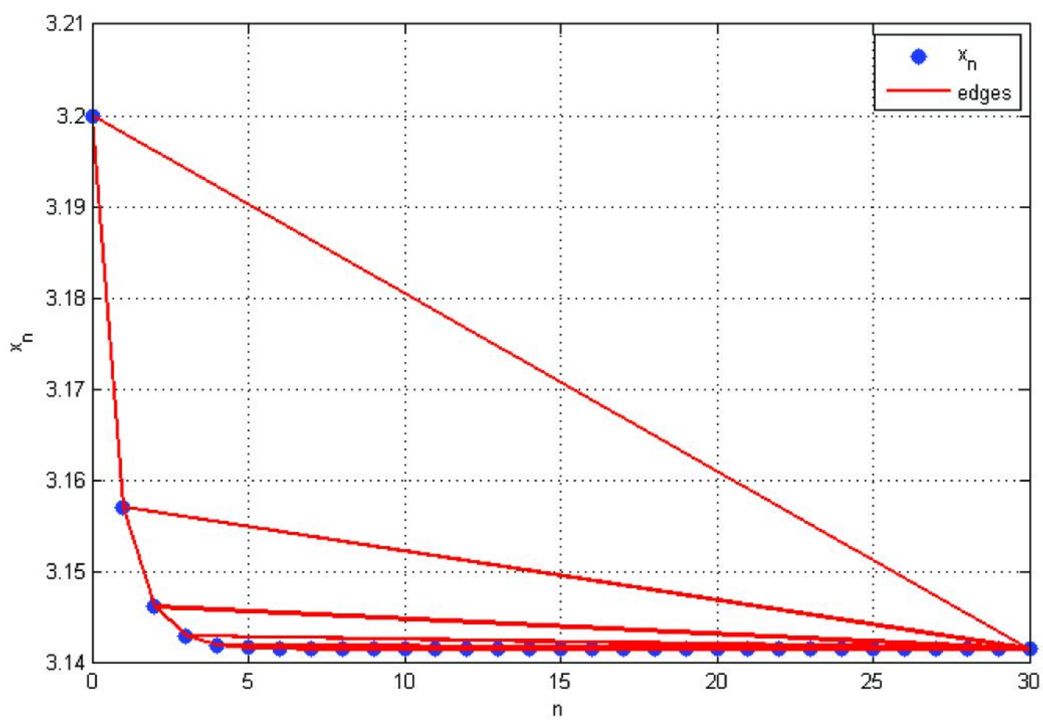
In this work, we introduce a viscosity approximation method of G-nonexpansive mapping defined on a Hilbert space endowed with a directed graph. We obtain a strong convergence theorem for the sequence generated by the proposed method under suitable conditions. However, we should like remark the following:

1. In Theorem 3.2, we use the concept of a viscosity approximation method and our result is proved with an assumption on a directed graph, which is a different result from Xu [14].
2. From Theorem 3.2, we can conclude that the sequence $\{x_n\}$, in Example 4.2, converges to π.
3. In Example 4.2, the sequence $\{x_n\}$ converges to π as shown in the Table 1 and Figure 1.
4. In order to gain more accuracy of π, the iterative approximation is depended on the number of n as shown in the Table 1.

Acknowledgement

The first author would like to thank the RMUTT Research Grant for New Scholar (NSF62D0601) for financial support. The second author would like to thank the Research and Innovation Services of KingMongkut's Institute of Tecnology Ladkrabang for financial support. We also would thank to the reviewers for their invaluable suggestions.


3.21
3.2
3.19
3.18
3.17
3.16
3.15
3.14
x_n
0
2
5
8
11
14
17
20
n
x_n
edges


3.21
3.2
3.19
3.18
3.17
3.16
3.15
3.14
x_n
0
5
10
15
20
25
30
n
x_n
edges

REFERENCES

1. F. Bojor, Fixed point of φ-contraction in metric spaces endowed with a graph, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **37**(4) (2010), 85-92.
2. F. Bojor, Fixed point theorems for Reich type contractions on metric spaces with a graph, *Nonlinear Anal.*, **75**(9) (2012), 3895-3901.
3. A. Sultana and V. Vetrivel, Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications, *J. Math. Anal. Appl.* **417** (2014), 336-344.
4. I. Beg, A. R. Butt, and S. Radojevic, The contraction principle for set valued mappings on a metric space with a graph, *Comput. Math. Appl.*, **60** (2010), 1214-1219.
5. S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fund. Math.*, **3** (1992), 133-181.
6. HS. Ding, L. Li, and S. Radenovic, Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces, *Fixed Point Theory Appl.*, **2012**, (2012). doi:10.1186/1687-1812-2012-96.
7. HS. Ding, Z. Kadelburg, and H. K. Nashine, Common fixed point theorems for weakly increasing mappings on ordered orbitally complete metric spaces, *Fixed Point Theory Appl.*, **2012**(85) (2012), https://doi.org/10.1186/1687-1812-2012-85.
8. J. Jachymski, The contraction principle for mappings on a metric space with a graph, *Proc. Am. Math. Soc.*, **136**(4) (2008), 1359-1373.
9. A. Hanjing and S. Suantai, Coincidence point and fixed point theorems for a new type of G-contraction multivalued mappings on a metric space endowed with a graph, *Fixed Point Theory Appl.*, **171** (2015). doi 10.1186/s13663-015-0420-4.
10. C. Klanarong and S. Suantai, Coincidence point theorems for some multi-valued mappings in complete metric spaces endowed with a graph, *Fixed Point Theory Appl.*, **129** (2015). doi 10.1186/s13663-015-0379-1.
11. T. Dinevari and M. Frigon, Fixed point results for multivalued contractions on a metric space with a graph, *J. Math. Anal. Appl.*, **405** (2013), 507-517.
12. M. R. Alfuraidan, Fixed points of monotone nonexpansive mappings with a graph, *Fixed Point Theory Appl.*, **49**(2015) (2015). doi:10.1186/s13663-015-0299-0.
13. J. Tiammee, A. Kaewkhao, and S. Suantai, On Browder's convergence theorem and Halpern iteration process for G-nonexpansive mappings in Hilbert spaces endowed with graphs, *Fixed Point Theory Appl.*, **187**(2015) (2015). doi:10.1186/s13663-015-0436-9.
14. H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, **298**

(2004), 279-291.

15. A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.*, **241** (2000), 46-55.

16. R. P. Agarwal, D. O'Regan, and D. R. Sahu, Fixed point theory for Lipschitzian type-mappings with applications, Springer, New York (2009).

17. H. K. Xu, An iterative approach to quadric optimization, *J. Optim. Theory Appl.*, **116** (2003), 659-678.

18. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama (2000).

19. Z. Opial, Weak convergence of the sequence of successive approximation of nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591-597.

20. C. Pang, R. Zhang, Q. Zhang, and J. Wang, Dominating sets in directed graphs, *Inf. Sci.*, **180** (2010), 3647-3652.

21. J. Bang-Jensen and G. Gutin, Digraphs theory, algorithms and applications, Springer Monographs in Mathematics, Springer, London (2007).

22. D. H. Bailey, The computation of Pi to 29360000 decimal digit using Borweins' quartically convergent algorithm, *Math. Comput.*, **50** (1988), 283-296.

23. J. M. Borwein, P. B. Borwein, and D. H. Bailey, Ramanujan, modular equations, and approximations to Pi, or how to compute one billion digits of Pi, *Amer. Math. Monthly*, **96** (1989), 201-219.