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A GENERALIZATION OF POSNER'S THEOREM ON DERIVATIONS IN RINGS

Fuad Ali Ahmed Almahdi*, Abdellah Mamouni** and Mohammed Tamekkante***

**Department of Mathematics, Faculty of Sciences, King Khalid University, P. O. Box* 9004, *Abha, Saudi Arabia*

***Department of Mathematics, Faculty of Science and Technology Box 509-Boutalamine, University Moulay Ismail, Errachidia, Morocco*

****Laboratory MACS, Faculty of Sciences, University Moulay Ismail,* 5000, *P.B.* 11201, *Zitoune, Meknes, Morocco*

e-mails: fuadalialmahdy@hotmail.com; a.mamouni.fste@gmail.com; tamekkante@yahoo.fr

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In this paper, we generalize the Posner's theorem on derivations in rings as follows: Let R be an arbitrary ring, P be a prime ideal of R, and d be a derivation of R. If $[[x, d(x)], y] \in P$ for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative. In particular, if R is semiprime and d is a centralizing derivation of R, we prove that either R is commutative or there exists a minimal prime ideal P of R such that $d(R) \subseteq P$. As a consequence, we show that for any semiprime ring with a centralizing derivation there exists at least a minimal prime ideal P such that $d(P) \subseteq P$.

Key words : Prime and semiprime rings; Posner's result.

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1. Introduction

Throughout, R denotes an associative ring with center $Z(R)$. Recall that an ideal P of R is said to be prime if $P \neq R$ and, for $a, b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. The ring R is called a prime ring if (0) is a prime ideal of R. The Lie product of two elements x and y of R is $[x, y] = xy - yx$. A mapping $f : R \to R$ is called centralizing (on R) if $[x, f(x)] \in Z(R)$ for all $x \in R$; in the special case where $[x, f(x)] = 0$ for all $x \in R$, the mapping f is said to be commuting on R. By a derivation of R, we mean an additive map $d : R \to R$ satisfying $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Several authors subsequently proved commutativity theorems for prime rings admitting derivations which are centralizing on R. This work was initiated by Posner [16] who proved that a prime ring R admitting a nonzero centralizing derivation is commutative. Since then a number of authors have extended the Posner's result in several directions. Let R be a prime ring. In [17], Vukman proved that if R admits a nonzero derivation d such that the mapping $x \to [d(x), x]$ is centralizing on R, then R is commutative provided the characteristic of R is different from 2 and 3. In [12], Deng and Bell gave an extension of the Vukman's result as follows: Let R be a 6-torsion-free semiprime ring and U a nonzero left ideal of R. If R admits a derivation d such that $d(U) \neq (0)$ and the map $x \to [d(x), x]$ is centralizing on U, then R contains a nonzero central ideal. An other generalization of the Posner's theorem to the case of semiprime without assuming that the ring is 6-torsion-free is given by Bell and Martindale [5] as follows: Let R be a semiprime ring and U a nonzero left ideal of R. If R admits a nonzero derivation d such that $d(U) \neq (0)$ and centralizing on U, then R contains a nonzero central ideal. Other works in this line can be found, for example, in [2, 7, 8, 11, 14]).

In this paper, we are interested in the study of prime rings given as a quotient R/P where R is an arbitrary ring and P is a prime ideal of R. The originality in this work is that we use a derivation on R (and not on R/P) which satisfies an algebraic property on R with respect to P. Let R be a ring, P be a prime ideal of R, and d be a derivation of R. The main result of this paper (Theorem 2.2) states that if $[[x, d(x)], y] \in P$ for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative. When R is a prime ring, the Posner's result is a consequence of Theorem 2.2 by letting $P = (0)$. Recall that a ring R is called semiprime if, for $a \in R$, $aRa = \{0\}$ implies that $a = 0$. An other consequence of our main result affirms that, if R is a semiprime ring and d is a centralizing derivation of R, then R is commutative or there exists a minimal prime ideal P such that $d(R) \subseteq P$ (Corollary 2.4).

In the sequel, we use frequently the basic commutator identities:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z \text{ for all } x, y, z \in R,$$

and some well-known facts about prime rings. Namely, (1) if R is a prime ring, $x \in Z(R)$, and $y \in R$, then $xy = 0$ implies $x = 0$ or $y = 0$, and (2) if R is a prime ring so R is either of characteristic two or $2x = 0$ implies $x = 0$ for x in R.

2. The Results

We begin with the following lemma.

Lemma 2.1 — Let R be a ring, P be a prime ideal of R, and d be a derivation of R. If $[x, d(x)] \in P$ for all $x \in R$, then $d(R) \subseteq P$ or R/P is commutative.

PROOF : Linearizing $[x, d(x)]$ gives

$$[x, d(y)] + [y, d(x)] \in P \quad \text{for all } x, y \in R. \tag{2.1}$$

Replacing y by yx in (2.1), we get

$$[x, d(y)]x + y[x, d(x)] + [x, y]d(x) + y[x, d(x)] + [y, d(x)]x \in P \quad \text{for all } x, y \in R. \tag{2.2}$$

Multiplying (2.1) by x on right we have

$$[x, d(y)]x + [y, d(x)]x \in P \quad \text{for all } x, y \in R. \tag{2.3}$$

Using the hypothesis, (2.2), and (2.3) we have

$$[x, y]d(x) \in P \quad \text{for all } x, y \in R. \tag{2.4}$$

Replacing y by zy we have

$$[x, z]yd(x) + z[x, y]d(x) \in P \quad \text{for all } x, y, z \in R. \tag{2.5}$$

By (2.4), we have

$$[x, z]yd(x) \in P \quad \text{for all } x, y, z \in R. \tag{2.6}$$

That is

$$[x, z]Rd(x) \in P \quad \text{for all } x, z \in R. \tag{2.7}$$

Since P is prime, we have

$$[x, z] \in P \text{ or } d(x) \in P \quad \text{for all } x, z \in R. \tag{2.8}$$

Suppose that $d(R) \nsubseteq P$. There exists $x \in R$ such that $d(x) \notin P$. Hence, by (2.8), $[x, z] \in P$ for all $z \in R$. Hence, $\overline{x} \in Z(R/P)$. Let $y \in R$ such that $\overline{y} \notin Z(R/P)$. Hence, there exists $z_0 \in R$ such that $[y, z_0] \notin P$. Therefore, by (2.8), $d(y) \in P$. Accordingly, $d(x + y) = d(x) + d(y) \notin P$ since $d(x) \notin P$. By (2.8) again, $[x + y, z] \in P$ for all $z \in R$, and so $[y, z] \in P$ for all $z \in R$ since $[x, z] \in P$ for all $z \in R$, a contradiction. □

***Theorem* 2.2** — *Let R be a ring, P be a prime ideal of R, and d be a derivation of R. If $[[x, d(x)], y] \in P$ for all $x, y \in R$, then $d(R) \subseteq P$ or R/P is commutative.*

PROOF : For all $x \in R$, we have

$$\begin{aligned} [x^2, d(x^2)] &= [x^2, d(x)x + xd(x)] \\ &= 2[x^2, xd(x)] - [x^2, [x, d(x)]] \\ &= 2x[x, xd(x)] + 2[x, xd(x)]x - [x^2, [x, d(x)]] \\ &= 2x^2[x, d(x)] + 2x[x, d(x)]x - [x^2, [x, d(x)]] \end{aligned}$$

By hypothesis, we have $[[x, d(x)], y] \in P$ for all $x, y \in R$ (which means that $\overline{[x, d(x)]}$ is central in R/P), and in particular $[x^2, [x, d(x)] \in P$ for all $x \in R$. Thus, from above

$$\overline{[x^2, d(x^2)]} = 4\overline{x^2[x, d(x)]} \quad \text{for all } x \in R. \tag{2.9}$$

Keeping in mind that $\overline{[x^2, d(x^2)]}$ is central in R/P and using (2.9) we have

$$4\overline{[x^2[x, d(x)], d(x)])} = \overline{0} \quad \text{for all } x \in R. \tag{2.10}$$

That is

$$4\overline{[x, d(x)][x^2, d(x)]} = \overline{0} \quad \text{for all } x \in R. \tag{2.11}$$

Hence,

$$8\overline{x[x, d(x)]^2} = \overline{0} \quad \text{for all } x \in R. \tag{2.12}$$

Suppose R/P is not of characteristic two. Then, by (2.12) and the hypothesis, we have $\overline{x} = \overline{0}$ or $\overline{[x, d(x)]} = \overline{0}$ since R/P is a prime ring. Lemma 2.1 finishes the proof.

In case R/P is of characteristic two, the symbol $\equiv$ will always mean $\equiv (modP)$ and, modulo P, the operation $(-)$ will be written $(+)$.

The hypothesis says

$$[[x, d(x)], z] \equiv 0 \quad \text{for all } x, z \in R, \tag{2.13}$$

which implies that $[[x, d(x)], x] \equiv 0$ showing that

$$xd(x)x + d(x)x^2 + x^2d(x) + xd(x)x \equiv [x^2, d(x)] \equiv 0.$$

Similarly $[x, d(x)^2] \equiv 0$ and, hence, $[x^2, d(x)^2] \equiv 0$.

On the other hand, linearizing the hypothesis yields

$$[[x, d(y)] + [y, d(x)], z] \equiv 0 \quad \text{for all } x, y, z \in R. \tag{2.14}$$

Moreover, for all $x, y \in R$,

$$[xy+yx, d(x)]+[x^2, d(y)] \equiv x([y, d(x)]+[x, d(y)])+([y, d(x)]+[x, d(y)])x+y[x, d(x)]+[x, d(x)]y. \tag{2.15}$$

As can be seen by expanding both sides. Then, (2.14) shows that the second expression is $\equiv 0$. Replacing y by $d(x)x$ in (2.15) yields

$$[xd(x)x + d(x)x^2, d(x)] + [x^2, d^2(x)x + d(x)^2] \equiv 0 \quad \text{for all } x \in R. \tag{2.16}$$

However,

$$[xd(x)x + d(x)x^2, d(x)] \equiv [[x, d(x)]x, d(x)] \equiv [x, d(x)]^2 \quad \text{for all } x \in R. \tag{2.17}$$

and

$$[x^2, d^2(x)x + d(x)^2] \equiv [x^2, d^2(x)] \equiv 0 \quad \text{for all } x \in R. \tag{2.18}$$

Replacing y by $d(x)$ and z by x in (2.14) shows

$$0 \equiv [[x, d^2(x)], x] \equiv [x^2, d^2(x)] \quad \text{for all } x \in R. \tag{2.19}$$

By putting (2.19) and (2.17) in (2.18), the conclusion follows. □

If R is a prime ring, we obtain the Posners's result by taking $P = (0)$ (which is a prime ideal).

Corollary 2.3 — ([16, Theorem 2]). Let R be a prime ring and f be a centralizing derivation of R. Then, $d = 0$ or R is commutative.

Given a derivation d of R, an ideal I of R is said to be invariant under d (or d-ideal for short) if $d(I) \subseteq I$. If I is such an ideal of R then d induces a derivation $\overline{d}$ of R/I which inherits many properties of d. If we have a family of d-ideals I_α with trivial intersection $\cap I_\alpha = \{0\}$, then the ring R can be embedded in the direct product of R/I_α and properties of R/I_α can usually be integrated to give properties of R. In this way, we wish that questions of derivations on semiprime rings may be reduced to questions of derivations on prime rings. This amounts to the existence of minimal prime d-ideals with trivial intersection for semiprime rings. Let R be a ring. An element $x \in R$ is said to be torsion if there exists a nonzero $n \in \mathbb{Z}$ such that $nx = 0$, and R is said to be torsion-free if the only torsion element of R is 0. It is well known that every minimal prime ideal of a torsion-free semiprime unital ring is invariant under all derivations [13]. Herstein raised the following problem:

Problem : Given a semiprime ring R, does $d(P) \subseteq P$ hold for any minimal prime ideal P of R and for any derivation d of R.

The problem was also mentioned in [6, 15]. If this problem had an affirmative answer, then Theorem 2.2 will be easily seen for minimal prime ideals. Indeed, the condition $[[x, d(x)], y] \in P$ for all $x, y \in R$ just means that $\overline{d}$ is centralizing on R/P. So, if $d(R) \nsubseteq P$, then $\overline{d}$ is a nonzero derivation, and hence by Posner's result R/P is commutative. However, this Problem turns out to be false in general. Chuang and Lee [10] constructed a semiprime ring R which possesses a minimal prime ideal not invariant under a derivation of the ring. Which gives a meaning to our main result.

A ring R is said to be of bounded index m if m is a positive integer such that $x^m = 0$ for all nilpotent elements $x \in R$. Beidar and Mikhalev proved the theorem: Let R be a ring of bounded index m such that the additive order of every nonzero torsion element of R, if any, is strictly larger than m. Then all minimal prime ideals of R are invariant under all derivations of R (see [3] or [4, Theorem 8.16]). As a special case of this, every minimal prime ideal of a reduced ring is invariant under derivations of the ring (see [10, p. 614]).

In the following result we prove that a semiprime ring having a centralizing derivation d must have at lest one a minimal prime d-ideal.

Corollary 2.4 — Let R be a semiprime ring and d a centralizing derivation of R. Then, R has a minimal prime ideal P such that $d(P) \subseteq P$.

PROOF : (1) Set $\mathrm{Min}(R)$ the set of minimal prime ideals of R. Suppose that, for any $P \in \mathrm{Min}(R)$, we have $d(P) \nsubseteq P$. Then, by Theorem 2.2, R/P is commutative (for all $P \in \mathrm{Min}(R)$). Then, for all $x, y \in R$, we have $[x, y] \in P$ (for all $P \in \mathrm{Min}(R)$). Since $\bigcap_{P \in \mathrm{Min}(R)} P = \{0\}$, we get $[x, y] = 0$ for all $x, y \in R$. So, R is commutative. Hence, R is a reduced ring, and so every minimal prime ideal of R is a d-ideal (see [6, p. 614]), a contradiction. □

Corollary 2.5 — Let R be a semiprime ring and d be a centralizing derivation of R. Then, $d(R) \subseteq Z(R)$.

PROOF : Let $x \in R$. Let also P be any minimal prime ideal of R. Using Theorem 2.2, we have $d(R) \subseteq P$ or R/P is commutative. Then, $d(x) \in P$ or $[d(x), y] \in P$ for all $y \in R$. In the both cases we have $[d(x), y] \in P$ for all $y \in R$. Hence, $[d(x), y] \in \bigcap_{P \in \mathrm{Min}(R)} P = \{0\}$ for all $y \in R$. Hence, $d(x) \in Z(R)$. □

Let R be a unital ring. Following [9], $a \in R$ is called a weak zero-divisor if there are $r, s \in R$ with $ras = 0$ and $rs \neq 0$. Hence, an element $a \in R$ is not weak zero divisor if, for all $r, s \in R$, $ras = 0$ implies $rs = 0$. It is shown, in [9, Theorem 2.2], that the elements of a minimal prime ideal of R are weak zero-divisors. Unlike the commutative case, Example 3.4 in [9] shows that a minimal prime ideal (even over a semiprime ring) may have regular elements (i.e., elements which are neither left nor right zero-divisors).

Corollary 2.6 — Let R be a unital semiprime ring and d a centralizing derivation of R. If the set $d(R)$ contains any regular element then R is commutative.

PROOF : Let a be a regular element in $d(R)$. Using Corollary 2.5, a is a central element of R. Now, let $r, s \in R$ such that $ras = 0$. Hence, $ars = 0$, and so $rs = 0$. Thus, a is not a weak zero divisor element. Hence, for any minimal prime ideal P of R, we have $d(R) \nsubseteq P$ since P consists only of weak zero divisors (by [9, Theorem 2.2]). Thus, using Theorem 2.2, R/P is commutative (for every minimal prime ideal P of R). As in the proof of the previous Corollary, we conclude that R is commutative. □

Corollary 2.7 — Let R be a semiprime ring, P be a prime ideal of R, and d be a nonzero centralizing derivation of R. If $P \cap Z(R) = \{0\}$ then R/P is commutative.

PROOF : Following Theorem 2.2, $d(R) \subseteq P$ or R/P is commutative. However, using Corollary 2.5, $d(R) \subseteq Z(R)$. Hence, if $d(R) \subseteq P$ we must have $d(R) \subseteq \{0\}$ since $P \cap Z(R) = \{0\}$. But that means that d is zero, a contradiction. Hence, R/P is commutative. □

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REFERENCES

1. A. Ali and M. Yasen, A note on automorphisms of prime and semiprime rings, *J. Math. Kyoto.*, **45**(2) (2005), 243-246.

2. K. I. Beidar, Y. Fong, P. H. Lee, and T. L. Wong, On additive maps in prime rings satisfying the Engel condition, *Commun. Algebra*, **25**(12) (1997), 3889-3902.

3. K. I. Beidar and A. V. Mikhalv, Orthogonal completeness and minimal prime ideals, *Trudy Sem. Petrovski*, **10** (1984), 227-234.

4. K. I. Beidar and A. V. Mikhalv, Orthogonal completeness and algebraic systems, *Russ. Math. Surv.*, **40**(6) (1986), 51-95; translation from *Usp. Mat. Nauk*, **40**(6) (246) (1985), 79-115.

5. H. E. Bell and W. S. Martindale, Centralizing mappings of semiprime rings, *Canad. Math. Bull.*, **30** (1987), 92-101.

6. J. Bergen, Automorphic-differential identities in rings, *Proc. Amer. Math. Soc.*, **106** (1989), 297-305.

7. M. Bresar and B. Hvala, On additive maps of prime rings, *Bull. Austral. Math. Soc.*, **51** (1995), 377-381.

8. M. Bresar, Centralizing mappings and derivations in prime rings, *J. Algebra*, **156** (1993), 385-394.

9. W. D. Burgess, A. Lashgari, and A. Mojiri, Elements of minimal prime ideals in general rings, *Trends in Mathematics, Advances in Ring Theory*, (2010), 69-81.

10. C. L. Chuang and T. K. Lee, Invariance of minimal prime ideals under derivations, *Proc. Amer. Math. Soc.*, **113** (1991), 613-616.

11. Q. Deng, On N-centralizing mappings of prime rings, *Proc. R. Irish Acad*., **93A**(2) (1993), 171-176.

12. Q. Deng and H. E. Bell, On derivations and commutativity in semiprime rings, *Comm. Algebra*, **23** (1995), 3705-3713.

13. K. R. Goodearl and R. B. Warfield Jr., Primitivity in differential operator rings, *Math. Z.*, **180** (1982), 503-523.

14. P. H. Lee and T. K. Lee, Lie ideals of prime rings with derivations, *Bull. Inst. Math. Acad. Sinica*, **11** (1983), 75-80.

15. G. Letzter, Derivations and nil ideals, *Rend. Circ. Mat. Palermo*, **37**(2) (1988), 174-176.

16. E. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8** (1957), 1093-1100.

17. J. Vukman, Commuting and centralizing mappings in prime rings, *Proc. Arner. Math. Soc.*, **109** (1990), 47-52.