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COMMUTATORS AND SEMI-COMMUTATORS OF TOEPLITZ OPERATORS ON THE FOCK SPACE[1]

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In this paper we completely characterize when the commutator and semi-commutator of two monomial-type Toeplitz operators on the Fock space over $\mathbb{C}^n$ have finite rank. In sharp contrast to Bergman space over the unit ball case, it turns out that there are many other cases on the Fock space for (semi-)commuting Toeplitz operators.

Key words : Toeplitz operator; Fock space; quasi-homogeneous function.

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1. Introduction

Let $\mathbb{C}^n$ be the complex n-space and dv be the usual Lebesgue measure on $\mathbb{C}^n$. If $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ are points in $\mathbb{C}^n$, we write $z \cdot \overline{w} = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ and $|z| = (z \cdot \overline{z})^{1/2}$. For any positive parameter α, we consider the normalized Gaussian measure on $\mathbb{C}^n$ given by

$$d\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z).$$

The Fock space $F^2_\alpha(\mathbb{C}^n)$ is then the space of all Gaussian square integrable entire functions on $\mathbb{C}^n$. The reproducing kernel in $F^2_\alpha(\mathbb{C}^n)$ is given by

$$K_\alpha(z, w) = e^{\alpha z\cdot\overline{w}}$$

for $z,\ w \in \mathbb{C}^n$, and the orthogonal projection P_α from $L^2(\mathbb{C}^n, d\lambda_\alpha)$ onto $F^2_\alpha(\mathbb{C}^n)$ is given by

$$P_\alpha f(z) = \int_{\mathbb{C}^n} f(w) K_\alpha(z, w)\, d\lambda_\alpha(w).$$

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For any measurable function u, the Toeplitz operator $T_u : F^2_\alpha(\mathbb{C}^n) \mapsto F^2_\alpha(\mathbb{C}^n)$ with symbol u is defined by $T_u(f) = P_\alpha(uf)$ for all $f \in F^2_\alpha(\mathbb{C}^n)$ for which fu belongs to $L^2(\mathbb{C}^n, d\lambda_\alpha)$. For such a function f and $z \in \mathbb{C}^n$, we have

$$T_u(f)(z) = \int_{\mathbb{C}^n} f(w)u(w)K_\alpha(z,w)\, d\lambda_\alpha(w).$$

The study of algebraic properties of k-quasi-homogeneous Toeplitz operators on the Bergman space over various domains has attracted the interest of several authors. As a natural extension of the classical unilateral shift operator, k-quasi-homogeneous Toeplitz operator on the Bergman space has very interesting structure and enjoys many very interesting properties. See [5, 7, 8, 10, 13, 14] and references there. Corresponding problems on the Fock space have also attracted great attention in recent years, and some results illustrate the essential difference between Toeplitz operators on the Fock space and on the Bergman space over the unit ball (see 1-4, 6]).

Recently, the author and Zhu [9] completely characterized finite rank commutator and semi-commutator of two monomial Toeplitz operators on the Bergman space over the unit ball. In this paper, we study the corresponding problems for monomial-type Toeplitz operators on $F^2_\alpha(\mathbb{C}^n)$. As a consequence, some unexpected results and new phenomena appear in higher dimensions.

In order to describe our main results, we first recall some definitions and standard notation. Recall that the commutator $[T_{u_1}, T_{u_2}]$ and the semi-commutator $(T_{u_1}, T_{u_2}]$ of two Toeplitz operators T_{u_1} and T_{u_2} are defined by $[T_{u_1}, T_{u_2}] = T_{u_1}T_{u_2} - T_{u_2}T_{u_1}$ and $(T_{u_1}, T_{u_2}] = T_{u_1}T_{u_2} - T_{u_1u_2}$, respectively. An operator on a Hilbert space is said to have finite rank if the closure of the range of the operator has finite dimension. For $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $p = (p_1, \cdots, p_n) \in \mathbb{N}^n$, where $\mathbb{N}$ is the set of all non-negative integers, we write $|p| = |p_1| + \cdots + |p_n|$ and $z^p = z_1^{p_1} \cdots z_n^{p_n}$. A function $u(z)$ on $\mathbb{C}^n$ is called monomial-type if it has the form

$$u(z) = \xi^p \overline{\xi}^q r^\mu, \qquad z = r\xi$$

for some $p,\ q \in \mathbb{N}^n$, and $\mu \in \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. In this case, the associated operator T_u is called a monomial-type Toeplitz operator. Obviously, if $\mu = |p| + |q|$, then $u(z)$ is just the ordinary monomial $z^p\overline{z}^q$. Let $p_i, q_i, s_i, t_i \in \mathbb{N}$, and $\mu, \nu \in \mathbb{R}_+$. Then we say that a tuple $(p_i, q_i, s_i, t_i, \mu, \nu)$ satisfies condition (I) if at least one of the following conditions holds:

(i) $p_i = q_i$ and $\mu = 0$,

(ii) $s_i = t_i$ and $\nu = 0$,

(iii) $p_i = q_i$ and $s_i = t_i$,

(iv) $p_i - q_i = s_i - t_i$ and $\mu = \nu$,

(v) $\mu = p_i - q_i$ and $\nu = s_i - t_i$,

(vi) $\mu = -p_i + q_i$ and $\nu = -s_i + t_i$.

This definition will significantly simplify our presentation later on.

Now we are ready to state one of our main results, which shows us exactly when two monomial-type Toeplitz operators commute on the Fock space.

***Theorem* 1** — *Let $p, q, s, t \in \mathbb{N}^n$ and $\mu, \nu \in \mathbb{R}_+$. Then the following statements are equivalent.*

(a) The commutator $\left[T_{\xi^p \bar{\xi}^q r^\mu}, T_{\xi^s \bar{\xi}^t r^\nu}\right]$ on $F^2_\alpha(\mathbb{C}^n)$ has finite rank.

(b) The operators $T_{\xi^p \bar{\xi}^q r^\mu}$ and $T_{\xi^s \bar{\xi}^t r^\nu}$ commute on $F^2_\alpha(\mathbb{C}^n)$.

(c) One of the following statements holds.

(c1) If $n = 1$, then (p, q, s, t, μ, ν) satisfies condition (I).

(c2) If $n \geq 2$, then $(p_i, q_i, s_i, t_i, p_i + q_i, s_i + t_i)$ satisfies condition (I) for each $i \in \{1, 2, \cdots, n\}$, and the tuple $(|p|, |q|, |s|, |t|, \mu, \nu)$ satisfies one of the following conditions:

- $|p| = |q|$ *and* $\mu = 2|p|$.
- $|s| = |t|$ *and* $\nu = 2|s|$.
- $|p| = |q|$ *and* $|s| = |t|$.
- $|p| = |s|$, $|q| = |t|$, *and* $\mu = \nu$.
- $\mu = |p| + |q|$ *and* $\nu = |s| + |t|$.
- $\mu = 3|q| - |p|$, $\nu = 3|t| - |s|$, *and* $|t| = |q|$.
- $\mu = 3|p| - |q|$, $\nu = 3|s| - |t|$, *and* $|s| = |p|$.
- $\mu = |s| + |t|$, $\nu = |p| + |q|$, *and* $|p| - |q| = -|s| + |t|$.

When $n = 1$, Theorem 1 shows that two Toeplitz operators $T_{e^{i(p-q)\theta} r^\mu}$ and $T_{e^{i(s-t)\theta} r^\nu}$ on the Fock space over the complex plane has a finite-rank commutator if and only if they commute if and only if one of the following five conditions holds:

(a) one of the two operators is the identity operator,

(b) the two operators are equal,

(c) both operators are ”diagonal” (induced by radial symbols),

(d) both operators have analytic symbols,

(e) both operators have conjugate analytic symbols.

This result is exactly same as that on the Bergman space over the unit disk, and these five conditions above are often called the trivial (or obvious) cases.

However, when $n > 1$, Theorem 1 produces an ample supply of non-trivial commuting Toeplitz operators on the Fock space. For example, fix $n = 6$, and let $p, q, s, t \in \mathbb{N}^6$ with

$$\begin{aligned} p &= (0, p_2, 0, p_4, a, c), \\ q &= (0, q_2, q_3, 0, a, d), \\ s &= (s_1, 0, 0, s_4, b, c), \\ t &= (t_1, 0, t_3, 0, b, d). \end{aligned}$$

Then it is easy to check that the tuple $(p_i, q_i, s_i, t_i, p_i + q_i, s_i + t_i)$ satisfies condition (I) for each $i \in \{1, 2, \cdots, 6\}$. Obviously, there are many non-trivial cases for the tuple $(|p|, |q|, |s|, |t|, \mu, \nu)$ satisfying each condition of (c2) of Theorem 1. In addition, we would like to mention that these monomial-type Toeplitz operators are not necessarily commutative in the case of Bergman space over the unit ball. Take the monomial Toeplitz operators $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ for example. Since $z^p\overline{z}^q$ and $z^s\overline{z}^t$ are special monomial-type functions with

$$\mu = |p| + |q| \text{ and } \nu = |s| + |t|,$$

it follows from Theorem 1 that $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on $F^2_\alpha(\mathbb{C}^n)$ if and only if $(p_i, q_i, s_i, t_i, p_i + q_i, s_i + t_i,)$ satisfies condition (I) for each $i \in \{1, 2, \cdots, n\}$. However, [9, Theorem 5] shows that $T_{z^p\overline{z}^q}$ and $T_{z^s\overline{z}^t}$ commute on the Bergman space over the unit ball if and only if $(|p|, |q|, |s|, |t|, |p| + |q|, |s| + |t|)$ and $(p_i, q_i, s_i, t_i, p_i + q_i, s_i + t_i)$ satisfy condition (I) for all $i \in \{1, 2, \cdots, n\}$.

After studying the bicommutant of the quasihomogeneous Toeplitz operator on the Bergman space over the unit disk, Louhichi and Rao [12] conjectured that if two Topelitz operators commute with a third one, none of them being the identity, then they commute with each other. Then [8] and [13] gave a counterexample to the conjecture on the Bergman space over the unit ball with $n > 1$, respectively. Recently, Bauer and Issa [2] show that a corresponding conjecture is wrong for Toeplitz operators on the Fock space over the complex plane. As another interesting application of Theorem 1, we can also construct many counterexamples to the conjecture when formulated for Toeplitz operators on the Fock space over $\mathbb{C}^n$ with $n > 1$. For example, we fix $p = (3, 0, 0, \cdots, 0)$ and $q = (2, 1, 0, \cdots, 0)$, and consider the monomial $z^p \overline{z}^q$. Notice that

$$|p| = |q| \text{ and } \mu = 2|p|.$$

Then by condition (c2) of Theorem 1 we get the monomial Toeplitz operator $T_{z^p\overline{z}^q}$ commutes with all Toeplitz operators $T_{\xi^s\overline{\xi}^t r^\nu}$ with $\nu \in \mathbb{R}_+$, $s = (0, 0, s_3, \cdots, s_n)$, and $t = (0, t_2, t_3, \cdots, t_n)$. Using condition (c2) of Theorem 1 again, it is clear that most of these monomial-type Toeplitz operators $T_{\xi^s\overline{\xi}^t r^\nu}$ cannot commute with each other.

Next, we introduce our second result about the semi-commutator of two monomial-type Toeplitz operators on the Fock space.

***Theorem* 2** — *Let $p, q, s, t \in \mathbb{N}^n$ and $\mu, \nu \in \mathbb{R}_+$. Then the following statements are equivalent:*

(a) The semi-commutator $\left(T_{\xi^p\overline{\xi}^q r^\mu}, T_{\xi^s\overline{\xi}^t r^\nu}\right]$ on $F^2_\alpha(\mathbb{C}^n)$ has finite rank.

(b) $T_{\xi^p\overline{\xi}^q r^\mu} T_{\xi^s\overline{\xi}^t r^\nu} = T_{\xi^{p+s}\overline{\xi}^{q+t} r^{\mu+\nu}}$ on $F^2_\alpha(\mathbb{C}^n)$.

(c) One of the following statements holds.

(c1) If $n = 1$, then either $\mu = -p + q$ or $\nu = s - t$.

(c2) If $n \geq 2$, then either $p_i = 0$ or $t_i = 0$ for each $i \in \{1, 2, \cdots, n\}$, and the tuple $(|p|, |q|, |s|, |t|, \mu, \nu)$ satisfies one of the following conditions:

- *$|p| = 0$ and $\mu = |q|$.*
- *$|p| = 0$ and $\nu = |s| - |t|$.*
- *$|t| = 0$ and $\nu = |s|$.*
- *$|t| = 0$ and $\mu = |q| - |p|$.*
- *$\mu = |p| + |q|$ and $\nu = |s| + |t|$.*
- *$\mu = 2|t| - |p| + |q|$ and $\nu = 2|p| + |s| - |t|$.*

When $n = 1$, Theorem 2 shows that the semi-commutator of two Toeplitz operators $T_{e^{i(p-q)\theta} r^\mu}$ and $T_{e^{i(s-t)\theta} r^\nu}$ on the Fock space over the complex plane has finite rank only in two trivial cases: either the first operator has conjugate analytic symbol or the second operator has analytic symbol.

But to our surprise, when $n > 1$, there also exist so many non-trivial semi-commuting monomial-type Toeplitz operators on the Fock space, which can be easily constructed by (c2) of Theorem 2.

This phenomenon is in sharp contrast with the result obtained in [9] on the Bergman space over the unit ball, which shows that the semi-commutator of two monomial Toeplitz operators has finite rank only in two trivial cases: either $p = 0$ or $t = 0$.

For any polynomials ψ and ϕ in z and $\overline{z}$, it follows from [6] that $T_\psi T_\phi = T_{\psi\sharp\phi}$ on the ordinary (unweighted) Fock space, where $\psi\sharp\phi$ is a polynomial given by

$$\psi\sharp\phi = \sum_{\beta\in\mathbb{N}^n} \frac{(-1)^{|\beta|}}{\beta!} \frac{\partial^{|\beta|}\psi}{\partial z^\beta} \frac{\partial^{|\beta|}\phi}{\partial \overline{z}^\beta}.$$

By a direct calculation it follows

$$(z^p\overline{z}^q)\,\sharp\,\left(z^s\overline{z}^t\right) = \sum_{\beta\preceq p,\beta\preceq t,\beta\in\mathbb{N}^n} \frac{(-1)^{|\beta|}}{\beta!} \frac{p!t!}{(p-\beta)!(t-\beta)!} z^{p+s-\beta}\overline{z}^{q+t-\beta}.$$

Thus $T_{z^p\overline{z}^q}T_{z^s\overline{z}^t} = T_{z^{p+s}\overline{z}^{q+t}}$ on the ordinary Fock space if and only if

$$(z^p\overline{z}^q)\,\sharp\,\left(z^s\overline{z}^t\right) = z^{p+s}\overline{z}^{q+t}$$

if and only if either $p_i = 0$ or $t_i = 0$ for each $i \in \{1, 2, \cdots, n\}$. It is clear that this result also follows immediately from condition (c2) of Theorem 2.

Due to the high dimensional combinatorial complexities, we haven't fully understood the conceptual hints of each condition for the tuple $(|p|, |q|, |s|, |t|, \mu, \nu)$ appeared in condition (c2) of Theorem 1 or 2. However, as opposed to the classical Bergman space case, many new high dimensional phenomena are found on the Fock space. As an application, some deeper difference and connection in operator theory between the Bergman space and the Fock space was given in Section 4.

2. Proof of Theorem 1

In this section, we will give the proof of Theorem 1. We first introduce some notation. For multi-index $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n$, we write $\beta! = \beta_1! \cdots \beta_n!$. For another multi-index $\rho = (\rho_1, \cdots, \rho_n) \in \mathbb{N}^n$, we write $\beta \succeq \rho$ if $\beta_i \geq \rho_i$ for any $i = 1, \cdots, n$.

With regard to the monomial-type Toeplitz operator on $F^2_\alpha(\mathbb{C}^n)$, we have the following basic fact.

Lemma 3 — Let $p, q \in \mathbb{N}^n$ and $\mu \in \mathbb{R}_+$. Then on $F^2_\alpha(\mathbb{C}^n)$, for each $\beta \in \mathbb{N}^n$ we have

$$T_{\xi^p\overline{\xi}^q r^\mu}(z^\beta) = \begin{cases} \alpha^{(|p|-|q|-\mu)/2} F_{p,q,\mu}(\beta)\, z^{\beta+p-q}, & \text{if } \beta + p \succeq q, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$F_{p,q,\mu}(\zeta)=\frac{\Gamma\left(\Sigma\zeta+(|p|-|q|+\mu)/2+n\right)}{\Gamma\left(\Sigma\zeta+|p|+n\right)}\prod_{i=1}^{n}\frac{\Gamma\left(\zeta_i+p_i+1\right)}{\Gamma\left(\zeta_i+p_i-q_i+1\right)}$$

is analytic and polynomially bounded in $\{\zeta\in\mathbb{C}^n:\operatorname{Re}\zeta_i\geq\max\{0,q_i-p_i\}\}$. Here, the notation $\Sigma\zeta$ stands for $\zeta_1+\cdots+\zeta_n$.

PROOF : Notice that the reproducing kernel $K_\alpha(z,w)$ can be also given by

$$K_\alpha(z,w)=\sum_{l=0}^{\infty}\frac{\alpha^l}{l!}(z\cdot\overline{w})^l=\sum_{\rho\in\mathbb{N}^n}\frac{\alpha^{|\rho|}}{\rho!}z^\rho\overline{w}^\rho,$$

so we see that for each $\beta\in\mathbb{N}^n$,

$$\begin{aligned}T_{\xi^p\overline{\xi}^q r^\mu}(z^\beta)&=\sum_{\rho\in\mathbb{N}^n}\frac{\alpha^{n+|\rho|}z^\rho}{\pi^n\rho!}\int_{\mathbb{C}^n}\xi^p\overline{\xi}^q r^\mu w^\beta\overline{w}^\rho e^{-\alpha|w|^2}\,dv(w)\\&=\sum_{\rho\in\mathbb{N}^n}\frac{\alpha^{n+|\rho|}z^\rho}{\pi^n\rho!}\int_0^{+\infty}r^{\mu+|\beta|+|\rho|+2n-1}e^{-\alpha r^2}\,dr\int_{\mathbb{S}^n}\xi^{\beta+p}\overline{\xi}^{\rho+q}\,dS(\xi),\end{aligned}$$

where dS is the surface measure on $\mathbb{S}^n$ before normalization. Then it follows from Formula (1.21) and Formula (1.22) of [15] that

$$T_{\xi^p\overline{\xi}^q r^\mu}(z^\beta)=\begin{cases}\frac{\Gamma(|\beta|+(|p|-|q|+\mu)/2+n)(\beta+p)!}{\alpha^{(\mu-|p|+|q|)/2}(|\beta|+|p|+n-1)!(\beta+p-q)!}\,z^{\beta+p-q}, & \text{if } \beta+p\succeq q,\\ 0, & \text{otherwise},\end{cases}$$

as desired.

For each $\zeta\in\mathbb{C}^n$ with $\operatorname{Re}\zeta_i\geq\max\{0,q_i-p_i\}$, $1\leq i\leq n$, since the gamma function is a zero-free analytic function in the right half-plane, it is clear that $F_{p,q,\mu}(\zeta)$ is analytic. Moreover, using well-known identity

$$\frac{\Gamma(\eta+a)}{\Gamma(\eta+b)}=\eta^{a-b}\left(1+O\left(\frac{1}{|\eta|}\right)\right)$$

for large values of $|\eta|$ with $\operatorname{Re}\eta>0$ and $a,b\geq 0$, one can easily check that $|F_{p,q}(\zeta)|\leq P(|\zeta|)$ for some polynomial P. This completes the proof. □

In order to simplify the proof of Theorem 1, we also need the following key lemma.

Lemma 4 — Let $a,b\in\mathbb{Z}$, $\sigma,\tau\in\mathbb{N}$, and $\mu,\nu\in\mathbb{R}_+$. If

$$\frac{\Gamma\left(\eta+b+\frac{a+\mu}{2}+n\right)\Gamma(\eta+a+\tau+n)}{\Gamma\left(\eta+\frac{a+\mu}{2}+n\right)\Gamma(\eta+\tau+n)}=\frac{\Gamma\left(\eta+a+\frac{b+\nu}{2}+n\right)\Gamma(\eta+b+\sigma+n)}{\Gamma\left(\eta+\frac{b+\nu}{2}+n\right)\Gamma(\eta+\sigma+n)}\qquad(1)$$

holds for any $\eta\in\mathbb{C}$ with $\operatorname{Re}\eta\geq\max\{0,-a,-b,-a-b\}$, then at least one of the following conditions holds:

1. $a=0$ and $\mu=2\sigma$,
2. $b=0$ and $\nu=2\tau$,
3. $a=b=0$,
4. $a=b$, $\tau=\sigma$, and $\mu=\nu$,
5. $\mu=2\sigma-a$ and $\nu=2\tau-b$,
6. $a+\mu=2(\sigma-a)$, $b+\nu=2(\tau-b)$, and $\tau-b=\sigma-a$,
7. $a+\mu=2(\sigma+a)$, $b+\nu=2(\tau+b)$, and $\tau=\sigma$,
8. $a+\mu=2(\tau-b)$, $b+\nu=2(\sigma-a)$, and $a=-b$.

PROOF : We will break the discussion into three cases. First we suppose $ab=0$. Without loss of generality, we may assume $a=0$. Then (1) becomes

$$\frac{\Gamma\left(\eta+b+\frac{\mu}{2}+n\right)}{\Gamma\left(\eta+\frac{\mu}{2}+n\right)}=\frac{\Gamma(\eta+b+\sigma+n)}{\Gamma(\eta+\sigma+n)},$$

which implies that either $\mu=2\sigma$ or $b=0$. Consequently, we infer that one of conditions (1)-(3) holds in this case.

We next suppose $ab>0$. If $a>0$ and $b>0$, then (1) can be written as

$$\prod_{j_1=1}^{b}\left(\eta+\frac{a+\mu}{2}+n-1+j_1\right)\prod_{j_2=1}^{a}(\eta+\tau+n-1+j_2) = \prod_{j_3=1}^{a}\left(\eta+\frac{b+\nu}{2}+n-1+j_3\right)\prod_{j_4=1}^{b}(\eta+\sigma+n-1+j_4).$$

By a basic deduction, one can see that the equation above holds if and only if one of the following statements holds:

- $\frac{a+\mu}{2}=\sigma$ and $\tau=\frac{b+\nu}{2}$.
- $\frac{a+\mu}{2}=\frac{b+\nu}{2}$, $\tau=\sigma$, and $a=b$.
- $\frac{a+\mu}{2}=\frac{b+\nu}{2}$, $\tau=\frac{b+\nu}{2}+b$, and $\tau+a=\sigma+b$.
- $\tau=\sigma$, $\frac{a+\mu}{2}+b=\frac{b+\nu}{2}+a$, and $\tau+b=\frac{b+\nu}{2}$.

Thus one of conditions (4)-(7) holds. If $a < 0$ and $b < 0$, then the same result can be got by a similar way as shown before.

We finally suppose $ab < 0$. Similarly, we may consider $a > 0, b < 0$. Then (1) can be written as

$$\prod_{j_4=1}^{-b} (\eta + b + \sigma + n - 1 + j_4) \prod_{j_2=1}^{a} (\eta + \tau + n - 1 + j_2)$$
$$= \prod_{j_3=1}^{a} \left(\eta + \frac{b+\nu}{2} + n - 1 + j_3\right) \prod_{j_1=1}^{-b} \left(\eta + b + \frac{a+\mu}{2} + n - 1 + j_1\right).$$

Notice that the equation above holds if and only if one of the following statements holds:

- $b + \sigma = b + \frac{a+\mu}{2}$ and $\tau = \frac{b+\nu}{2}$.
- $b + \sigma = \frac{b+\nu}{2}$, $\tau = b + \frac{a+\mu}{2}$, and $a = -b$.
- $b + \sigma = \frac{b+\nu}{2}$, $\tau = \frac{b+\nu}{2} - b$, and $\tau + a = \frac{a+\mu}{2}$.
- $\tau = b + \frac{a+\mu}{2}$, $\tau - b = \frac{b+\nu}{2}$, and $\sigma = \frac{b+\nu}{2} + a$.

Thus one of conditions (5)-(8) holds. This completes the proof. □

Throughout the rest part of this section, we will use the following notation for brevity. We denote

$$\gamma_i = \max\{0, -p_i + q_i, -s_i + t_i, -p_i + q_i - s_i + t_i\}$$

for each $i \in \{1, 2, \cdots, n\}$. Obviously, $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}^n$ such that $\gamma + p \succeq q$, $\gamma + s \succeq t$ and $\gamma + p + s \succeq q + t$.

We are now ready to prove Theorem 1. First we show that (a) implies (c). For each $\beta \in \mathbb{N}^n$ with $\beta \succeq \gamma$, Lemma 3 shows that

$$\left[T_{\xi^p \bar{\xi}^q r^\mu}, T_{\xi^s \bar{\xi}^t r^\nu}\right](z^\beta) = 0$$
$$\Longleftrightarrow F_{s,t,\nu}(\beta) F_{p,q,\mu}(\beta + s - t) - F_{p,q,\mu}(\beta) F_{s,t,\nu}(\beta + p - q) = 0. \tag{2}$$

Note that $\left\{\sqrt{\frac{\alpha^{|\beta|}}{\beta!}} z^\beta\right\}_{\beta \in \mathbb{N}^n}$ forms an orthonormal basis for $F^2_\alpha(\mathbb{C}^n)$ and $\left[T_{\xi^p \bar{\xi}^q r^\mu}, T_{\xi^s \bar{\xi}^t r^\nu}\right]$ has finite rank on $F^2_\alpha(\mathbb{C}^n)$, then there exists some $\gamma' \in \mathbb{N}^n$ such that (2) holds for every $\beta \succeq \gamma'$. Thus combining Lemma 3 and [11, Proposition 3.2] we have

$$F_{s,t,\nu}(\zeta) F_{p,q,\mu}(\zeta + s - t) - F_{p,q,\mu}(\zeta) F_{s,t,\nu}(\zeta + p - q) = 0$$

for all $\zeta \in \mathbb{C}^n$ with $\operatorname{Re}\zeta_i \geq \gamma_i$, $1 \leq i \leq n$, which is equivalent to

$$\begin{aligned} &\frac{\Gamma\left(\Sigma\zeta + \frac{|s|-|t|+\nu}{2} + n\right)}{\Gamma\left(\Sigma\zeta + |s| + n\right)} \prod_{i=1}^{n} \frac{\Gamma\left(\zeta_i + s_i + 1\right)}{\Gamma\left(\zeta_i + s_i - t_i + 1\right)} \\ &\times \frac{\Gamma\left(\Sigma\zeta + |s| - |t| + \frac{|p|-|q|+\mu}{2} + n\right)}{\Gamma\left(\Sigma\zeta + |s| - |t| + |p| + n\right)} \prod_{i=1}^{n} \Gamma\left(\zeta_i + s_i - t_i + p_i + 1\right) \\ &= \frac{\Gamma\left(\Sigma\zeta + \frac{|p|-|q|+\mu}{2} + n\right)}{\Gamma\left(\Sigma\zeta + |p| + n\right)} \prod_{i=1}^{n} \frac{\Gamma\left(\zeta_i + p_i + 1\right)}{\Gamma\left(\zeta_i + p_i - q_i + 1\right)} \\ &\times \frac{\Gamma\left(\Sigma\zeta + |p| - |q| + \frac{|s|-|t|+\nu}{2} + n\right)}{\Gamma\left(\Sigma\zeta + |p| - |q| + |s| + n\right)} \prod_{i=1}^{n} \Gamma\left(\zeta_i + p_i - q_i + s_i + 1\right). \end{aligned} \tag{3}$$

Suppose $n = 1$ and denote $a = p - q$, $b = s - t$. Then (3) becomes

$$\frac{\Gamma\left(\zeta + b + \frac{a+\mu}{2} + 1\right)\Gamma(\zeta + a + 1)}{\Gamma\left(\zeta + \frac{a+\mu}{2} + 1\right)} = \frac{\Gamma\left(\zeta + a + \frac{b+\nu}{2} + 1\right)\Gamma(\zeta + b + 1)}{\Gamma\left(\zeta + \frac{b+\nu}{2} + 1\right)}$$

for any $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta \geq \max\{0, -a, -b, -a-b\}$. Thus by Lemma 4 with $\sigma = \tau = 0$, it follows that at least one of the following conditions holds:

(i) $a = \mu = 0$,

(ii) $b = \nu = 0$,

(iii) $a = b = 0$,

(iv) $a = b$ and $\mu = \nu$,

(v) $\mu = a$ and $\nu = b$,

(vi) $\mu = -a$ and $\nu = -b$.

Or equivalently, (p, q, s, t, μ, ν) satisfies condition (I), and so (c1) holds.

We now suppose $n \geq 2$. If we write

$$\begin{aligned} G(\eta) = &\frac{\Gamma\left(\eta + \frac{|s|-|t|+\nu}{2} + n\right)\Gamma\left(\eta + |s| - |t| + \frac{|p|-|q|+\mu}{2} + n\right)}{\Gamma\left(\eta + |s| + n\right)\Gamma\left(\eta + |s| - |t| + |p| + n\right)} \\ &\times \frac{\Gamma\left(\eta + |p| + n\right)\Gamma\left(\eta + |p| - |q| + |s| + n\right)}{\Gamma\left(\eta + \frac{|p|-|q|+\mu}{2} + n\right)\Gamma\left(\eta + |p| - |q| + \frac{|s|-|t|+\nu}{2} + n\right)}, \end{aligned}$$

where $\eta \in \mathbb{C}$ and $\operatorname{Re}\eta \geq |\gamma|$, then it follows from (3) that

$$G(\Sigma\zeta)\prod_{i=1}^{n}\frac{\Gamma(\zeta_i+s_i+1)\Gamma(\zeta_i+s_i-t_i+p_i+1)}{\Gamma(\zeta_i+s_i-t_i+1)} = \prod_{i=1}^{n}\frac{\Gamma(\zeta_i+p_i+1)\Gamma(\zeta_i+p_i-q_i+s_i+1)}{\Gamma(\zeta_i+p_i-q_i+1)} \quad (4)$$

for any $\zeta \in \mathbb{C}^n$ with $\operatorname{Re}\zeta_i \geq \gamma_i$, $1 \leq i \leq n$.

For any $i_0 \in \mathbb{N}$ such that $1 \leq i_0 \leq n$, we fix $\widetilde{\omega} = \{\omega_1, \cdots, \omega_{i_0-1}, 0, \omega_{i_0+1}\cdots, \omega_n\} \in \mathbb{N}^n$ with $\omega_i \geq \gamma_i$ for any $i \neq i_0$. Since $n \geq 2$, we can also adjust it as needed so that $N_{i_0} + |\widetilde{\omega}| \geq |\gamma|$ for a given constant N_{i_0}. Let

$$C(\widetilde{\omega}) = \prod_{i\neq i_0}\frac{\Gamma(\omega_i+p_i+1)\Gamma(\omega_i+p_i-q_i+s_i+1)\Gamma(\omega_i+s_i-t_i+1)}{\Gamma(\omega_i+p_i-q_i+1)\Gamma(\omega_i+s_i+1)\Gamma(\omega_i+s_i-t_i+p_i+1)} \neq 0,$$

and make the substitution $\zeta = \{\omega_1, \cdots, \omega_{i_0-1}, \zeta_{i_0}, \omega_{i_0+1}, \cdots, \omega_n\}$, then (4) becomes

$$\frac{\Gamma(\zeta_{i_0}+s_{i_0}+1)\Gamma(\zeta_{i_0}+s_{i_0}-t_{i_0}+p_{i_0}+1)}{\Gamma(\zeta_{i_0}+s_{i_0}-t_{i_0}+1)}G(\zeta_{i_0}+|\widetilde{\omega}|) = \frac{\Gamma(\zeta_{i_0}+p_{i_0}+1)\Gamma(\zeta_{i_0}+p_{i_0}-q_{i_0}+s_{i_0}+1)}{\Gamma(\zeta_{i_0}+p_{i_0}-q_{i_0}+1)}C(\widetilde{\omega}).$$

Using the same argument as employed in the proof of [9, Theorem 5], we can show that one of the following conditions holds:

- $p_{i_0} = q_{i_0} = 0$,
- $s_{i_0} = t_{i_0} = 0$,
- $p_{i_0} = s_{i_0} = 0$,
- $q_{i_0} = t_{i_0} = 0$,
- $p_{i_0} = q_{i_0}$ and $s_{i_0} = t_{i_0}$,
- $p_{i_0} = s_{i_0}$ and $q_{i_0} = t_{i_0}$.

Thus we have derived that $(p_{i_0}, q_{i_0}, s_{i_0}, t_{i_0}, p_{i_0}+q_{i_0}, s_{i_0}+t_{i_0})$ satisfies condition (I) for each $i_0 \in \mathbb{N}$ such that $1 \leq i_0 \leq n$.

Furthermore, since $(p_{i_0}, q_{i_0}, s_{i_0}, t_{i_0}, p_{i_0}+q_{i_0}, s_{i_0}+t_{i_0})$ satisfies condition (I), it can easily be checked that

$$\frac{\Gamma(\zeta_{i_0}+s_{i_0}+1)\Gamma(\zeta_{i_0}+s_{i_0}-t_{i_0}+p_{i_0}+1)}{\Gamma(\zeta_{i_0}+s_{i_0}-t_{i_0}+1)} = \frac{\Gamma(\zeta_{i_0}+p_{i_0}+1)\Gamma(\zeta_{i_0}+p_{i_0}-q_{i_0}+s_{i_0}+1)}{\Gamma(\zeta_{i_0}+p_{i_0}-q_{i_0}+1)}.$$

Therefore, it follows from (4) that $G(\Sigma\zeta)=1$, which is equivalent to

$$\frac{\Gamma\left(\eta+\frac{|s|-|t|+\nu}{2}+n\right)\Gamma\left(\eta+|s|-|t|+\frac{|p|-|q|+\mu}{2}+n\right)}{\Gamma\left(\eta+|s|+n\right)\Gamma\left(\eta+|s|-|t|+|p|+n\right)} = \frac{\Gamma\left(\eta+\frac{|p|-|q|+\mu}{2}+n\right)\Gamma\left(\eta+|p|-|q|+\frac{|s|-|t|+\nu}{2}+n\right)}{\Gamma\left(\eta+|p|+n\right)\Gamma\left(\eta+|p|-|q|+|s|+n\right)}$$

for any $\eta\in\mathbb{C}$ and $\operatorname{Re}\eta\geq|\gamma|$. If we denote

$$\sigma=|p|,\tau=|s|,\ a=|p|-|q|,\ b=|s|-|t|,$$

then the assertion above exactly becomes (1). Consequently, according to Lemma 4 it follows that $(|p|,|q|,|s|,|t|,\mu,\nu)$ satisfies one of the conditions appeared in (c2). We have thus proved that (a) implies (c).

Next we will prove that (c) implies (b). If p,q,s,t,μ,ν satisfies (c), then by a direct calculation, it is obvious that (3) holds for any ζ with $\operatorname{Re}\zeta_i\geq\gamma_i$, $1\leq i\leq n$, and so $\left[T_{\xi^p\bar{\xi}^q r^\mu}, T_{\xi^s\bar{\xi}^t r^\nu}\right](z^\beta)=0$ for each $\beta\in\mathbb{N}^n$ with $\beta\succeq\gamma$. For $\beta\in\mathbb{N}^n$ with $\beta\not\succeq\gamma$, we have $\gamma_{i_0}>\beta_{i_0}\geq 0$ for some $i_0\in\{1,2,\cdots,n\}$. If $n=1$, then combining (c1) and the fact

$$\gamma_{i_0}=\max\{0,-p_{i_0}+q_{i_0},-s_{i_0}+t_{i_0},-p_{i_0}+q_{i_0}-s_{i_0}+t_{i_0}\}>0,$$

we have that one of the following conditions holds:

- $p_{i_0}=q_{i_0}$, $\mu_{i_0}=0$, and $s_{i_0}<t_{i_0}$.
- $s_{i_0}=t_{i_0}$, $\nu_{i_0}=0$, and $p_{i_0}<q_{i_0}$.
- $p_{i_0}-q_{i_0}=s_{i_0}-t_{i_0}<0$ and $\mu_{i_0}=\nu_{i_0}$.
- $\mu_{i_0}=-p_{i_0}+q_{i_0}>0$ and $\nu_{i_0}=-s_{i_0}+t_{i_0}>0$.

In each case we have that

$$\gamma_{i_0} = -p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0}.$$

Similarly, if $n \geq 2$, then using the fact that $(p_{i_0}, q_{i_0}, s_{i_0}, t_{i_0}, p_{i_0} + q_{i_0}, s_{i_0} + t_{i_0})$ satisfies condition (I), we can also get that $\gamma_{i_0} = -p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0}$. Thus we have that $-p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0} > \beta_{i_0}$, and further by Lemma 3

$$\left[T_{\xi^p \bar{\xi}^q r^\mu}, T_{\xi^s \bar{\xi}^t r^\nu}\right](z^\beta) = T_{\xi^p \bar{\xi}^q r^\mu} T_{\xi^s \bar{\xi}^t r^\nu}(z^\beta) - T_{\xi^s \bar{\xi}^t r^\nu} T_{\xi^p \bar{\xi}^q r^\mu}(z^\beta) = 0.$$

Consequently, $\left[T_{\xi^p \bar{\xi}^q r^\mu}, T_{\xi^s \bar{\xi}^t r^\nu}\right] = 0$ and condition (b) holds.

It is trivial that condition (b) implies (a). The proof of Theorem 1 is now completed.

3. Proof of Theorem 2

In this section we will give the proof of Theorem 2. Similarly, we begin with the following key lemma.

Lemma 5 — Let $a, b \in \mathbb{Z}$, $\sigma, \tau \in \mathbb{N}$, and $\mu, \nu \in \mathbb{R}_+$. If

$$\frac{\Gamma\left(\eta + \frac{b+\nu}{2} + n\right)\Gamma\left(\eta + b + \frac{a+\mu}{2} + n\right)}{\Gamma(\eta + \tau + n)\Gamma(\eta + b + \sigma + n)} = \frac{\Gamma\left(\eta + \frac{a+\mu}{2} + \frac{b+\nu}{2} + n\right)}{\Gamma(\eta + \sigma + \tau + n)} \tag{5}$$

holds for any $\eta \in \mathbb{C}$ with $\operatorname{Re}\eta \geq \max\{0, -b, -a - b\}$, then at least one of the following conditions holds:

1. $\sigma = 0$ and $\mu = -a$,
2. $\sigma = 0$ and $\nu = b$,
3. $\tau = b$ and $\nu = b$,
4. $\tau = b$ and $\mu = -a$,
5. $\mu = 2\sigma - a$ and $\nu = 2\tau - b$,
6. $a + \mu = 2(\tau - b)$ and $b + \nu = 2(\sigma + b)$.

Proof : If $\sigma = 0$, then (5) becomes

$$\frac{\Gamma\left(\eta + b + \frac{a+\mu}{2} + n\right)}{\Gamma(\eta + b + n)} = \frac{\Gamma\left(\eta + \frac{b+\nu}{2} + \frac{a+\mu}{2} + n\right)}{\Gamma\left(\eta + \frac{b+\nu}{2} + n\right)},$$

which implies that $\mu = -a$ or $\nu = b$. Thus either condition (1) or condition (2) holds.

We now assume $\sigma \neq 0$. Then (5) can be written as

$$\frac{\Gamma(\eta+b+\sigma+n)\Gamma\left(\eta+\frac{a+\mu}{2}+\frac{b+\nu}{2}+n\right)}{\Gamma\left(\eta+b+\frac{a+\mu}{2}+n\right)\Gamma\left(\eta+\frac{b+\nu}{2}+n\right)}=\prod_{j_1=1}^{\sigma}(\eta+\tau+n-1+j_1). \tag{6}$$

Notice that the right-hand side function of the equation above is analytic on $\mathbb{C}$. Thus $-\frac{a+\mu}{2}-\frac{b+\nu}{2}-N$ cannot be the pole of the left-hand side function. Therefore, either $\frac{b+\nu}{2}-b$ or $\frac{a+\mu}{2}$ is an integer.

We first suppose $\frac{b+\nu}{2}-b=l$ for some $l \in \mathbb{Z}$. Then (6) becomes

$$\frac{\Gamma(\eta+b+\sigma+n)\Gamma\left(\eta+\frac{a+\mu}{2}+b+l+n\right)}{\Gamma\left(\eta+b+l+n\right)\Gamma\left(\eta+\frac{a+\mu}{2}+b+n\right)}=\prod_{j_1=1}^{\sigma}(\eta+\tau+n-1+j_1).$$

Obviously, if $l=0$ then $\tau=b$, so condition (3) holds, and if $l=\sigma$ then $\frac{a+\mu}{2}+b=\tau$, so condition (6) holds. If $l<0$, then it follows that

$$\prod_{j_2=1}^{\sigma-l}(\eta+b+l+n-1+j_2)$$
$$=\prod_{j_3=1}^{-l}\left(\eta+\frac{a+\mu}{2}+b+l+n-1+j_3\right)\prod_{j_1=1}^{\sigma}(\eta+\tau+n-1+j_1),$$

which implies either

$$b+l=(a+\mu)/2+b+l,\ b+\sigma=\tau+\sigma$$

or

$$b+l=\tau,\ b+\sigma=(a+\mu)/2+b.$$

Recall that $b+l=\frac{b+\nu}{2}$, so we have either condition (4) or condition (5) holds. Similarly, if $0<l<\sigma$ or $l>\sigma$, then in each case we can both get either condition (4) or condition (5) holds.

Next, we suppose $\frac{a+\mu}{2}=l$ for some $l \in \mathbb{Z}$, then (5) becomes

$$\frac{\Gamma(\eta+b+\sigma+n)\Gamma\left(\eta+l+\frac{b+\nu}{2}+n\right)}{\Gamma\left(\eta+b+l+n\right)\Gamma\left(\eta+\frac{b+\nu}{2}+n\right)}=\prod_{j_1=1}^{\sigma}(\eta+\tau+n-1+j_1).$$

Thus by the same way as shown before, we get one of conditions (3)-(6) holds. This completes the proof. □

In what follows we will use the notation

$$\delta_i=\max\{0,-s_i+t_i,-p_i+q_i-s_i+t_i\}.$$

Obviously, we have $\delta = (\delta_1, \cdots, \delta_n) \in \mathbb{N}^n$, $\delta + s \succeq t$, and $\delta + p + s \succeq q + t$.

We are now ready to prove Theorem 2. First we show that (a) implies (c). For each $\beta \in \mathbb{N}^n$ with $\beta \succeq \delta$, Lemma 3 shows that

$$\left(T_{\xi^p \overline{\xi}^q r^\mu}, T_{\xi^s \overline{\xi}^t r^\nu}\right] (z^\beta) = 0 \Longleftrightarrow F_{p,q,\mu}(\beta + s - t) F_{s,t,\nu}(\beta) - F_{p+s,q+t,\mu+\nu}(\beta) = 0.$$

Since the semicommutator $\left(T_{\xi^p \overline{\xi}^q r^\mu}, T_{\xi^s \overline{\xi}^t r^\nu}\right]$ has finite rank, we have

$$F_{s,t,\nu}(\zeta) F_{p,q,\mu}(\zeta + s - t) - F_{p+s,q+t,\mu+\nu}(\zeta) = 0$$

for all $\zeta \in \mathbb{C}^n$ with $\operatorname{Re} \zeta_i \geq \delta_i$, $1 \leq i \leq n$, which is equivalent to

$$\begin{aligned} &\frac{\Gamma\left(\Sigma\zeta + \frac{|s|-|t|+\nu}{2} + n\right)}{\Gamma\left(\Sigma\zeta + |s| + n\right)} \prod_{i=1}^n \frac{\Gamma\left(\zeta_i + s_i + 1\right)}{\Gamma\left(\zeta_i + s_i - t_i + 1\right)} \\ &\times \frac{\Gamma\left(\Sigma\zeta + |s| - |t| + \frac{|p|-|q|+\mu}{2} + n\right)}{\Gamma\left(\Sigma\zeta + |s| - |t| + |p| + n\right)} \prod_{i=1}^n \Gamma\left(\zeta_i + s_i - t_i + p_i + 1\right) \\ &= \frac{\Gamma\left(\Sigma\zeta + \frac{|p|-|q|+\mu}{2} + \frac{|s|-|t|+\nu}{2} + n\right)}{\Gamma\left(\Sigma\zeta + |p| + |s| + n\right)} \prod_{i=1}^n \Gamma\left(\zeta_i + p_i + s_i + 1\right). \end{aligned} \tag{7}$$

Suppose $n = 1$ and let $a = p - q$ and $b = s - t$. Then it follows from (7) that

$$\frac{\Gamma\left(\zeta + b + \frac{a+\mu}{2} + 1\right)}{\Gamma(\zeta + b + 1)} = \frac{\Gamma\left(\zeta + \frac{b+\nu}{2} + \frac{a+\mu}{2} + 1\right)}{\Gamma\left(\zeta + \frac{b+\nu}{2} + 1\right)}.$$

Then combining this and Lemma 5 with $\sigma = \tau = 0$, we get either $\mu = -p + q$ or $\nu = s - t$, which implies that (c1) holds.

We now suppose $n \geq 2$. If we write

$$\begin{aligned} G(\eta) = &\frac{\Gamma\left(\eta + \frac{|s|-|t|+\nu}{2} + n\right) \Gamma\left(\eta + |s| - |t| + \frac{|p|-|q|+\mu}{2} + n\right)}{\Gamma\left(\eta + |s| + n\right) \Gamma\left(\eta + |s| - |t| + |p| + n\right)} \\ &\times \frac{\Gamma(\eta + |p| + |s| + n)}{\Gamma\left(\eta + \frac{|p|-|q|+\mu}{2} + \frac{|s|-|t|+\nu}{2} + n\right)}, \end{aligned}$$

where $\eta \in \mathbb{C}$ and $\operatorname{Re} \eta \geq |\delta|$, then it follows from (7) that

$$G\left(\Sigma\zeta\right) \prod_{i=1}^n \frac{\Gamma(\zeta_i + s_i + 1)}{\Gamma(\zeta_i + s_i - t_i + 1)} = \prod_{i=1}^n \frac{\Gamma(\zeta_i + p_i + s_i + 1)}{\Gamma(\zeta_i + s_i - t_i + p_i + 1)} \tag{8}$$

for any $\zeta \in \mathbb{C}^n$ with $\operatorname{Re}\zeta_i \geq \delta_i$, $1 \leq i \leq n$.

For any $i_0 \in \mathbb{N}$ such that $1 \leq i_0 \leq n$, we now proceed as in the proof of Theorem 1. One can first get that either $p_{i_0} = 0$ or $t_{i_0} = 0$, and then it follows from (8) that $G(\Sigma\zeta) = 1$, which is equivalent to

$$\frac{\Gamma\left(\eta + \frac{|s|-|t|+\nu}{2} + n\right)\Gamma\left(\eta + |s| - |t| + \frac{|p|-|q|+\mu}{2} + n\right)}{\Gamma(\eta + |s| + n)\,\Gamma(\eta + |s| - |t| + |p| + n)} = \frac{\Gamma\left(\eta + \frac{|p|-|q|+\mu}{2} + \frac{|s|-|t|+\nu}{2} + n\right)}{\Gamma(\eta + |p| + |s| + n)}.$$

Denote $\sigma = |p|, \tau = |s|, a = |p| - |q|$ and $b = |s| - |t|$. Then the equation above becomes exactly (5), and so according to Lemma 5 it follows that $(|p|, |q|, |s|, |t|, \mu, \nu)$ satisfies one of the conditions appeared in (c2). We have thus proved that (a) implies (c).

Next we will prove that (c) implies (b). According to (c), it is easy to check that (7) holds for any ζ with $\operatorname{Re}\zeta_i \geq \delta_i$, $1 \leq i \leq n$, and so $\left(T_{\xi^p\bar{\xi}^q r^\mu}, T_{\xi^s\bar{\xi}^t r^\nu}\right](z^\beta) = 0$ for each $\beta \in \mathbb{N}^n$ with $\beta \succeq \delta$. For $\beta \in \mathbb{N}^n$ with $\beta \not\succeq \delta$, we have $\delta_{i_0} > \beta_{i_0} \geq 0$ for some $i_0 \in \{1, 2, \cdots, n\}$. If $n = 1$, then (c1) implies that either $\mu_{i_0} = -p_{i_0} + q_{i_0} \geq 0$ or $\nu_{i_0} = s_{i_0} - t_{i_0} \geq 0$. Recall that $\delta_{i_0} = \max\{0, -s_{i_0} + t_{i_0}, -p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0}\} > 0$, so we have that $\delta_{i_0} = -p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0}$. Similarly, if $n \geq 2$, then (c2) implies that either $p_{i_0} = 0$ or $t_{i_0} = 0$, and so $\delta_{i_0} = -p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0}$. Thus $-p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0} > \beta_{i_0}$, and by Lemma 3,

$$\left(T_{\xi^p\bar{\xi}^q r^\mu}, T_{\xi^s\bar{\xi}^t r^\nu}\right](z^\beta) = T_{\xi^p\bar{\xi}^q r^\mu} T_{\xi^s\bar{\xi}^t r^\nu}(z^\beta) - T_{\xi^{p+s}\bar{\xi}^{q+t} r^{\mu+\nu}}(z^\beta) = 0.$$

Consequently, $\left(T_{\xi^p\bar{\xi}^q r^\mu}, T_{\xi^s\bar{\xi}^t r^\nu}\right] = 0$ and condition (b) holds.

It is trivial that condition (b) implies (a). The proof of Theorem 2 is now completed.

4. Further Results

In this section we will further illustrate the difference and connection in operator theory between the Bergman space and the Fock space. We do this with the help of k-quasi-homogeneous Toeplitz operators. Define the space of measurable functions of at most polynomial growth at infinity to be

$$\mathcal{S} := \{u : \mathbb{C}^n \to \mathbb{C} : \exists C, c > 0 \text{ s.t. } |u(z)| \leq C(1 + |z|)^c\}.$$

Then one can consider any finite products $T_{u_1} \cdots T_{u_N}$ with $u_i \in \mathcal{S}$ as densely defined operators on $F^2_\alpha(\mathbb{C}^n)$. See [1, 4] for details and related facts.

In order to introduce the definition of k-quasi-homogeneous Toeplitz operators, we recall some notation from [13]. Let $k = (k_1, \cdots, k_m)$ be a tuple of positive integers with $|k| = k_1 + \cdots + k_m = n$.

Then we can interpret $\mathbb{C}^n$ as a product space $\mathbb{C}^n = \mathbb{C}^{k_1} \times \cdots \times \mathbb{C}^{k_m}$, and we use the notation $z = (z_{(1)}, \cdots, z_{(m)}) \in \mathbb{C}^n$, where $z_{(j)} = (z_{j,1}, \cdots, z_{j,k_j}) \in \mathbb{C}^{k_j}$ for $j \in \{1, \cdots, m\}$. Then we represent each $z_{(j)} \in \mathbb{C}^{k_j}$ in the form $z_{(j)} = r_j \xi_{(j)}$ with $r_j = |z_{(j)}|$ and $\xi_{(j)} \in \mathbb{S}^{k_j}$. For any multi-index $p = (p_1, \cdots, p_n) \in \mathbb{N}^n$, we also write $p = (p_{(1)}, \cdots, p_{(m)})$ with $p_{(j)} = (p_{j,1}, \cdots, p_{j,k_j}) \in \mathbb{N}^{k_j}$, and denote $|p_{(j)}| = |p_{j,1}| + \cdots + |p_{j,k_j}|$.

Fix $p,\ q \in \mathbb{N}^n$, a function $f(z) \in \mathcal{S}$ on $\mathbb{C}^n$ is called k-quasi-homogeneous if it has the form

$$f(z_{(1)}, \cdots, z_{(m)}) = \xi^p \overline{\xi}^q \varphi(r) = \xi_{(1)}^{p_{(1)}} \cdots \xi_{(m)}^{p_{(m)}} \overline{\xi}_{(1)}^{q_{(1)}} \cdots \overline{\xi}_{(m)}^{q_{(m)}} \varphi(r_1, \cdots, r_m),$$

where $\varphi(r) = \varphi(r_1, \cdots, r_m)$ is known as a k-quasi-radial function. In this case the associated Toeplitz operator T_f is also called k-quasi-homogeneous Toeplitz operator. By the same argument in the proof of Lemma 3, we have

$$T_{\xi^p \overline{\xi}^q \varphi}(z^\beta) = \begin{cases} \widetilde{\gamma}_{\varphi,k,p,q,\alpha}(\beta)\, z^{\beta+p-q}, & \text{if } \beta + p \succeq q, \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

on $F^2_\alpha(\mathbb{C}^n)$, where

$$\begin{aligned}\widetilde{\gamma}_{\varphi,k,p,q,\alpha}(\beta) = {} & \frac{2^m \alpha^{n+|\beta|+|p|-|q|}(\beta+p)!}{\prod\limits_{j=1}^{m} (|\beta_{(j)}| + |p_{(j)}| + k_j - 1)!(\beta+p-q)!} \\ & \times \int_{[0,+\infty)^m} \varphi(r_1, \cdots, r_m) \prod_{j=1}^{m} r_j^{|2\beta_{(j)}+p_{(j)}-q_{(j)}|+2k_j-1} e^{-\alpha r_j^2}\, dr_j.\end{aligned}$$

Using (9) and [13, Lemma 3.3], one can easily get the following two non-trivial sufficient conditions for the k-quasi-homogeneous Toeplitz operators $T_{\xi^p \overline{\xi}^q \varphi}$ and $T_{\xi^s \overline{\xi}^t \psi}$ commuting on both the Fock space and Bergman space over the unit ball:

(i) $(p_i, q_i, s_i, t_i, p_i + q_i, s_i + t_i)$ satisfies condition (I) for each $i \in \{1, 2, \cdots, n\}$, $|p_{(j)}| = |q_{(j)}|$ and $|s_{(j)}| = |t_{(j)}|$ for each $j \in \{1, 2, \cdots, m\}$.

(ii) $(p_i, q_i, s_i, t_i, p_i + q_i, s_i + t_i)$ satisfies condition (I) for each $i \in \{1, 2, \cdots, n\}$, $|p_{(j)}| = |s_{(j)}|$ and $|q_{(j)}| = |t_{(j)}|$ for each $j \in \{1, 2, \cdots, m\}$, and $\varphi(r_1, \cdots, r_m) = \psi(r_1, \cdots, r_m)$.

So some conditions appeared in (c2) of Theorem 1 and [9, Theorem A] is natural and understandable. However, different from the case on the Bergman space, Theorems 1 and 2 also give many new commuting or semi-commuting Toeplitz operators on $F^2_\alpha(\mathbb{C}^n)$ induced by the monomial-type symbols, respectively. Next, we present some interesting specific examples.

Example 6 : Notice that $z_1^3\overline{z_1}r^4$ and $z_2^3\overline{z_2}^2r^2$ are monomial-type with $p = (3,0,0,\ldots,0)$, $q = (1,0,0,\ldots,0)$, $s = (0,3,0,\ldots,0)$, $t = (0,2,0,\ldots,0)$,

$$\mu = 3|p| - |q| = 8,\ \nu = 3|s| - |t| = 7, \text{ and } |s| = |p| = 3.$$

It follows from Theorem 1 that monomial-type Toeplitz operators $T_{z_1^3\overline{z_1}r^4}$ and $T_{z_2^3\overline{z_2}^2r^2}$ commute on $F_\alpha^2(\mathbb{C}^n)$. However, on the Bergman space over the unit ball, a calculation using [14, Lemma 4.3] shows that $T_{z_1^3\overline{z_1}r^4}T_{z_2^3\overline{z_2}^2r^2}(1) = \frac{36}{(n+1)(n+4)(2n+3)}z_1^2z_2$, and $T_{z_2^3\overline{z_2}^2r^2}T_{z_1^3\overline{z_1}r^4}(1) = \frac{36}{(n+3)^2(2n+7)}z_1^2z_2$. Consequently, $T_{z_1^3\overline{z_1}r^4}$ does not commute with $T_{z_2^3\overline{z_2}^2r^2}$ on the Bergman space over the unit ball.

Example 7 : Consider the monomial-type symbols $z_1\overline{z_1}^2\overline{z_2}r^2$ and $z_1z_2\overline{\xi_2}^2$, namely, $p = (1,0,0,\ldots,0)$, $q = (2,1,0,\ldots,0)$, $s = (1,1,0,\ldots,0)$, $t = (0,2,0,\ldots,0)$,

$$\mu = 2|t| - |p| + |q| = 6, \text{ and } \nu = 2|p| + |s| - |t| = 2.$$

Then it follows from Theorem 2 that $T_{z_1\overline{z_1}^2\overline{z_2}r^2}T_{z_1z_2\overline{\xi_2}^2} = T_{|z_1|^4|z_2|^2\overline{z_2}^2}$ on $F_\alpha^2(\mathbb{C}^n)$. However, a calculation shows that

$$T_{z_1\overline{z_1}^2\overline{z_2}r^2}T_{z_1z_2\overline{\xi_2}^2}(z_2^2) \neq T_{|z_1|^4|z_2|^2\overline{z_2}^2}(z_2^2)$$

on the Bergman space over the unit ball.

Moreover, contrary to the Bergman space over the unit ball, the Fock space admits a tensor product structure, i.e.,

$$F_\alpha^2(\mathbb{C}^n) = F_\alpha^2(\mathbb{C}^{k_1}) \otimes F_\alpha^2(\mathbb{C}^{k_2}) \otimes \ldots \otimes F_\alpha^2(\mathbb{C}^{k_m}).$$

For $\mu = (\mu_1, \cdots, \mu_m) \in \mathbb{R}_+^m$, if we write

$$r^\mu := r_1^{\mu_1} \cdots r_m^{\mu_m},$$

and call the Toeplitz operator $T_{\xi^p\overline{\xi}^q r^\mu}$ as a k-monomial-type Toeplitz operator, then the k-monomial-type Toeplitz operator $T_{\xi^p\overline{\xi}^q r^\mu}$ is, in fact, the tensor products of the operators T_{u_j}, with $u_j(z_{(j)}) = \xi_{(j)}^{p(j)}\overline{\xi}_{(j)}^{q(j)}r_j^{\mu_j}$, which act on $F_\alpha^2(\mathbb{C}^{k_j})$, $j = 1, \ldots, m$, respectively. As a consequence, one can also obtain the following complicated results concerning k-monomial-type Toeplitz operators.

Corollary 8 — Let $p, q, s, t \in \mathbb{N}^n$ and $\mu, \nu \in \mathbb{R}_+^m$. Then the following statements are equivalent.

(a) The commutator $\left[T_{\xi^p\overline{\xi}^q r^\mu}, T_{\xi^s\overline{\xi}^t r^\nu}\right]$ on $F_\alpha^2(\mathbb{C}^n)$ has finite rank.

(b) The operators $T_{\xi^p\overline{\xi}^q r^\mu}$ and $T_{\xi^s\overline{\xi}^t r^\nu}$ commute on $F_\alpha^2(\mathbb{C}^n)$.

(c) For any k_j, $j \in \{1, 2, \cdots, m\}$, the following two statements hold.

(c1) If $k_j = 1$, then $(p_i, q_i, s_i, t_i, \mu_i, \nu_i)$ satisfies condition (I) for $i = k_1 + \cdots + k_j$.

(c2) If $k_j \geq 2$, then $(p_i, q_i, s_i, t_i, p_i + q_i, s_i + t_i)$ satisfies condition (I) for each $i \in \{1, 2, \cdots, n\}$ such that $k_1 + \cdots + k_{j-1} + 1 \leq i \leq k_1 + \cdots + k_{j-1} + k_j$, and the tuple $\left(|p_{(j)}|, |q_{(j)}|, |s_{(j)}|, |t_{(j)}|, \mu_j, \nu_j\right)$ satisfies one of the following conditions:

- $|p_{(j)}| = |q_{(j)}|$ and $\mu_j = 2|p_{(j)}|$.
- $|s_{(j)}| = |t_{(j)}|$ and $\nu_j = 2|s_{(j)}|$.
- $|p_{(j)}| = |q_{(j)}|$ and $|s_{(j)}| = |t_{(j)}|$.
- $|p_{(j)}| = |s_{(j)}|$, $|q_{(j)}| = |t_{(j)}|$, and $\mu_j = \nu_j$.
- $\mu_j = |p_{(j)}| + |q_{(j)}|$ and $\nu_j = |s_{(j)}| + |t_{(j)}|$.
- $\mu_j = 3|q_{(j)}| - |p_{(j)}|$, $\nu_j = 3|t_{(j)}| - |s_{(j)}|$, and $|t_{(j)}| = |q_{(j)}|$.
- $\mu_j = 3|p_{(j)}| - |q_{(j)}|$, $\nu_j = 3|s_{(j)}| - |t_{(j)}|$, and $|s_{(j)}| = |p_{(j)}|$.
- $\mu_j = |s_{(j)}| + |t_{(j)}|$, $\nu_j = |p_{(j)}| + |q_{(j)}|$, and $|p_{(j)}| - |q_{(j)}| = -|s_{(j)}| + |t_{(j)}|$.

In addition, we can also consider $z^p \overline{z}^q$ and $z^s \overline{z}^t$ as k-monomial-type functions with

$$k_i = 1,\ \mu_i = p_i + q_i,\ \text{and}\ \nu_i = s_i + t_i$$

for all $i \in \{1, 2, \cdots, n\}$, then the same necessary and sufficient condition for commuting monomial Toeplitz operators as mentioned in Introduction follows immediately from condition (c1) of Corollary 8.

Corollary 9 — Let $p, q, s, t \in \mathbb{N}^n$ and $\mu, \nu \in \mathbb{R}^m_+$. Then the following statements are equivalent:

(a) The semi-commutator $\left(T_{\xi^p \overline{\xi}^q r^\mu}, T_{\xi^s \overline{\xi}^t r^\nu}\right]$ on $F^2_\alpha(\mathbb{C}^n)$ has finite rank.

(b) $T_{\xi^p \overline{\xi}^q r^\mu} T_{\xi^s \overline{\xi}^t r^\nu} = T_{\xi^{p+s} \overline{\xi}^{q+t} r^{\mu+\nu}}$ on $F^2_\alpha(\mathbb{C}^n)$.

(c) For any k_j, $j \in \{1, 2, \cdots, m\}$, the following two statements hold.

(c1) If $k_j = 1$, then either $\mu_i = -p_i + q_i$ or $\nu_i = s_i - t_i$ for $i = k_1 + \cdots + k_j$.

(c2) If $k_j \geq 2$, then either $p_i = 0$ or $t_i = 0$ for each $i \in \{1, 2, \cdots, n\}$ such that $k_1 + \cdots + k_{j-1} + 1 \leq i \leq k_1 + \cdots + k_{j-1} + k_j$, and the tuple $\left(|p_{(j)}|, |q_{(j)}|, |s_{(j)}|, |t_{(j)}|, \mu_j, \nu_j\right)$ satisfies one of the following conditions:

- $|p_{(j)}| = 0$ and $\mu_j = |q_{(j)}|$.
- $|p_{(j)}| = 0$ and $\nu_j = |s_{(j)}| - |t_{(j)}|$.
- $|t_{(j)}| = 0$ and $\nu_j = |s_{(j)}|$.

- $|t_{(j)}| = 0$ and $\mu_j = |q_{(j)}| - |p_{(j)}|$.
- $\mu_j = |p_{(j)}| + |q_{(j)}|$ and $\nu_j = |s_{(j)}| + |t_{(j)}|$.
- $\mu_j = 2|t_{(j)}| - |p_{(j)}| + |q_{(j)}|$ and $\nu_j = 2|p_{(j)}| + |s_{(j)}| - |t_{(j)}|$.

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