

ON THE NORM ATTAINMENT SET OF A BOUNDED LINEAR OPERATOR AND SEMI-INNER-PRODUCTS IN NORMED SPACES

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Dedicated to Dr. Dwijendra Nath Sain

We obtain a complete characterization of the norm attainment set of a bounded linear operator between normed spaces, in terms of semi-inner-product(s) defined on the space. In particular, this answers an open question raised recently in [D. Sain, On the norm attainment set of a bounded linear operator, *J. Math. Anal. Appl.*, **457** (2018), 67-76]. Our results illustrate the applicability of semi-inner-products towards a better understanding of the geometry of normed spaces.

Key words : Normed space; linear operator; norm attainment; semi-inner-product.

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1. INTRODUCTION

Norm of a vector is of fundamental importance in operator theory and functional analysis. The concept of norm of a bounded linear operator between normed spaces gives rise to the following two natural queries. First, whether the norm of the concerned operator is attained. And, if the answer to

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this question is in the affirmative, the second question is how to identify the points at which the norm of the operator is attained. While the first question has been explored and answered in great detail, the second question, surprisingly enough, had laid dormant for a long time. The primary purpose of this short note is to explore this question. As a matter of fact, we obtain a complete characterization of the norm attainment set of a bounded linear operator between normed spaces.

Another major highlight of the present work is that through our study, we also illustrate the applicability of the concept of semi-inner-products in the study of geometry of normed spaces. In recent times, the structure and properties of the norm attainment set of a bounded linear operator between normed spaces has been explored in [7, 8]. The present author obtained an interesting necessary condition for operator norm attainment in [8], in terms of Birkhoff-James orthogonality: If a bounded linear operator between normed spaces attains norm at a point on the unit sphere of the domain space then the operator preserves Birkhoff-James orthogonality at the said point. In the same article, complete characterization of the norm attainment set of a bounded linear operator between normed spaces was identified as an important open question in the geometry of Banach spaces. In this article, we obtain a solution to this problem by applying the concept of semi-inner-products. Without further ado, let us establish the relevant notations and terminologies to be used throughout the article.

Let \mathbb{X} , \mathbb{Y} be normed spaces of dimension strictly greater than 1. In this article, without mentioning any further, we work with only *real* normed spaces. It is easy to observe that the *complex* case can be treated similarly. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ denote the unit ball and the unit sphere of \mathbb{X} respectively. Let \mathbb{X}^* be the dual space of \mathbb{X} . Given any $x \in S_{\mathbb{X}}$, a functional $\psi_x \in S_{\mathbb{X}^*}$ is said to be a supporting functional at x if $\psi_x(x) = 1$. It is worth noticing that the Hahn-Banach theorem guarantees the existence of at least one supporting functional at each point of $S_{\mathbb{X}}$. We say that \mathbb{X} is smooth if given any $x \in S_{\mathbb{X}}$, there exists a unique supporting functional at x . Let $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ denote the normed space of all bounded linear operators from \mathbb{X} to \mathbb{Y} , endowed with the usual operator norm. Given $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, let M_T denote the norm attainment set of T , i.e., $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. We recall that T is said to be a similarity if T is a non-zero scalar multiple of an isometry. Clearly, if T is a similarity then $M_T = S_{\mathbb{X}}$. We next mention the concept of semi-inner-products in normed spaces, which is integral to the theme of this article.

Definition 1.1 — Let \mathbb{X} be a normed space. A function $[\cdot, \cdot] : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{R}$ is a semi-inner-product (s.i.p.) if for any $\alpha, \beta \in \mathbb{R}$ and for any $x, y, z \in \mathbb{X}$, it satisfies the following:

- (a) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$,
- (b) $[x, x] > 0$, whenever $x \neq 0$,

(c) $|[x, y]|^2 \leq [x, x][y, y]$,

(d) $[x, \alpha y] = \alpha[x, y]$.

Giles proved in [3] that every normed space $(\mathbb{X}, \|\cdot\|)$ can be represented as an s.i.p. space $(\mathbb{X}, [\cdot, \cdot])$ such that for all $x \in \mathbb{X}$, we have, $[x, x] = \|x\|^2$. Whenever we speak of a s.i.p. $[\cdot, \cdot]$ in context of a normed space \mathbb{X} , we implicitly assume that $[\cdot, \cdot]$ is compatible with the norm, i.e., for all $x \in \mathbb{X}$, we have, $[x, x] = \|x\|^2$. In general, there can be many compatible s.i.p. corresponding to a given norm. Lumer stated in [6] that there exists a unique s.i.p. on a normed space if and only if the space is smooth. In particular, for a normed space whose norm is induced by an inner product, the unique s.i.p. on the space is the inner product itself. In this connection, it should be noted that the homogeneity condition (d) was proved in Theorem 1 of the seminal paper [3]. However, although the statement of Theorem 1 of [3] is correct, the proof of the result, as presented in [3], contains a subtle logical gap. We point out this fact and also provide a rigorous proof of the concerned result.

We make use of the notion of Birkhoff-James orthogonality [1, 4] in normed spaces to relate s.i.p. with norm attainment of a bounded linear operator. An element $x \in \mathbb{X}$ is said to be orthogonal to another element $y \in \mathbb{X}$ in the sense of Birkhoff-James, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all scalars λ . While studying Birkhoff-James orthogonality of bounded linear operators, the author introduced the following notations in [7]: For any two elements x, y in \mathbb{X} , let us say that $y \in x^+$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \geq 0$. Accordingly, we say that $y \in x^-$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \leq 0$. Let $x^\perp = \{y \in \mathbb{X} : x \perp_B y\}$.

Using the concept of s.i.p., we obtain a complete characterization of the norm attainment set of a bounded linear operator between normed spaces. This evidently illustrates the strength of s.i.p. type arguments in studying the geometry of normed spaces. The following open question was posed in [8]:

Let T be a bounded linear operator defined on a Banach space \mathbb{X} . Find a necessary and sufficient condition for $x \in S_{\mathbb{X}}$ to be such that $x \in M_T$.

Our result gives a complete solution to this problem. We would also like to mention that our answer is not restricted by the additional condition of completeness assumed in the above mentioned problem. Furthermore, we do not require the domain space and the range space to be identical. Our characterization is particularly illuminating when the domain space and the range space are smooth. We deduce that in particular, it is possible to completely describe the action of a bounded linear functional on a smooth normed space provided we know that the value of the functional equals its norm at some particular point of the unit sphere of the domain space. As another immediate application of

our approach towards the operator norm attainment problem, we observe that it is possible to obtain a complete characterization of similarity mappings between any two normed spaces. We would like to note that the same characterization was obtained by Chmieliński in Theorem 2.1 of [2], by using arguments from [5]. However, our method is completely different from that in [2] in the sense that our focus is on the local problem of operator norm attainment from a geometric point of view. Indeed, Theorem 2.2 of the present article should be viewed as a local version of Theorem 2.1 of [2], from which the global version follows easily.

2. MAIN RESULTS

As mentioned in the introduction, the major highlight of this section is to obtain a complete characterization of the norm attainment set of a bounded linear operator between normed spaces. However, before proceeding in that direction, let us first obtain a rigorous proof of Theorem 1 of [3]. We require the following lemma for that purpose.

Lemma 2.1 — Let \mathbb{X} be a normed linear space. Then there exists a subset \mathcal{A} of $S_{\mathbb{X}}$ with the following property: given any $x \in S_{\mathbb{X}}$, exactly one element of the set $\{x, -x\}$ belongs to \mathcal{A} .

PROOF : In order to prove the result, we require an application of the axiom of choice: for every function F acting from a non-empty set D to the collection of non-empty subsets of another set B , there is a function $f : D \rightarrow B$ such that $f(d) \in F(d)$ for every $d \in D$. We note that such an F is called a multifunction and f is called a selector of F . Let us now consider the following equivalence relation on $S_{\mathbb{X}} : x \sim y$ if x is linearly dependent with y . Let $S_{\mathbb{X}} / \sim$ denote the corresponding set of equivalence classes. Consider the multifunction F acting from $S_{\mathbb{X}} / \sim$ to the collection of non-empty subsets of $S_{\mathbb{X}}$ as follows: $F(d)$ is the set of all elements of $S_{\mathbb{X}}$ that belong to the equivalence class d . As mentioned before, it follows from the axiom of choice that there exists a selector f of F . Let \mathcal{A} be the range of f , i.e., $\mathcal{A} = f(S_{\mathbb{X}} / \sim)$. It is immediate that \mathcal{A} is a subset of $S_{\mathbb{X}}$ with the desired property. \square

Now we are in a position to obtain a rigorous proof of Theorem 1 of [3].

Theorem 2.1 — *Every normed space \mathbb{X} can be represented as an s.i.p. space with the homogeneity property $[x, \lambda z] = \lambda[x, z]$, for any $x, z \in \mathbb{X}$ and for any $\lambda \in \mathbb{R}$.*

PROOF : It follows from Lemma 2.1 that there exists a subset \mathcal{A} of $S_{\mathbb{X}}$ with the following property: given any $u \in S_{\mathbb{X}}$, exactly one element of the set $\{u, -u\}$ belongs to \mathcal{A} . Now, by the Hahn-Banach theorem, given any $y \in \mathcal{A}$, there exists at least one $f_y \in S_{\mathbb{X}^*}$ such that $f_y(y) = 1$. For each $y \in \mathcal{A}$, we choose exactly one such $f_y \in S_{\mathbb{X}^*}$ by using the axiom of choice. Given any $y \in \mathcal{A}$ and any $\lambda \in \mathbb{R}$,

we define $f_{\lambda y} = \lambda f_y$. Now, given any $x, z \in \mathbb{X}$, let us define $[x, z] = f_z(x)$. It is easy to verify that $[,]$ is indeed an s.i.p. with the desired homogeneity property $[x, \lambda z] = \lambda[x, z]$, for any $x, z \in \mathbb{X}$ and for any $\lambda \in \mathbb{R}$. This completes the proof of the theorem. \square

In the next remark, we further illustrate the importance of Lemma 2.1 in order to obtain a rigorous proof of Theorem 1 of [3].

Remark 2.1 : In the proof of Theorem 1 of [3], exactly one supporting functional f_x has been chosen at each point x of the unit sphere $S_{\mathbb{X}}$. Now, if $x_0 \in S_{\mathbb{X}}$ is not a smooth point then there are two distinct supporting functionals $f_0, g_0 \in S_{\mathbb{X}^*}$ at the point x_0 . Clearly, $-f_0, -g_0$ are distinct supporting functionals at the point $-x_0 \in S_{\mathbb{X}}$. Therefore, while choosing the supporting functional at each point of the unit sphere, if f_0 is chosen for the point x_0 and $-g_0$ is chosen for the point $-x_0$, then the required homogeneity property of the corresponding s.i.p. is already violated on the unit sphere of the space. This explains why we need to consider a special subset of $S_{\mathbb{X}}$, instead of the whole of $S_{\mathbb{X}}$. Lemma 2.1 guarantees the existence of such a subset of $S_{\mathbb{X}}$ in any normed space \mathbb{X} . We would like to note that the proof presented in Theorem 1 of [3] is valid if the normed space is smooth.

Before proving our main result on operator norm attainment, we need the following two lemmas. The proof of the first lemma is omitted, as it is rather obvious.

Lemma 2.2 — Let \mathbb{X} be a normed space and let A, B be two non-empty disjoint open subsets of \mathbb{X} . Then there does not exist a path in \mathbb{X} that starts at a point in A , ends at a point in B and lies entirely within $A \cup B$.

Lemma 2.3 — Let \mathbb{X}, \mathbb{Y} be normed spaces and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Let $z \in S_{\mathbb{X}}$ be such that $z \in M_T$. Then there exists $y \in \mathbb{X} \setminus \{0\}$ such that $z \perp_B y$ and $Tz \perp_B Ty$.

PROOF : Clearly, the result is true if T is the zero operator. Let us assume that T is non-zero. It follows from Proposition 2.1 of [7] that $\mathbb{X} = (z^+ \setminus z^\perp) \cup (z^- \setminus z^\perp) \cup (z^\perp)$. Let us choose $u \in (z^+ \setminus z^\perp)$ and $v \in (z^- \setminus z^\perp)$ such that the line segment joining u and v does not pass through origin. We note that such a choice of u and v is always possible. We further note that the first two statements of Theorem 2.3 of [8] does not depend on the smoothness of the concerned spaces. Therefore, we have, $Tu \in ((Tz)^+ \setminus (Tz)^\perp)$ and $Tv \in ((Tz)^- \setminus (Tz)^\perp)$. Therefore, applying the continuity of T , it is easy to see that we have the following:

$\{T((1-t)u + tv) : t \in [0, 1]\}$ is a path in \mathbb{Y} , that begins at $Tu \in ((Tz)^+ \setminus (Tz)^\perp)$ and ends at $Tv \in ((Tz)^- \setminus (Tz)^\perp)$.

Since $(Tz)^+ \cap (Tz)^- = (Tz)^\perp$, it follows that $((Tz)^+ \setminus (Tz)^\perp) \cap ((Tz)^- \setminus (Tz)^\perp) = \emptyset$. It is

also easy to observe that since T is non-zero, both $A = ((Tz)^+ \setminus (Tz)^\perp)$ and $B = ((Tz)^- \setminus (Tz)^\perp)$ are non-empty open subsets of \mathbb{Y} . Therefore, by Lemma 2.2, there cannot exist a path that starts at a point in A , ends at a point in B and lies completely within $A \cup B$. In other words, there exists $t_0 \in (0, 1)$ such that $T((1 - t_0)u + t_0v) \notin A \cup B$. On the other hand, since $T(\mathbb{X}) \subseteq A \cup B \cup (Tz)^\perp$, it follows that $T((1 - t_0)u + t_0v) \in (Tz)^\perp$. Once again, it follows from (i) and (ii) of Theorem 2.3 of [8], that $(1 - t_0)u + t_0v \notin (z^+ \setminus z^\perp) \cup (z^- \setminus z^\perp)$. Since $\mathbb{X} = (z^+ \setminus z^\perp) \cup (z^- \setminus z^\perp) \cup (z^\perp)$, we must have, $(1 - t_0)u + t_0v \in z^\perp$. Let us choose $y = (1 - t_0)u + t_0v$. It follows from our choice of u and v that $y \neq 0$. Furthermore, we have already proved that $z \perp_B y$ and $Tz \perp_B Ty$. This completes the proof of the lemma. \square

Now we obtain the promised characterization of the norm attainment set of a bounded linear operator between normed spaces. We would like to remark that the crux of our argument is practically dependent on Theorem 2.3 of [8], that states that if a bounded linear operator between smooth normed spaces attains norm at a particular point of the unit sphere then the operator preserves Birkhoff-James orthogonality at that point. Lemma 2.3 allows us to apply our argument in case of any normed space, not necessarily smooth.

Theorem 2.2 — *Let \mathbb{X}, \mathbb{Y} be normed spaces and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Let $z \in S_{\mathbb{X}}$. Then $z \in M_T$ if and only if given any $x \in \mathbb{X}$, there exists two s.i.p. $[\cdot, \cdot]_{\mathbb{X}}$ and $[\cdot, \cdot]_{\mathbb{Y}}$ on \mathbb{X} and \mathbb{Y} respectively such that*

$$[Tx, Tz]_{\mathbb{Y}} = \|T\|^2[x, z]_{\mathbb{X}}.$$

PROOF : We observe that the sufficient part of the theorem follows trivially. Indeed, choosing $x = z$, we obtain that $[Tz, Tz]_{\mathbb{Y}} = \|T\|^2[z, z]_{\mathbb{X}}$. This is clearly equivalent to the fact that $\|Tz\| = \|T\|$, i.e., $z \in M_T$. Let us now prove the necessary part of the theorem. If T is the zero operator then the result follows immediately. Let us assume that T is non-zero. Let $z \in M_T$. Let $x \in \mathbb{X}$ be arbitrary. We may and do assume that x, z are linearly independent. Let \mathbb{Z} be the two-dimensional normed space spanned by $\{x, z\}$, with the induced norm. By Lemma 2.3, there exists $y \in \mathbb{Z} \setminus \{0\}$ such that $z \perp_B y$ and $Tz \perp_B Ty$. We would like to construct two s.i.p. $[\cdot, \cdot]_{\mathbb{X}}$ and $[\cdot, \cdot]_{\mathbb{Y}}$ on \mathbb{X} and \mathbb{Y} respectively such that the desired condition holds.

Clearly, $x = \lambda_0 z + \mu_0 y$, for some scalars $\lambda_0, \mu_0 \in \mathbb{R}$. Since $z \perp_B y$, there exists a functional $l_z : \mathbb{Z} \rightarrow \mathbb{R}$ such that $\|l_z\| = l_z(z) = \|z\| = 1$ and $l_z(y) = 0$. By the Hahn-Banach theorem, l_z has a norm preserving extension $\psi_z : \mathbb{X} \rightarrow \mathbb{R}$. Clearly, $\psi_z(y) = 0$. Let us now define a suitable s.i.p. on \mathbb{X} . It follows from Lemma 2.1 that there exists a subset \mathcal{A} of $S_{\mathbb{X}}$ with the following property: given any $x \in S_{\mathbb{X}}$, exactly one element of the set $\{x, -x\}$ belongs to \mathcal{A} . Moreover, we assume without loss of any generality that $z \in \mathcal{A}$. Now, by virtue of the Hahn-Banach theorem, given any $w \in \mathcal{A}$, there

exists at least one bounded linear functional $g_w \in \mathbb{X}^*$ such that $g_w(w) = \|g_w\| = 1$. Applying the axiom of choice, we choose exactly one such g_w for each $w \in \mathcal{A}$. For $\lambda w \in \mathbb{X}$, where $\lambda \in \mathbb{R}$ and $w \in \mathcal{A}$, we choose $g_{\lambda w} \in \mathbb{X}^*$ such that $g_{\lambda w} = \lambda g_w$. Our only modification is the following:

It is easy to see that ψ_z is a candidate for g_z . We choose $g_z = \psi_z$.

We now define a s.i.p. $[\cdot, \cdot]_{\mathbb{X}}$ on \mathbb{X} by $[v, w] = g_w(v)$ for any $v, w \in \mathbb{X}$. We note that it is easy to check that $[\cdot, \cdot]_{\mathbb{X}}$ is indeed a s.i.p. on \mathbb{X} . Moreover, we have,

$$[x, z]_{\mathbb{X}} = [\lambda_0 z + \mu_0 y, z]_{\mathbb{X}} = \lambda_0 [z, z]_{\mathbb{X}} + \mu_0 [y, z]_{\mathbb{X}} = \lambda_0 \|z\|^2 = \lambda_0.$$

On the other hand, since T is non-zero and $z \in M_T$, we must have, Tz is non-zero in \mathbb{Y} . Furthermore, we have already proved that $Tz \perp_B Ty$. Therefore, following similar method, we can define a s.i.p. $[\cdot, \cdot]_{\mathbb{Y}}$ on \mathbb{Y} such that $[Ty, Tz]_{\mathbb{Y}} = 0$. Thus, we have,

$$[Tx, Tz]_{\mathbb{Y}} = [\lambda_0 Tz + \mu_0 Ty, Tz]_{\mathbb{Y}} = \lambda_0 [Tz, Tz]_{\mathbb{Y}} + \mu_0 [Ty, Tz]_{\mathbb{Y}} = \lambda_0 \|T\|^2 = \|T\|^2 [x, z]_{\mathbb{X}}.$$

Since $x \in \mathbb{X}$ was chosen arbitrarily, this completes the proof of the necessary part of the theorem and establishes it completely. □

Remark 2.2 : As mentioned in the introduction, Theorem 2.2 gives a complete answer to the open question posed in [8], regarding a necessary and sufficient condition for a bounded linear operator to attain norm at a given point of the unit sphere. Moreover, we do not require the additional assumption of completeness, nor do we require the normed spaces to be finite-dimensional or identical.

We would like to draw a series of corollaries from the above theorem, in order to illustrate its applicability towards studying the geometry of normed spaces from the point of view of operator norm attainment. Our first observation is that while Theorem 2.2 is undoubtedly of theoretical importance, it is particularly interesting if the corresponding domain space and the codomain space are smooth. It follows from [6] that in this case there exists a unique s.i.p. on both the domain space and the codomain space. Therefore, the following result follows directly from Theorem 2.2.

Corollary 2.2.1 — Let \mathbb{X}, \mathbb{Y} be smooth normed spaces and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. Let $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$ be the unique s.i.p. on \mathbb{X} and \mathbb{Y} respectively. Let $z \in S_{\mathbb{X}}$. Then $z \in M_T$ if and only if $[Tx, Tz]_2 = \|T\|^2 [x, z]_1$ for any $x \in \mathbb{X}$.

On the other hand, considering the codomain space to be \mathbb{R} in Theorem 2.2, we obtain a complete characterization of the norm attainment set of a bounded linear functional on a normed space.

Corollary 2.2.2 — Let \mathbb{X} be a normed space and let $f \in \mathbb{X}^*$. Let $z \in S_{\mathbb{X}}$. Then $z \in M_f$ if and only if there exists a s.i.p. $[\cdot, \cdot]$ on \mathbb{X} such that $|fx| = |[x, \|f\|z]|$ for any $x \in \mathbb{X}$.

PROOF : The sufficient part follows trivially, as before. Let us prove the necessary part. Let $z \in M_f$. Then $fz = \pm\|f\|$. Let us first assume that $fz = \|f\|$. We note that the unique s.i.p. on \mathbb{R} is the usual scalar product. Therefore, it follows from Theorem 2.2 that there exists a s.i.p. $[\cdot, \cdot]$ on \mathbb{X} such that $\|f\|^2[x, z] = [fx, fz] = fx.fz = \|f\|fx$. In other words, $fx = [x, \|f\|z]$. Similarly, if we assume that $fz = -\|f\|$ then it can be shown that $fx = -[x, \|f\|z]$. In either case, we have that $|fx| = |[x, \|f\|z]|$. This completes the proof of the corollary. \square

Remark 2.3 : We would like to note that it follows from Corollary 2.2.2 that given any bounded linear functional f of unit norm, on a smooth normed space \mathbb{X} , if we know beforehand that $z \in S_{\mathbb{X}}$ is such that $f(z) = \|f\|$ then the value of f at any point of \mathbb{X} can be evaluated by using the unique s.i.p. on \mathbb{X} .

As another example of the extensive application of s.i.p. in the study of geometry of normed spaces, we observe that it is possible to characterize the set of all similarity mappings between any two real normed spaces. Indeed, the following result is immediate from Theorem 2.2:

Corollary 2.2.3 — Let \mathbb{X}, \mathbb{Y} be real normed spaces and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be non-zero. Then T is a similarity if and only if given any $x \in \mathbb{X}$ and any $z \in S_{\mathbb{X}}$, there exists two s.i.p. $[\cdot, \cdot]_{\mathbb{X}}$ and $[\cdot, \cdot]_{\mathbb{Y}}$ on \mathbb{X} and \mathbb{Y} respectively such that

$$[Tx, Tz]_{\mathbb{Y}} = \|T\|^2[x, z]_{\mathbb{X}}.$$

In view of the results obtained in the present article, it is apparent that s.i.p. type arguments can be extensively applied to explore the geometry of bounded linear operators between normed spaces.

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