Indian J. Pure Appl. Math., **51**(1): 251-264, March 2020

DOI: 10.1007/s13226-020-0398-4

UNCERTAINTY PRINCIPLES FOR THE HANKEL-GABOR TRANSFORM

Aymen Hammami

SAMM EA 4543, Université Paris 1 Panthéon-Sorbonne,

Fédération CNRS FP2M, FR 2036

e-mails: Aymen.Hammami@univ-paris1.fr

(*Received* 2 *February* 2018; *after final revision* 22 *December* 2018;

accepted 22 *January* 2019)

In this paper, we prove an analogue of a time-frequency localization theorem for orthonormal sequences in $L^2_{\mu_\alpha}(\mathbb{R}_+)$. As a consequence, we obtain an analogue of Shapiro's Umbrella theorem for the Hankel-Gabor transform $\mathscr{V}_g$. We also prove a mean dispersion inequality for $\mathscr{V}_g$. Finally, we get a strong version of the uncertainty inequality for orthonormal sequences of $L^2_{\mu_\alpha}(\mathbb{R}_+)$.

Key words : Uncertainty inequality; The windowed Hankel transform; time-frequency localization theorem; mean dispersion inequality.

2010 Mathematics Subject Classification : 42A38, 44A35

1. INTRODUCTION

Shapiro proved in [16] a number of uncertainty inequalities for orthonormal sequences that are stronger then corresponding inequalities for a single function. Quantitative versions of Shapiro's results appeared in a recent article by Jaming and Powell [10], where in particular the following sharp Mean-Dispersion inequality is obtained. Let $\{e_k\}_{k\geqslant 0}$ be an orthonormal sequence in $L^2(\mathbb{R})$ then for all $N \geqslant 0$

$$\sum_{k=0}^{N}(M(e_k)^2 + \triangle(e_k)^2 + M(\mathscr{F}(e_k))^2 + \triangle(\mathscr{F}(e_k))^2 \geqslant \frac{(N+1)(2N+1)}{4\pi}. \tag{1.1}$$

The equality is attained for the sequence of Hermite function.

Here, $M(e_k) = \int_{\mathbb{R}} t|e_k|^2 dt$ and $\triangle(e_k) = \big(\int_{\mathbb{R}}(t - M(e_k))^2|e_k|^2 dt\big)^{1/2}$, which are called the time mean of e_k, the variance of e_k respectively and $\mathscr{F}$ is the Fourier transform defined for $f \in$

$L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$\mathscr{F}(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-ixy} dy \tag{1.2}$$

and extended from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ in the usual way.

Next Malinnikova in [11] gives the following Shapiro type inequality which is a generalization of the Mean-Dispersion principle :
let $s > 0$ and $\{\phi_n\}_n$ be an orthonormal sequence in $L^2(\mathbb{R}^d)$, then

$$\sum_{n=1}^{N} \big(\tau_s^s(\phi_n) + \tau_s^s(\mathscr{F}(\phi_n)) \big) \geqslant C\, N^{1+s/2d}, \tag{1.3}$$

where C depends only on d and s, here $\tau_s^s(\phi_n) = \displaystyle\int_{\mathbb{R}^d} |x|^s |\phi_n|^2 dx$.

The purpose of this paper is to extend these type of inequalities to the Hankel-Gabor transform. In order to describe our paper, we first need to introduce some notations.
Let $f \in L^1(\mathbb{R}^d)$, it is well known that if $f(x) = F(||x||)$ is a radial function on $\mathbb{R}^d$ then the Fourier transform $\mathscr{F}(f)$ is also a radial function on $\mathbb{R}^d$ and we have

$$\forall y \in \mathbb{R}^d, \mathscr{F}(f)(y) = \int_0^{+\infty} F(y) j_{\frac{d}{2}-1}(r||y||) \frac{r^{d-1} dr}{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2})} = \mathscr{H}_{\frac{d}{2}-1}(F)(||y||), \tag{1.4}$$

where for $\alpha > -\frac{1}{2}$, $\mathscr{H}_\alpha$ is the Hankel transform (also known as the Fourier-Bessel transform) defined by (see e.g. [15, 17]):

$$\mathscr{H}_\alpha(f)(\xi) = \int_0^{+\infty} f(x) j_\alpha(x\xi) d\mu_\alpha(x),\ \xi \in \mathbb{R}_+. \tag{1.5}$$

Here $d\mu_\alpha(x) = \dfrac{x^{2\alpha+1}\, dx}{2^\alpha\, \Gamma(\alpha+1)}$ and j_α (see e.g. [17, 19]) is the spherical Bessel function given by

$$j_\alpha(x) = \frac{2^\alpha\, \Gamma(\alpha+1)}{x^\alpha} J_\alpha(x) = \Gamma(\alpha+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} (\frac{x}{2})^{2k}, \tag{1.6}$$

with J_α is the Bessel function of first kind and index α ([14, P.184]).

Relation (1.4) can be extended to every function $f \in L^2(\mathbb{R}^d)$ by the standard density argument.

For $\alpha > -\frac{1}{2}$, let us recall the Poisson representation formula

$$j_\alpha(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} \exp(-ixt) dt. \tag{1.7}$$

Therefore, j_α is bounded with $\big|j_\alpha(x)\big| \leqslant 1$. As a consequence,

$$||\mathscr{H}_\alpha(f)||_{\infty,\mu_\alpha} \leqslant ||f||_{1,\mu_\alpha} \tag{1.8}$$

where $||.||_{\infty,\mu_\alpha}$ is the usual essential supremum norm and for $1 \leqslant p < \infty$, we denote by $L^p_{\mu_\alpha}(\mathbb{R}_+)$, the Banach space consisting of measurable functions f on $\mathbb{R}_+$ equipped with the norms:

$$||f||_{p,\mu_\alpha} = \Big(\int_0^{+\infty} \big|f(x)\big|^p d\mu_\alpha(x)\Big)^{\frac{1}{p}}.$$

It is also well-known (see [17, 20]) that the Fourier-Bessel transform extends to an isometry on $L^2_{\mu_\alpha}(\mathbb{R}_+)$:

$$||\mathscr{H}_\alpha(f)||_{2,\mu_\alpha} = ||f||_{2,\mu_\alpha}. \tag{1.9}$$

One way one may hope to overcome the lack of localization is to use the Hankel-Gabor transform, also known as the windowed Hankel transform introduced in [3, 5, 7]. We note that the Shapiro's type uncertainty principle for the Hankel transform, was also studied by Ghobber in [8].

Precisely, for every function $g \in L^2_{\mu_\alpha}(\mathbb{R}_+)$ the modulation of g by $\xi \in \mathbb{R}_+$ is defined by:

$$M_\xi(g) = \mathscr{H}_\alpha\Big(\sqrt{\tau_\xi\big(|\mathscr{H}_\alpha(g)|^2\big)}\Big). \tag{1.10}$$

Here τ_ξ denotes the Hankel translation operator defined on $L^p_{\mu_\alpha}(\mathbb{R}_+);\ p \in [1, +\infty]$ by

$$\tau^\alpha_\xi(f)(y) \;=\; \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f(\sqrt{\xi^2+y^2+2\xi y\cos\theta})(sin\theta)^{2\alpha}d\theta. \tag{1.11}$$

Due to Plancheral theorem and the invariance of the measure μ_α under the Hankel translation τ_ξ, we have for all $g \in L^2_{\mu_\alpha}(\mathbb{R}_+)$

$$||M_\xi(g)||_{2,\mu_\alpha} = ||g||_{2,\mu_\alpha}. \tag{1.12}$$

For a non-zero window function g in $L^2_{\mu_\alpha}(\mathbb{R}_+)$; and all $x,\ \xi \in [0, +\infty[$; we consider the family $g_{x,\xi}$ defined by:

$$g_{x,\xi} \;=\; \tau_x\big(M_\xi(g)\big). \tag{1.13}$$

Then, for any function $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$, the Hankel-Gabor transform with respect to the window g is given by [7]:

$$\mathscr{V}_g(f)(x,\xi) \;=\; \int_0^\infty f(y)\,\overline{g_{x,\xi}(y)}\,d\mu_\alpha(y) = \langle f|g_{x,\xi}\rangle_{\mu_\alpha},\ (x,\xi)\in(\mathbb{R}_+)^2. \tag{1.14}$$

Moreover, from Cauchy-Schwarz's inequality and relation (1.12), we get

$$||\mathscr{V}_g(f)||_{\infty,\omega_\alpha} \quad \leqslant \quad ||f||_{2,\mu_\alpha}\,||g||_{2,\mu_\alpha}, \tag{1.15}$$

where ω_α is the product measure on $\mathbb{R}_+ \times \mathbb{R}_+$ defined by

$$d\omega_\alpha(x,y) = d\mu_\alpha(x) \otimes d\mu_\alpha(y),$$

then $L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+)$ is the Hilbert space of square integrable functions on $\mathbb{R}_+ \times \mathbb{R}_+$ with respect to the measure ω_α equipped with the inner product

$$\langle f|g\rangle_{\omega_\alpha} = \int\int_{([0,+\infty[)^2} f(x,y)\,\overline{g(x,y)}\,d\omega_\alpha(x,y)$$

and the norm $||f||_{2,\omega_\alpha} = \sqrt{\langle f|f\rangle_{\omega_\alpha}}$.

The Hankel- Gabor transform satisfies a Plancherel-type formula,

$$||\mathscr{V}_g(f)||_{2,\omega_\alpha} \quad = \quad ||f||_{2,\mu_\alpha}\,||g||_{2,\mu_\alpha}. \tag{1.16}$$

and

$$\langle \mathscr{V}_g(f)|\mathscr{V}_g(h)\rangle_{\omega_\alpha} \quad = \quad ||g||^2_{2,\mu_\alpha}\,\langle f|h\rangle_{\mu_\alpha}. \tag{1.17}$$

Bowie [2] and then Rösler and Voit [14] proved the analogue of Heisenberg's uncertainty inequality for the Fourier-Bessel transform which can be written for unit-norm functions in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ of the form:

$$||xf||_{2,\mu_\alpha}||\xi\,\mathscr{H}_\alpha(f)||_{2,\mu_\alpha} \geqslant (\alpha+1) \quad \text{or} \quad ||xf||^2_{2,\mu_\alpha} + ||\xi\,\mathscr{H}_\alpha(f)||^2_{2,\mu_\alpha} \geqslant 2(\alpha+1) \tag{1.18}$$

with equality, if and only if f is a multiple of a suitable Gaussian function.

In [7], the authors have established the Heisenberg-type uncertainty inequality for the Hankel-Gabor transform, that is for every $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$ and $s > 0$, we have

$$||\,x^s\mathscr{V}_g(f)||_{2,\omega_\alpha}||\,\xi^s\mathscr{V}_g(f)||_{2,\omega_\alpha} \geqslant c(\alpha,s)||f||^2_{2,\mu_\alpha}||g||^2_{2,\mu_\alpha}, \tag{1.19}$$

and a dilation argument (see [7, Lemma 2.2]) implies that inequality (1.19) is equivalent to,

$$||\,x^s\mathscr{V}_g(f)||^2_{2,\omega_\alpha} + ||\,\xi^s\mathscr{V}_g(f)||^2_{2,\omega_\alpha} \geqslant 2c(\alpha,s)||f||^2_{2,\mu_\alpha}||g||^2_{2,\mu_\alpha}, \tag{1.20}$$

where $c(\alpha,s)$ depends only on s and α.

The last two inequalities limit the concentration of the Hankel-Gabor transform in the time-frequency plane $\mathbb{R}_+ \times \mathbb{R}_+$. These state in particular that if one concentrates $\mathscr{V}_g(f)$ in time (with respect to the x-variable), then one looses concentration in frequency (with respect to the ξ-variable).

Considerable attention has been devoted recently to discovering new mathematical formulations and new contexts for the uncertainty principle (see the surveys [4, 6, 13] and the book [9] for other forms of the uncertainty principle). This paper will adopt the broader view that the uncertainty principle can be seen not only as a statement about the phase space (or time-frequency) localization of a single function but also as a statement on the degradation of localization when one considers successive elements of an orthonormal basis. In particular, Heisenberg's inequality (1.18) states that a unit-norm function in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ cannot occupy an arbitrarily small region in the phase space plane and the results that we consider show that the elements of an orthonormal basis as well as their Fourier-Bessel transforms cannot be uniformly concentrated in the time-frequency plane.

For some of the well-known results related to uncertainty principles for orthonormal sequences, Shapiro proved a number of uncertainty inequalities that are stronger than corresponding inequalities for a single function. For example, using compactness argument, see [16], one can conclude that for any orthonormal sequence $\{f_n\}_{n=0}^{+\infty}$ in $L^2(\mathbb{R})$

$$\sup_n \left(||xf_n||^2_{L^2(\mathbb{R})} + ||\xi\mathscr{F}(f_n)||^2_{L^2(\mathbb{R})}\right) \quad = \quad +\infty. \tag{1.21}$$

Some other results on time-frequency localization of orthonormal sequences and bases have been obtained by Benedetto [1] and Powell [12] and the quantitative version of Shapiro's result has been proved by Jaming and Powell [10] which states that, if $\{f_n\}_{n=0}^{+\infty}$ is an orthonormal sequence in $L^2(\mathbb{R})$ then for all $N \geqslant 0$:

$$\sum_{k=0}^{N} \left(||xf_n||^2_{L^2(\mathbb{R})} + ||\xi\mathscr{F}(f_n)||^2_{L^2(\mathbb{R})}\right)^2 \quad \geqslant \quad (N+1)^2. \tag{1.22}$$

This theorem implies in particular that, if the elements of an orthonormal sequence and their Fourier transforms have uniformly bounded dispersions then the sequence is finite.

The equality in (1.22) is attained for the sequence of Hermite functions and the higher dimensional version of this result involving generalized dispersions $|||x|^s f_n||^2_{L^2(\mathbb{R}^d)}$ and $|||\xi|^s\mathscr{F}(f_n)||^2_{L^2(\mathbb{R}^d)}$, $s > 0$, for orthonormal sequences in $L^2(\mathbb{R}^d)$ was obtained by Malinnikova [11].

As we have mentioned above, we will here concentrate on Shapiro-type uncertainty inequality for the Hankel-Gabor transform.

Our first result is the following Shapiro's Umbrella Theorem for the Hankel-Gabor transform :

If $\{\varphi_n\}$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$ such that the sequence $\{\mathscr{V}_g(\varphi_n)\}$ is bounded by a given square integrable function $\psi \in L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+)$, then the sequence $\{\varphi_n\}$ is finite.

Next, we will prove that Heisenberg's inequality (1.20) can be refined for orthonormal sequences.

More precisely, we show the following result :

Let $s > 0$ and let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$. Then for all $N \geqslant 1$;

$$\sum_{n=1}^{N} \left(|| \, x^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha} + || \, \xi^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha} \right) \geqslant K \; N^{1+\frac{s}{2\alpha+2}}. \tag{1.23}$$

where K depends only on s and α.

This result is known as Shapiro's uncertainty principle and the proof of this theorem is inspired from the paper of Malinnikova [11], who proved a similar result for the usual Fourier transform (1.22).

Consequently, we obtain the following strong uncertainty principle improving the Heisenberg-type inequality (1.21) for orthonormal sequences in $L^2_{\mu_\alpha}(\mathbb{R}_+)$,

$$\sup_n (|| \, |x|^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha} + || \, |\xi|^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha}) \quad = \quad +\infty. \tag{1.24}$$

2. Quantitative Dispersion Inequality for Orthonormal Sequences

In this section, g will be a fixed non-zero window function in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ such that $||g||^2_{2,\mu_\alpha} = 1$ and Σ be a subset of the time-frequency plane $\mathbb{R}_+ \times \mathbb{R}_+$ of finite measure $0 < \omega_\alpha(\Sigma) < \infty$. We introduce a pair of orthogonal projections on $L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+)$. The first, denoted $\mathscr{P}_g$ is the orthogonal projection defined by :

$$\mathscr{P}_g : L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+) \to \mathscr{V}_g(L^2_{\mu_\alpha}(\mathbb{R}_+)),$$

$$\mathscr{P}_g(F)(x,\xi) = \int_{\mathbb{R}_+ \times \mathbb{R}_+} F(x',\xi') \mathscr{K}_g((x',\xi'),(x,\xi)) d\omega_\alpha(x',\xi'), \tag{2.1}$$

where

$$\mathscr{K}_g((x',\xi'),(x,\xi)) = \frac{1}{||g||^2_{2,\mu_\alpha}} \mathscr{V}_g(g_{x,\xi})(x',\xi'). \tag{2.2}$$

The second is the time-frequency limiting operator defined by :

$$\mathscr{P}_\Sigma : L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+) \to L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+), \; \mathscr{P}_\Sigma(F) = \mathbf{1}_\Sigma F.$$

Then from [7, Inequality (4.8)], the operator $\mathscr{P}_\Sigma \mathscr{P}_g$ is a Hilbert-Schmidt operator with norm satisfying

$$||\mathscr{P}_\Sigma \mathscr{P}_g||^2_{HS} \leqslant \omega_\alpha(\Sigma). \tag{2.3}$$

Definition 2.1 — Let $0 < \varepsilon < 1$ and $f \in L^2_{\mu_\alpha}(\mathbb{R}_+)$ be a nonzero function. We say that $\mathscr{V}_g(f)$ is ε-time-frequency-concentrated on Σ, if

$$||\mathbf{1}_{\Sigma^c}\, \mathscr{V}_g(f)||_{2,\omega_\alpha} \leqslant \varepsilon ||f||_{2,\mu_\alpha} ||g||_{2,\mu_\alpha}. \tag{2.4}$$

Remark 2.2 : If we take $\varepsilon = 0$ in inequality (2.4), then Σ will be the exact support of $\mathscr{V}_g(f)$ and when $0 < \varepsilon < 1$, inequality (2.4) means that $\mathscr{V}_g(f)$ is ”practically zero” outside Σ. Indeed Σ may be considered as the ”essential” support of $\mathscr{V}_g(f)$.

Now, we prove an analogous of a time-frequency localization inequality for orthonormal sequences in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ which is similar to Theorem 2 in [11].

***Theorem* 2.3** — *Let* $\{\varphi_n\}_{n=1}^N$ *be an orthonormal system of* $L^2_{\mu_\alpha}(\mathbb{R}_+)$. *If* $\mathscr{V}_g(\varphi_n)$ *is* ε_n*-time-frequency-concentrated on* Σ*, then*

$$\sum_{n=1}^N (1-\varepsilon_n) \quad \leqslant \quad \omega_\alpha(\Sigma). \tag{2.5}$$

PROOF : Let $\{\varphi_n\}_{n=1}^N$ be an orthonormal sequence in $L^2_{\mu_\alpha}(\mathbb{R}_+)$, by relation (1.17) we deduce that $\{\mathscr{V}_g(\varphi_n)\}_{n=1}^N$ is an orthonormal sequence in $L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+)$.

Moreover, as $\mathscr{P}_\Sigma \mathscr{P}_g$ is a Hilbert-Schmidt operator and $\mathscr{P}_g \mathscr{V}_g(\varphi_n) = \mathscr{V}_g(\varphi_n)$, then by [18, Theorems 2.6 and 2.7], it is easy to see that

$$\begin{aligned}\sum_{n=1}^N \langle \mathscr{P}_\Sigma \mathscr{V}_g(\varphi_n) \mid \mathscr{V}_g(\varphi_n)\rangle_{\omega_\alpha} &= \sum_{n=1}^N \langle \mathscr{P}_\Sigma \mathscr{P}_g \mathscr{V}_g(\varphi_n) \mid \mathscr{P}_g \mathscr{V}_g(\varphi_n)\rangle_{\omega_\alpha} \\ &= \sum_{n=1}^N \langle \mathscr{P}_g \mathscr{P}_\Sigma \mathscr{P}_g \mathscr{V}_g(\varphi_n) \mid \mathscr{V}_g(\varphi_n)\rangle_{\omega_\alpha} \\ &\leqslant tr(\mathscr{P}_g \mathscr{P}_\Sigma \mathscr{P}_g) \\ &= ||\mathscr{P}_\Sigma \mathscr{P}_g||^2_{HS}. \end{aligned} \tag{2.6}$$

Then by (2.3) we obtain

$$\sum_{n=1}^N \langle \mathscr{P}_\Sigma \mathscr{V}_g(\varphi_n) \mid \mathscr{V}_g(\varphi_n)\rangle_{\omega_\alpha} \quad \leqslant \quad \omega_\alpha(\Sigma) \tag{2.7}$$

Then by Cauchy-Schwartz's inequality,

$$\begin{aligned}\langle P_\Sigma \mathscr{V}_g(\varphi_n) \mid \mathscr{V}_g(\varphi_n)\rangle_{\omega_\alpha} &= 1 - \langle \mathscr{P}_{\Sigma^c}\mathscr{V}_g(\varphi_n) \mid \mathscr{V}_g(\varphi_n)\rangle_{\omega_\alpha} \\ &\geqslant 1 - ||1_{\Sigma^c}\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}. \qquad (2.8)\end{aligned}$$

Therefore by (2.7), we deduce the desired result. □

This result provides a quantitative estimate for the Shapiro's umbrella theorem for the Hankel-Gabor transform as well as a number of inequalities for orthonormal sequences in $L^2_{\mu_\alpha}(\mathbb{R}_+)$. More precisely:

Given $\psi \in L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+)$ and $\varepsilon > 0$, define

$$\mathscr{C}_\psi(\varepsilon) = \inf\big\{\omega_\alpha(\Sigma) \setminus \int\int_{\mathbb{R}_+\times\mathbb{R}_+\setminus\Sigma} \mid \psi(x,\xi)\mid^2 d\omega_\alpha(x,\xi) \leqslant \varepsilon^2\big\}.$$

Note that if ψ is not identically zero then for all $0 < \varepsilon < 1$ one has $0 < \mathscr{C}_\psi(\varepsilon) < \infty$.

***Theorem* 2.4** — *(Quantitative version of Shapiro's Umbrella Theorem)*

Let $\psi \in L^2_{\omega_\alpha}(\mathbb{R}_+ \times \mathbb{R}_+)$ *and let* $\{\varphi_n\}_{n=1}^N$ *be an orthonormal system of* $L^2_{\mu_\alpha}(\mathbb{R}_+)$ *such that for all* $1 \leqslant n \leqslant N$ *and for almost all* $(x,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\mid \mathscr{V}_g(\varphi_n)\big(x,\xi\big) \mid \leqslant \psi\big(x,\xi\big).$$

Then

$$N \leqslant \frac{\mathscr{C}_\psi(\varepsilon)}{1-\varepsilon},\ \forall\varepsilon \in (0,1). \qquad (2.9)$$

PROOF : Following [11, Corollary 2], for every positive real number $0 < \varepsilon < 1$, there is a subset $\Delta_{\psi,\varepsilon} \subset \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\omega_\alpha(\Delta_{\psi,\epsilon}) = \mathscr{C}_\psi(\varepsilon) \text{ and } \int\int_{\mathbb{R}_+\times\mathbb{R}_+\setminus\Delta_{\psi,\varepsilon}} \mid \psi(x,\xi)\mid^2 d\omega_\alpha(x,\xi) = \varepsilon^2.$$

Then for every n, we obtain

$$\int\int_{\mathbb{R}_+\times\mathbb{R}_+\setminus\Delta_{\psi,\varepsilon}} \mid \mathscr{V}_g(\varphi_n)(x,\xi)\mid^2 d\omega_\alpha(x,\xi) \leqslant \varepsilon^2. \qquad (2.10)$$

We deduce that for every n ; $\mathscr{V}_g(\varphi_n)$ is ε-time-frequency-concentrated in $\Delta_{\psi,\varepsilon}$. Then by Theorem 2.3,

$$(1-\varepsilon)N \leqslant \omega_\alpha(\Delta_{\psi,\epsilon}).\square \qquad (2.11)$$

The following result shows that, if the Hankel-Gabor transform of an orthonormal sequence is ε-time-frequency concentrated in a given centred ball of $\mathbb{R}_+ \times \mathbb{R}_+$, then such sequence is necessary finite.

Corollary 2.5 — Let $0 < \varepsilon < 1$ and let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$ such that $\mathscr{V}_g(\varphi_n)$ is ε-time-frequency-concentrated on the ball $B_\rho^+ = \{(r,x) \in \mathbb{R}_+ \times \mathbb{R}_+ ; r^2 + x^2 \leqslant \rho^2\}$. Then

$$N \quad \leqslant \quad \frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\,\Gamma(2\alpha+3)(1-\varepsilon)}. \tag{2.12}$$

Applying Theorem 2.3, we deduce that

$$\sum_{n=1}^{N}(1-\varepsilon) \quad \leqslant \quad \omega_\alpha(B_\rho^+). \tag{2.13}$$

However, for every $1 \leqslant n \leqslant N,\ ||\mathbf{1}_{(B_\rho^+)^c}\,\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha} \leqslant \varepsilon$, and

$$\begin{aligned}
\omega_\alpha(B_\rho^+) &= \mu_\alpha \otimes \mu_\alpha\big(B_\rho^+\big) = \int_{r^2 \leqslant \rho^2}\Big(\int_{x^2 \leqslant \rho^2 - r^2} d\mu_\alpha(x)\Big) d\mu_\alpha(r)\\
&= \int_{r^2 \leqslant \rho^2}\Big(\frac{1}{2^\alpha\,\Gamma(\alpha+1)}\int_0^{\sqrt{\rho^2-r^2}} x^{2\alpha+1}\,dx\Big) d\mu_\alpha(r)\\
&= \int_{r^2 \leqslant \rho^2}\Big(\frac{1}{2^{\alpha+1}\,\Gamma(\alpha+2)}\,(\rho^2-r^2)^{\alpha+1}\Big) d\mu_\alpha(r)\\
&= \frac{1}{2^{2\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+2)}\int_0^\rho (\rho^2-r^2)^{\alpha+1} r^{2\alpha+1} dr\\
&= \frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\,\Gamma(\alpha+1)\Gamma(\alpha+2)}\int_0^1 (1-u)^{\alpha+1}\,u^\alpha\,du\\
&= \frac{\rho^{4\alpha+4}}{2^{2\alpha+2}\,\Gamma(2\alpha+3)}.
\end{aligned} \tag{2.14}$$

We get the result by combining relations (2.13) and (2.14). □

Therefore, if the generalized dispersion of the elements of an orthonormal sequence, $|||(x,\xi)|^s \mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{1/s},\ s > 0$, is uniformly bounded then this sequence is finite and we can give a bound on the number of elements in that sequence. More precisely:

Lemma 2.6 — Let $s, a > 0$ and $\{\varphi_n\}_{n=1}^N$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$ that satisfies

$$||\,|(x,\xi)|^s \mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{1/s} \leqslant a.$$

Then

$$N \leqslant \frac{2^{(2\alpha+2)(\frac{2}{s}-1)+1}a^{4\alpha+4}}{\Gamma(2\alpha+3)}. \tag{2.15}$$

PROOF : Since

$$||\mathbf{1}_{(B_\rho^+)^c}\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha} \leqslant \rho^{-s}||\,|(x,\xi)|^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha} \leqslant \Big(\frac{a}{\rho}\Big)^s,$$

then if we choose $\rho = a\,2^{1/s}$, we deduce that for every $1 \leqslant n \leqslant N$; $\mathscr{V}_g(\varphi_n)$ is $\frac{1}{2}$-time-concentrated in the ball B_ρ^+. Applying Corollary 2.5, we obtain the desired result. □

The previous Lemma may be expressed by using separately the time dispersion and the frequency dispersion of the sequence $\mathscr{V}_g(\varphi_n)$, defined respectively by

$$||x^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{\frac{1}{s}} \text{ and } ||\xi^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{\frac{1}{s}}.$$

Indeed, we have

Corollary 2.7 — Under the hypothesis of Corollary 2.6 with $s \geqslant 1$, we assume that for every $1 \leqslant n \leqslant N$,

$$||x^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{\frac{1}{s}} \leqslant a \text{ and } ||\xi^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{\frac{1}{s}} \leqslant a.$$

Then

$$N \leqslant \frac{2^{\frac{2}{s}(2\alpha+2)+1}a^{4\alpha+4}}{\Gamma(2\alpha+3)}. \tag{2.16}$$

PROOF : By a convexity argument $(s \geqslant 1)$ and for every $1 \leqslant n \leqslant N$, we get

$$||\,|(x,\xi)|^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^2 \leqslant 2^{s-1}\big(||\,x^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^2 + ||\,\xi^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^2\big), \tag{2.17}$$

which implies that for every $1 \leqslant n \leqslant N$,

$$||\,|(x,\xi)|^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{\frac{1}{s}} \leqslant a\sqrt{2}.$$

The result follows from Corollary 2.6. □

To prove our main result, we need the following lemma :

Lemma 2.8 — Let $s > 0$ and let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$. Then there exists $j_0 \in \mathbb{Z}$; $j_0 = E\left(\dfrac{\ln\big(\frac{\Gamma(2\alpha+3)}{2^{(2\alpha+2)(\frac{2}{s}-1)+1}}\big)}{4(\alpha+1)\ln(2)}\right) - 1$ such that

$$||\,|(x,\xi)|^s\mathscr{V}_g(\varphi_n)||_{2,\omega_\alpha}^{1/s} \geqslant 2^{j_0-1} \tag{2.18}$$

PROOF : For every $j \in \mathbb{Z}$, let

$$P_j = \Big\{ n \in \mathbb{N};\ 2^{j-1} \leqslant || \, |(x,\xi)|^s \mathscr{V}_g(\varphi_n)||^{1/s}_{2,\omega_\alpha} < 2^j \Big\}.$$

Then, $\mathbb{N} = \bigcup_{j \in \mathbb{Z}} P_j$, $P_{j_1} \cap P_{j_2} = \emptyset$ if $j_1 \neq j_2$ and for every $n \in P_j$,

$$|| \, |(x,\xi)|^s \mathscr{V}_g(\varphi_n)||^{1/s}_{2,\omega_\alpha} \leqslant 2^j. \tag{2.19}$$

Applying Lemma 2.6, we deduce that P_j is finite and

$$\mathrm{card}(P_j) \quad \leqslant \quad \frac{2^{(2\alpha+2)(\frac{2}{s}-1)+1}(2^j)^{4\alpha+4}}{\Gamma(2\alpha+3)}. \tag{2.20}$$

Thus for $j \leqslant j_0 = E\left(\dfrac{\ln\big(\frac{\Gamma(2\alpha+3)}{2^{(2\alpha+2)(\frac{2}{s}-1)+1}}\big)}{4(\alpha+1)\ln(2)} \right) - 1$, we get $\mathrm{card}(P_j) = 0$ or $P_j = \emptyset$. So,

$$\mathbb{N} = \bigcup_{j \in \mathbb{Z}} P_j = \bigcup_{j=j_0}^{+\infty} P_j. \tag{2.21}$$

***Theorem* 2.9** — *(Shapiro's dispersion theorem for $\mathscr{V}_g$) Let $s > 0$ and let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$. Then for all $N \geqslant 1$;*

$$\sum_{n=1}^{N} || \, |(x,\xi)|^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha} \geqslant \left(\frac{(2^{4\alpha+4}-1)\Gamma(2\alpha+3)}{2^{(2\alpha+2)(3+\frac{3}{s})+2}} \right)^{\frac{s}{2\alpha+2}} N^{1+\frac{s}{2\alpha+2}}. \tag{2.22}$$

PROOF : Let j_0 be defined in Lemma 2.8, we can see that P_j is empty for all $j < j_0$. Let $k \geqslant j_0$. Then for all $n \in P_j$, we have

$$|| \, |(x,\xi)|^s \mathscr{V}_g(\varphi_n)||^{\frac{1}{s}}_{2,\omega_\alpha} \leqslant 2^j.$$

Let N_j be the number of elements in P_j, then by Lemma 2.6,

$$N_j \quad \leqslant \quad \frac{2^{(2\alpha+2)(\frac{2}{s}-1)+1} 2^{j(4\alpha+4)}}{\Gamma(2\alpha+3)}. \tag{2.23}$$

The number of elements in $\bigcup_{j=j_0}^{k} P_j$ is less then $A_{s,\alpha}\, 2^{(4\alpha+4)(k+1)}$ where

$$A_{s,\alpha} = \frac{2^{(2\alpha+2)(\frac{2}{s}-1)+1}}{(2^{4\alpha+4}-1)\Gamma(2\alpha+3)}$$

is a constant that does not depend on k.

Now, if $N > 2\ A_{s,\alpha}\ 2^{(4\alpha+4)(j_0+1)}$, then there exists $k > j_0$ such that

$$2\ A_{s,\alpha}\ 2^{(4\alpha+4)k} \leqslant N < 2\ A_{s,\alpha}\ 2^{(4\alpha+4)(k+1)}. \tag{2.24}$$

Therefore at least half elements of the set $\{1, ..., N\}$ does not belong to $\bigcup\limits_{j=j_0}^{k-1} P_j$, then

$$\begin{aligned} \sum_{n=1}^{N} \| \, |(x,\xi)|^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha} \geqslant & \quad \frac{N}{2}\ 2^{2s(k-1)} \\ \geqslant & \quad \left(2^{(4\alpha+4)(k+1)+1} A_{s,\alpha}\right)^{\frac{-s}{2\alpha+2}}\ N^{1+\frac{s}{2\alpha+2}} \\ = & \quad \left(\frac{(2^{4\alpha+4}-1)\Gamma(2\alpha+3)}{2^{(2\alpha+2)(3+\frac{3}{s})+2}}\right)^{\frac{s}{2\alpha+2}}\ N^{1+\frac{s}{2\alpha+2}}. \end{aligned} \tag{2.25}$$

If $N \leqslant 2\ A_{s,\alpha}\ 2^{(4\alpha+4)(j_0+1)}$, then from Lemma 2.8, we have

$$\begin{aligned} \sum_{n=1}^{N} \| \, |(x,\xi)|^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha} \geqslant & \quad N\ 2^{2s(j_0-1)} \\ \geqslant & \quad \left(2^{8\alpha+9} A_{s,\alpha}\right)^{\frac{-s}{2\alpha+2}}\ N^{1+\frac{s}{2\alpha+2}} \\ = & \quad \left(\frac{(2^{4\alpha+4}-1)\Gamma(2\alpha+3)}{2^{(2\alpha+2)(3+\frac{2}{s})+2}}\right)^{\frac{s}{2\alpha+2}}\ N^{1+\frac{s}{2\alpha+2}}. \end{aligned} \tag{2.26}$$

Then, we get the desired result. □

Consequently we obtain the following corollary.

Corollary 2.10 — (Mean Dispersion principle) Let $s > 0$ and let $\{\varphi_n\}_{n=1}^N$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$. Then for all $N \geqslant 1$;

$$\sum_{n=1}^{N} \left(\| \, x^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha} + \| \, \xi^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha}\right) \geqslant \left(\frac{(2^{4\alpha+4}-1)\Gamma(2\alpha+3)}{2^{(2\alpha+2)(4+\frac{3}{s})+2}}\right)^{\frac{s}{2\alpha+2}}\ N^{1+\frac{s}{2\alpha+2}}. \tag{2.27}$$

PROOF : The result follows immediately from the previous theorem and the fact that

$$\| \, |(x,\xi)|^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha} \leqslant 2^s \left(\| \, x^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha} + \| \, \xi^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha}\right).\square \tag{2.28}$$

The last Dispersion inequality (2.27) implies in particular that, there does not exist an infinite sequence $\{\varphi_n\}_{n=1}^\infty$ in $L^2_{\mu_\alpha}(\mathbb{R}_+)$ such that the two sequences

$$\left\{\|x^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha}\right\}_{n=1}^\infty \text{ and } \left\{\| \, \xi^s \mathscr{V}_g(\varphi_n)\|^2_{2,\omega_\alpha}\right\}_{n=1}^\infty$$

are simultaneously bounded. More precisely:

Corollary 2.11 — Let $\{\varphi_n\}_{n=1}^{\infty}$ be an orthonormal system of $L^2_{\mu_\alpha}(\mathbb{R}_+)$. Then for every $N \geqslant 1$,

$$\sup_{1\leqslant n\leqslant N} \left\{||x^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha} + ||\xi^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha}\right\} \geqslant \left(\frac{(2^{4\alpha+4}-1)\Gamma(2\alpha+3)}{2^{(2\alpha+2)(4+\frac{3}{s})+2}}\right)^{\frac{s}{2\alpha+2}} N^{\frac{s}{2\alpha+2}}. \quad (2.29)$$

In particular, we get a strong version of the uncertainty inequality for orthonormal sequences of $L^2_{\mu_\alpha}(\mathbb{R}_+)$:

$$\sup_n \left(||x^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha} + ||\xi^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha}\right) = \infty. \quad (2.30)$$

Remark 2.12 : i. The relation (2.27) shows in particular that for every orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of $L^2_{\mu_\alpha}(\mathbb{R}_+)$ and for every $s > 0$,

$$\sum_{n\in\mathbb{N}} \left(||\, x^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha} + ||\, \xi^s \mathscr{V}_g(\varphi_n)||^2_{2,\omega_\alpha}\right) = \infty. \quad (2.31)$$

ii. By taking $N = 1$, relation (6.6) appears as a general version of the Heisenberg-type uncertainty inequality for the Hankel-Gabor transform (1.20).

Acknowledgement

The author is deeply indebted to the referees for providing constructive comments and helps in improving the contents of this article.

References

1. J. J. Benedetto, On frame decompositions, sampling and uncertainty principle inequalities, in: Wavelets: Mathematics and Applications, in: Studies in Advanced Mathematics, CRC Press, Boca Raton, (1994), pp. 247-304.

2. P. C. Bowie, Uncertainty inequalities for Hankel transforms, *SIAM J. Math. Anal.*, **2** (1971), 601-606.

3. A. Bonami, B. Demange, and P. Jaming, Hermite functions and uncertainty principles for the Fourier and the Gabor transforms, *Rev Mat lberoamericana*, **19** (2003), 23-55.

4. A. Bonami and B. Demange, A survey on uncertainty principles related to quadratic forms, *Collect. Math.*, **2** (2006), 136.

5. W. Czaja and G. Gigante, Continuous Gabor transform for strong hypergroups, *J. Fourier Anal. Appl.*, **9** (2003), 321-339.

6. G. B. Folland and A. Sitaram, The uncertainty principle: A mathematical survey, *J. Fourier Anal. Appl.*, **3** (1997), 207-238.

7. S. Ghobber and S. Omri, Time-frequency concentration of the windowed Hankel transform, *Integral Transforms Spec. Funct.*, **25** (2014), 481-496.

8. S. Ghobber, Phase space localization of orthonormal sequences in $L^2_\alpha(\mathbb{R}_+)$, *Journal of Approximation Theory*, **189** (2015), 123-136.

9. V. Havin and B. Jöricke, The uncertainty principle in harmonic analysis, Springer-Verlag, Berlin, 1994.

10. Ph. Jaming and A. M. Powell, Uncertainty principles for orthonormal sequences, *J. Funct. Anal.*, **243** (2007), 611-630.

11. E. Malinnikova, Orthonormal sequences in $L^2(\mathbb{R}^d)$ and time frequency localization, *J. Fourier Anal. Appl.*, **16** (2010), 983-1006.

12. A. M. Powell, Time-frequency mean and variance sequences of orthonormal bases, *J. Fourier Anal. Appl.*, **11** (2005), 375-387.

13. B. Ricaud and B. Torrsani, A survey of uncertainty principles and some signal processing applications, *Adv. Comput. Math.*, **40** (2014), 629-650.

14. M. Rösler and M. Voit, An uncertainty principle for Hankel transforms, *Proc. Amer. Math. Soc.*, **127** (1999), 183-194.

15. A. L. Schwartz, An inversion theorem for Hankel transforms, *Pro. Amer. Math. Soc.*, **22**(3) (1969), 713-717.

16. H. S. Shapiro, Uncertainty principles for bases in $L^2(\mathbb{R})$, Unpublished Manuscript.

17. K. Stempak, A note on Zemanian spaces, *Extracta Math.*, **12** (1997), 33-40.

18. M. W. Wong, *Wavelet transforms and localization operators*, Birkhauser (2002).

19. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, Cambridge, 1944.

20. Y. I. Zitomirski, Cauchys problem for systems of linear partial differential equations with differential operators of Bessel type, *Mat. Sb. (N.S.)*, **36** (1955), 299-310.