Indian J. Pure Appl. Math., **51**(1): 277-288, March 2020
 DOI: 10.1007/s13226-020-0400-1

ON TETRAVALENT VERTEX-TRANSITIVE BI-CIRCULANTS

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(*Received* 9 *September* 2016; *after final revision* 6 *August* 2018;
accepted 25 *January* 2019)

A graph Γ is called a *bi-circulant* if it admits a cyclic group as a group of automorphisms acting semiregularly on the vertices of Γ with two orbits. The characterization of tetravalent edge-transitive bi-circulants was given in several recent papers. In this paper, a classification is given of connected tetravalent vertex-transitive bi-circulants of order twice an odd integer.

Key words : Bi-Cayley graph; vertex-transitive graph; Cayley graph.

2010 Mathematics Subject Classification : 05C25, 20B25

1. Introduction

Throughout this paper, groups are assumed to be finite, and graphs are assumed to be finite, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [4, 22].

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular.

A graph Γ is called a *Cayley graph* if it admits a group G of automorphisms acting regularly on its vertex-set $V(\Gamma)$. In that case, Γ is isomorphic to the graph $\mathrm{Cay}(G, S)$ with vertex-set G and edge-set $\{\{g, xg\} : g \in G,\ x \in S\}$, where S is the subset of elements of G taking the identity element to one of its neighbours (see [3, Lemma 16.3]); and then the automorphism group of Γ contains a subgroup $R(G) = \{R(g) : g \in G\}$, where $R(g)$ is the permutation of G given by $x \mapsto xg$ for $\forall x \in G$.

If, instead, we require the graph Γ to admit a group H of automorphisms acting semi-regularly on $V(\Gamma)$ with two orbits, then we call Γ a *bi-Cayley graph* (for H). In this case, H acts regularly on each of its two orbits on $V(\Gamma)$, and the two corresponding induced subgraphs are Cayley graphs for H. In particular, we may label the vertices of these two subgraphs with elements of two copies H_0 and H_1 of H, and find that there are subsets R, L and S of H such that the edges of those two induced subgraphs are of the form $\{h_0, (xh)_0\}$ with $h_0 \in H_0$ and $x \in R$, and $\{h_1, (yh)_1\}$ with $h_1 \in H_1$ and $y \in L$, while all remaining edges are of the form $\{h_0, (zh)_1\}$ with $z \in S$ and where h_0 and h_1 are the elements of H_0 and H_1 that represent a given $h \in H$.

Conversely, if H is any group, and R, L and S are subsets of H such that $1_H \notin R = R^{-1}$ and $1_H \notin L = L^{-1}$, then the graph Γ with vertex set being the union $H_0 \cup H_1$ of two copies of H and with edges of the form $\{h_0, (xh)_0\}$, $\{h_1, (yh)_1\}$ and $\{h_0, (zh)_1\}$ with $x \in R$, $y \in L$ and $z \in S$, and $h_0 \in H_0$ and $h_1 \in H_1$ representing a given $h \in H$, is a bi-Cayley graph for H. Indeed H acts as a semi-regular group of automorphisms by right multiplication, with H_0 and H_1 as its orbits on vertices. We denote this graph by BiCay(H, R, L, S), and denote the group of automorphisms induced by H on the graph as $R(H)$. When $R(H)$ is normal in Aut(X), the bi-Cayley graph $X =$ BiCay(H, R, L, S) will be called a *normal bi-Cayley graph* over H.

A bi-Cayley graph over a cyclic group, an abelian group or a dihedral group is also simply called a *bicirculant*, *bi-abelian* or *bi-dihedrant*, respectively.

Bi-Cayley graphs form an extensively studied class of graphs (see, for instance, [2, 8, 12-15, 17, 18, 24]. In the study of bi-Cayley graphs, one of the natural problems is: for a given group H, classify bi-Cayley graphs with specific symmetry properties over H. Some partial answers have been obtained for this problem. For example, Marušič *et al*. [20, 21] classified all trivalent vertex-transitive bi-circulants, and Zhou and Feng [25] extended this to the classification of trivalent vertex-transitive bi-abelians. Recently, all tetravalent edge-transitive bi-circulants were classified in [13], and all pentavalent arc-transitive bi-circulants were classified in [1]. The works listed above provide a motivation for us to consider the tetravalent vertex-transitive non-Cayley bi-circulants. In this paper, we shall classify tetravalent vertex-transitive bi-circulant graphs of order $2n$ for each odd integer n.

The study in the literature on the construction of vertex-transitive non-Cayley graphs also provides a motivation for us to consider vertex-transitive non-Cayley bi-circulants. As one of the most important finite graphs, the Petersen graph is a bi-Cayley graph over a cyclic group of order 5. Note that the Petersen graph is vertex-transitive but not Cayley. We call a vertex-transitive graph which is not Cayley a *vertex-transitive non-Cayley graph*, or a *VNC-graph* for short. There have been a lot of work on the classification and construction of VNC-graphs (see, for instance, [16, 10]). In [23] and [7], Zhou *et al*. classified all tetravalent VNC-graphs of order $4p$ and $2p^2$ for each prime p, in

[11] Hujdurovic *et al*. constructed two infinite families of VNC-graphs by using generalized Cayley graphs and in [6] Conder *et al*. constructed an infinite family of trivalent VNC-graphs as Haar graphs (namely, 0-type bi-Cayley graphs) over some non-abelian groups. In this paper, by considering tetravalent vertex-transitive bi-circulants graphs of order $2n$ with n an odd integer, we construct a new infinite family of VNC-graphs.

2. Preliminaries

In this section, we introduce basic concepts and terminology as well as some preliminary results which will be used later in the paper. For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$, $A(X)$, $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. Let X be a connected vertex-transitive graph, and let $G \leq \mathrm{Aut}(X)$. For a *G-invariant* partition $\mathcal{B}$ of $V(X)$, the *quotient graph* $X_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two vertices B, $C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in X. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(X)$ is a G-invariant partition of $V(X)$. In this case, the symbol $X_{\mathcal{B}}$ will be replaced by X_N. Note that, if X is vertex-, edge- or arc-transitive, then so is X_N, respectively.

Let X and Y be two graphs. The *lexicographic product* $X[Y]$ is defined as the graph with vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = (x_1, y_1)$ and $v = (x_2, y_2)$ in $V(X[Y])$, u is adjacent to v in $X[Y]$ whenever $\{x_1, x_2\} \in \mathrm{E}(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in \mathrm{E}(Y)$.

For a positive integer n, denote by $\mathbb{Z}_n$ the cyclic group of order n as well as the ring of integers modulo n, *by* $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, by D_{2n} the dihedral group of order $2n$, and by C_n and K_n the cycle and the complete graph of order n, respectively. We call C_n an *n-cycle*. Finally, for two groups M and N, $N \rtimes M$ denotes a semidirect product of N by M. For a subgroup H of a group G , denote by $C_G(H)$ the centralizer of H in G and by $N_G(H)$ the normalizer of H in G. For a permutation group G over a set Ω and for a subset $\triangle$ of Ω, $G_{\triangle}$ is the subgroup of G fixing $\triangle$ pointwise.

In the following, we introduce some results regarding bi-Cayley graphs. We always assume that $X = \mathrm{BiCay}(H, R, L, S)$ is a connected bi-Cayley graph over a group H. It is easy to get the following obvious basic facts of bi-Cayley graphs.

Proposition 2.1 - [26]. The following hold.

(1) H is generated by $R \cup L \cup S$.

(2) Up to graph isomorphism, S can be chosen to contain the identity of H.

(3) For any automorphism α of H, BiCay $(H, R, L, S) \cong$ BiCay $(H, R^\alpha, L^\alpha, S^\alpha)$.

(4) BiCay $(H, R, L, S) \cong$ BiCay (H, L, R, S^{-1}).

The triple (R, L, S) of three subsets R, L, S of a group H is called *bi-Cayley triple* if $R = R^{-1}$, $L = L^{-1}$, and $1 \in S$. Two bi-Cayley triples (R, L, S) and (R', L', S') of a group H are said to be equivalent, denoted by $(R, L, S) \equiv (R', L', S')$, if either $(R', L', S') = (L, R, S^{-1})$ or $(R', L', S') = (R, L, S)^\alpha$ for some automorphism α of H. The bi-Cayley graphs corresponding to two equivalent bi-Cayley triples of the same group are isomorphic (see Proposition 2.1 (3)-(4)).

Next we give a result about the automorphisms of the bi-Cayley graph $X =$BiCay(H, R, L, S). Recall that for each $g \in H$, $R(g)$ is a permutation on $V(X)$ defined by the rule

$$h_i^{R(g)} = (hg)_i,\ \forall i \in \mathbb{Z}_2,\ h,\ g \in H, \tag{1}$$

and $R(H) = \{R(g) \mid g \in H\} \le \text{Aut}(X)$. For an automorphism α of H and $x, y, g \in H$, define two permutations on $V(X) = H_0 \cup H_1$ as following:

$$\begin{aligned} \delta_{\alpha,x,y}: &\ \ h_0 \mapsto (xh^\alpha)_1, h_1 \mapsto (yh^\alpha)_0, \forall h \in H, \\ \sigma_{\alpha,g}: &\ \ h_0 \mapsto (h^\alpha)_0, h_1 \mapsto (gh^\alpha)_1, \forall h \in H. \end{aligned} \tag{2}$$

Set

$$\begin{aligned} \text{I} &= \{\delta_{\alpha,x,y} \mid \alpha \in \text{Aut}(H)\ s.t.\ R^\alpha = x^{-1}Lx, L^\alpha = y^{-1}Ry, S^\alpha = y^{-1}S^{-1}x\}, \\ \text{F} &= \{\sigma_{\alpha,g} \mid \alpha \in \text{Aut}(H)\ s.t.\ R^\alpha = R, L^\alpha = g^{-1}Lg, S^\alpha = g^{-1}S\}. \end{aligned} \tag{3}$$

Proposition 2.2 — [26, Theorem 1.1]. Let $\Gamma = \text{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph over the group H. Then $N_{\text{Aut}(\Gamma)}(R(H)) = R(H) \rtimes F$ if $I = \emptyset$ and $N_{\text{Aut}(\Gamma)}(R(H)) = R(H)\langle F, \delta_{\alpha,x,y}\rangle$ if $I \neq \emptyset$ and $\delta_{\alpha,x,y} \in I$. Furthermore, for any $\delta_{\alpha,x,y} \in I$, we have the following:

(1) $\langle R(H), \delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$.

(2) if α has order 2 and $x = y = 1$, then Γ is isomorphic to the Cayley graph Cay$(\bar{H}, R \cup \alpha S)$, where $\bar{H} = H \rtimes \langle \alpha \rangle$.

3. Construction

In this section, we shall construct an infinite family of tetravalent vertex-transitive non-Cayley graphs.

Construction 3.1 — Let $m_1, m_2 > 1$ be two odd integers such that $(m_1, m_2) = 1$. Let $t \in \mathbb{Z}_{m_2}^*$ be such that $t^2 \equiv -1 \pmod{m_2}$. Let $H = \langle r \rangle \times \langle s \rangle \cong \mathbb{Z}_{m_2} \times \mathbb{Z}_{m_1} (\cong \mathbb{Z}_{m_1 m_2})$. Set $R = \{r, r^{-1}\}$, $L = \{r^t, r^{-t}\}$ and $S = \{1, s\}$. Let

$$X_{m_1,m_2,t} = \text{BiCay}(H, R, L, S).$$

It will be shown in Theorem 3.3 that the graph $X_{m_1,m_2,t}$ is a connected tetravalent vertex-transitive non-Cayley graph. Note that the smallest graph in this infinite family of graphs is $X_{3,5,2}$ with order 30. Clearly,

$$X_{3,5,2} = \text{BiCay}(H, \{r, r^{-1}\}, \{r^2, r^{-2}\}, \{1, s\}),$$

where $H = \langle r\rangle \times \langle s\rangle \cong \mathbb{Z}_5 \times \mathbb{Z}_3$. Furthermore, one may see Appendix I for the adjacency matrix of $X_{3,5,2}$. Below we shall prove a lemma which is useful in the proof of Theorems 3.3 and 4.3.

Lemma 3.2 — Let $n > 7$ be an odd integer, and let X be a connected tetravalent vertex-transitive bi-Cayley graph over a cyclic group of order n. If $3 \mid |\text{Aut}(X)_v|$ for some $v \in V(X)$, then either X is edge-transitive or is of 0-type.

PROOF : Let $A = \text{Aut}(X)$. Suppose that $3 \mid |A_v|$ for some $v \in V(X)$. Assume that X is not edge-transitive. We may let $X = \text{BiCay}(H, R, L, S)$, where $H = \langle a\rangle \cong \mathbb{Z}_n$ and $1 \in S$. Suppose that X is not of 0-type. Then X is of 2-type since $|H_0| = |H_1| = n$ is odd.

For any $v \in V(X)$, let A_v^* be the subgroup of A_v fixing the neighborhood $N(v)$ of v pointwise. Let $T \in \text{Syl}_3(A_v)$. Suppose $T \leq A_v^*$. Then $T \leq A_u$ for every $u \in N(v)$. Since X is vertex-transitive, one has $A_v/A_v^* \cong A_u/A_u^*$, implying that $T \leq A_u^*$. The connectedness of X implies that T fixes every vertex of X, forcing $T = 1$, a contradiction. Thus $T \nleq A_v^*$ and hence $\mathbb{Z}_3 \cong TA_v^*/A_v^* \leq A_v/A_v^* \leq S_3$ considering X is not edge-transitive. Then $A_v/A_v^* \cong \mathbb{Z}_3$ or S_3. It follows that for any $v \in V(X)$, there is a unique vertex $u \in N(v)$ such that $A_v = A_u$.

Set $F = \{\{u, v\} \in E(X) \mid A_v = A_u\}$ and $\Gamma = X - F$. Then Γ is a cubic graph. For any $g \in A$ and $\{u, v\} \in F$, one has $\{u, v\}^g = \{u^g, v^g\}$. Furthermore, $A_{u^g} = A_u^g = A_v^g = A_{v^g}$. It follows that $\{u, v\}^g = \{u^g, v^g\} \in F$ and hence $F^g = F$. Consequently, A is a vertex-transitive automorphism group of Γ. Since $3 \mid |A_v/A_v^*|$, A is also arc-transitive on Γ.

If Γ is connected, then Γ is a cubic arc-transitive bi-Cayley graph over $H \cong \mathbb{Z}_n$. Since $n > 7$, by [12, Theorem 1.1], Γ is a bipartite graph with H_0 and H_1 as its two partition sets, forcing X is of type 1, a contradiction.

If Γ is disconnected, then since Γ is cubic and $|V(\Gamma)| = 2n$, each component of Γ has order $2m$ with $m \mid n$. Let Γ_i $(0 \leq i \leq n/m - 1)$ be the n/m components of Γ, and let $B_i = V(\Gamma_i)$. Set $\Omega = \{B_i \mid 0 \leq i \leq n/m - 1\}$. Then Ω is an A-invariant partition of $V(\Gamma) = V(X)$. Consider the quotient graph X_Ω of X relative to Ω, and let K be the kernel of A acting on Ω. Recall that H_0 and

H_1 are the two orbits induced by $R(H)$ acting on $V(\Gamma)$. Since X is of type 2, $X[H_0] \cong X[H_1]$ are of valency 2. So for each $0 \le i \le n/m - 1$, $B_i \cap H_0 \neq \emptyset$ and $B_i \cap H_1 \neq \emptyset$. Then $R(H)$ is transitive on Ω. Since $R(H) \cong \mathbb{Z}_n$ and $|\Omega| = n/m$, the kernel $K \cap R(H)$ of $R(H)$ acting on Ω is isomorphic to $\mathbb{Z}_m$. It follows that for each $0 \le i \le n/m - 1$, $|B_i \cap H_0| = |B_i \cap H_1| = m$. If K is transitive on some B_i, then K is transitive on all B_j. Since Γ_i is of valency 3, the quotient graph X_Ω would have valency 1, and so $X_\Omega \cong K_2$. This forces that $n/m = 2$, contradicting that n is odd. Thus, K is intransitive on each B_i. Note that $B_i \cap H_0$ and $B_i \cap H_1$ are two orbits of $K \cap R(H)$ acting on B_i. So, $B_i \cap H_0$ and $B_i \cap H_1$ are also the two orbits of K on B_i. Let A_{B_i} be the subgroup of A fixing B_i setwise. Then A_{B_i} is arc-transitive on Γ_i. Clearly, $K \trianglelefteq A_{B_i}$. Then $B_i \cap H_0$ and $B_i \cap H_1$ are two independent sets. This implies that each Γ_i is bipartite with $B_i \cap H_0$ and $B_i \cap H_1$ as its two bipartition sets. Again this would force that X is of type 1, a contradiction. □

Now we give the main result of this section.

***Theorem* 3.3** — *The graph $X_{m_1,m_2,t}$ is a connected tetravalent VNC-graph.*

PROOF : Let $X = X_{m_1,m_2,t}$ and $A = \mathrm{Aut}(X)$. It is easy to see that H has an automorphism β such that $r^\beta = r^t$ and $s^\beta = s^{-1}$. Furthermore, we have $R^\beta = L$, $L^\beta = R$ and $S^\beta = S^{-1}$. By Proposition 2.2, we have $1 \neq \delta_{\beta,1,1} \in A$, and so X is vertex-transitive.

From [13, Theorem 1.1], one can obtain that $X_{m_1,m_2,t}$ is not edge-transitive. Since X is of valency 4, the vertex-stabilizer A_{1_0} is a $\{2,3\}$-group. It is easy to see that $m_1 m_2 \ge 15$. If $3 \mid |A_{1_0}|$, then by Lemma 3.2, X is of type 0, a contradiction. Thus, A_{1_0} is a 2-group. Hence $R(H)$ is a cyclic Hall $2'$-subgroup of A. By Wielandt theorem, every Hall $2'$-subgroup of A is conjugate to $R(H)$.

Suppose that X is a Cayley graph. Then A has a subgroup G acting regularly on $V(X)$, and so $|G| = 2|H| = 2m_1m_2$. Let $J \le G$ be such that $|J| = m_1m_2$. Then there exists $a \in A$ such that $R(H)^a = J$. It follows that $R(H)^a = J \le G$, and hence $R(H) \le G^{a^{-1}}$. Clearly, $G^{a^{-1}}$ is also regular on $V(X)$. Clearly, $G^{a^{-1}} \cong R(H) \rtimes \mathbb{Z}_2$. By Proposition 2.2, there exists an involution $\delta_{\alpha,x,y} \in G$ for some $\alpha \in \mathrm{Aut}(H)$, x, $y \in H$. By the definition of $\delta_{\alpha,x,y}$, α swaps R and L, and $S^\alpha = y^{-1}xS^{-1}$. Noting that $1_0^{\delta_{\alpha,x,y}} = x_1$, $x_1^{\delta_{\alpha,x,y}} = 1_0$ since $\delta_{\alpha,x,y}$ has order 2. It follows that $(yx^\alpha)_0 = 1_0$, and hence $yx^\alpha = 1$.

As α swaps R and L, we have $r^\alpha = r^t$ or r^{-t}, and hence $r^{\alpha^2} = r^{t^2} = r^{-1}$. Then

$$r_0 = r_0^{\delta^2_{\alpha,x,y}} = (xr^\alpha)_1^{\delta_{\alpha,x,y}} = (yx^\alpha r^{\alpha^2})_0 = (r^{-1})_0,$$

which forces that $r = r^{-1}$, a contradiction. Thus, X is a VNC-graph, as required. □

4. Classification

Throughout this section, we shall let $n > 1$ be an odd integer.

Lemma 4.1 — Let X be a connected tetravalent vertex-transitive graph of order $2n$. If $\mathrm{Aut}(X)$ has a non-trivial normal 2-subgroup, then X is a Cayley graph.

PROOF : Assume that M is a non-trivial normal 2-subgroup of $\mathrm{Aut}(X)$. Since n is odd, each orbit of M has length 2, and so the quotient graph X_M has order n. This implies that X_M has valency 4 or 2. For the former, one may see that M is semiregular on $V(X)$, and so $M \cong \mathbb{Z}_2$. Again, considering $n > 1$, $R(H)M$ must be regular on $V(X)$, and so X is a Cayley graph on $R(H)M$. For the latter, we have $X_M \cong C_n$ and so $\mathrm{Aut}(X_M) \cong D_{2n}$. Let K be the kernel of $\mathrm{Aut}(X)$ acting on $V(X_M)$. Then $\mathrm{Aut}(X)/K$ is a vertex-transitive automorphism group of X_M, and since n is odd, we have either $\mathrm{Aut}(X)/K \cong \mathbb{Z}_n$ or $\mathrm{Aut}(X)/K \cong D_{2n}$. It then follows that $\mathrm{Aut}(X)/K$ is edge-transitive on X_M, and so the subgraphs of X induced by any two adjacent orbits of M are isomorphic. Consequently, the subgraph induced by any two adjacent orbits of M is isomorphic to $K_{2,2}$, and so $X \cong C_n[2K_1]$, which is a Cayley graph, as required. □

Lemma 4.2 — Let X be a connected tetravalent vertex-transitive bi-Cayley graph over an abelian group H of order n. If $\mathrm{Aut}(X)$ has no non-trivial normal 2-subgroups and the vertex stabilizer $\mathrm{Aut}(X)_v$ of $v \in V(X)$ is a 2-group, then X is normal.

PROOF : Let $A = \mathrm{Aut}(X)$. Let P be a Sylow 2-subgroup of A such that $A_v \leq P$. Clearly, $|A| = 2|A_v||H|$. Since n is odd, one has $|P| = 2|A_v|$, and so $|A| = |H||P|$. It follows that $A = R(H)P$. According to a theorem of Kegel and Wielandt (see [9, VI, 4.3]), a product two pair-wise commutative nilpotent groups is solvable. It follows that A is solvable. Then every minimal normal subgroup of A is an elementary abelian p-group for some prime divisor p of n because A has no non-trivial normal 2-subgroups. Let $M = \mathrm{Core}_A(R(H)) = \bigcap_{g\in A} R(H)^g$. The argument above implies that $M \neq 1$. Note that M is the maximum normal subgroup of A contained in $R(H)$. Since $R(H)$ is an abelian group, one has $1 \neq M \leq R(H) \leq C_A(M)$. Suppose that $M < R(H)$. Let N/M be a minimal normal subgroup of A/M contained in $C_A(M)/M$. Again since A is solvable, one has $N/M \cong \mathbb{Z}_2^r$ or $\mathbb{Z}_p^s$ for some integers r, s. For the former, since $N \leq C_A(M)$, one has $N = M \times Q$ with $Q \cong \mathbb{Z}_2^r$. So Q is characteristic in N and so normal in A considering $N \trianglelefteq A$. This is contrary to our assumption that A has no non-trivial normal 2-subgroups. For the latter, we have $M < N \leq R(H)$ and $N \trianglelefteq A$. This is contrary to our assumption that M is a maximum normal subgroup of A. Thus $M = R(H)$, and hence $R(H) \trianglelefteq A$, as desired. □

The following is the main result of this paper.

***Theorem* 4.3** — *Let X be a connected tetravalent vertex-transitive bi-Cayley graph over a cyclic group of odd order n. Then X is a VNC-graph if and only if $X \cong X_{m_1,m_2,t}$.*

PROOF : By Theorem 3.3 we can get the sufficiency. We only need to prove the necessity.

Let $H = \langle a \rangle \cong \mathbb{Z}_n$ and let $X = \text{BiCay}(H, R, L, S)$ be a connected tetravalent vertex-transitive non-Cayley graph. By Proposition 2.1, we will assume that $1 \in S$. Since H is abelian, H has an automorphism that inverses every element of H, and we shall always use α to denote this automorphism.

Since n is odd, the induced subgraphs $X[H_0]$ and $X[H_1]$ are of even valency, and so X is of type 0 or 2. Suppose that X is of type 0. We have $S^\alpha = S^{-1}$, and by Proposition 2.2, X is a Cayley graph, a contradiction. Thus, X is of type 2. If X is edge-transitive, then by [13, Theorem 1.1], we have $X \cong \text{BiCay}(H, \{a, a^{-1}\}, \{a, a^{-1}\}, \{1, a^2\})$. For convenience of the statement, we may let $R = L = \{a, a^{-1}\}$ and $S = \{1, s\}$. Then $R^\alpha = L$, $L^\alpha = R$, $S^\alpha = S^{-1}$, and again by Proposition 2.2, X is also a Cayley graph, a contradiction.

In the remainder of the proof, we will assume that X is of type 2 and is not edge-transitive. Let $A = \text{Aut}(X)$. Then $|A_v| = 2^s 3^t$ for each $v \in V(X)$. If $t > 0$, then by Lemma 3.2, we must have $n = 3, 5$ or 7. If $n = 3$, then by [19], X is a Cayley graph, and if $n = 5$ or 7, then by MAGMA [5], X is a Cayley graph. A contradiction occurs.

Thus, $t = 0$, and hence each vertex-stabilizer A_v is a 2-group. If A has a non-trivial normal 2-subgroup, then by Lemma 4.1, X is a Cayley graph. In the remainder of the proof, assume that A has no non-trivial normal 2-subgroups. Since A_v is a 2-group, from Lemma 4.2 it follows that $R(H) \trianglelefteq A$. Recall that X is of type 2. Then $|R| = |L| = |S| = 2$. Assume that $S = \{1, s\}$ for some $s \in H$. Since X is vertex-transitive, by Proposition 2.2, there exist $\beta \in \text{Aut}(H)$ and $y \in H$ such that $1_0^{\delta_{\beta,1,y}} = 1_1$, $R^\beta = L$, $L^\beta = R$ and $S^\beta = y^{-1}S^{-1}$. The last equality implies that $\{1, s\}^\beta = S^\beta = \{y^{-1}, y^{-1}s^{-1}\}$. It follows that either $y = 1$ and $s^\beta = s^{-1}$, or $y^{-1} = s$ and $s^\beta = s$.

Since X is connected, by Proposition 2.1, we have $\langle R, L, S \rangle = H$. In the remainder of the proof, we will assume that $\langle S \rangle \cong \mathbb{Z}_{m_1}$ and $\langle R \rangle = \langle L \rangle \cong \mathbb{Z}_{m_2}$ for some $m_1, m_2 \in \mathbb{Z}_n$. Let $R = \{r, r^{-1}\}$. Then $\langle L \rangle = \langle R \rangle = \langle r \rangle$, and so $L = \{r^t, r^{-t}\}$ for some $t \in \mathbb{Z}_{m_2}^*$. Recall that $S = \{1, s\}$. Then $H = \langle r, s \rangle = \langle r \rangle \langle s \rangle$. Suppose that $m = (m_1, m_2)$. Then $\langle s^{\frac{m_1}{m}} \rangle = \langle r^{\frac{m_2}{m}} \rangle \cong \mathbb{Z}_m$.

As β swaps R and L, it follows that $r^\beta = r^t$ or r^{-t} and

$$t^2 \equiv \pm 1 \ (\text{mod } m_2). \quad (4)$$

It follows that $(r^{\frac{m_2}{m}})^\beta = (r^{\frac{m_2}{m}})^t$ or $(r^{\frac{m_2}{m}})^{-t}$. Since $s^\beta = s$ or s^{-1}, one has $(s^{\frac{m_1}{m}})^\beta = s^{\frac{m_1}{m}}$ or

$s^{-\frac{m_1}{m}}$. As $\langle s^{\frac{m_1}{m}}\rangle = \langle r^{\frac{m_2}{m}}\rangle \cong \mathbb{Z}_m$, it follows that

$$t \equiv \pm 1 \pmod{m}. \tag{5}$$

Suppose that $t^2 \equiv 1 \pmod{m_2}$. If $m = 1$, then we have $H = \langle r\rangle \times \langle s\rangle$, and hence there exists $\beta_t \in \mathrm{Aut}(H)$ such that

$$r^{\beta_t} = r^t, s^{\beta_t} = s^{-1}.$$

An easy computation shows that $R^{\beta_t} = L, L^{\beta_t} = R$ and $S^{\beta_t} = S^{-1}$. Clearly, β_t has order 2. By Proposition 2.2 (2), X would be a Cayley graph, a contradiction.

Let $m > 1$. Assume that $m_2 = p_1^{\ell_1} p_2^{\ell_2} \dots p_k^{\ell_k}$, where $p_1, p_2, \dots, p_k$ are pairwise distinct primes. Without loss of generality, we may assume that $m = p_1^{\ell'_1} p_2^{\ell'_2} \dots p_j^{\ell'_j}$ with $j \le k$ and $1 \le \ell'_l \le \ell_l (1 \le l \le j)$. Let $m'_2 = p_1^{\ell_1} p_2^{\ell_2} \dots p_j^{\ell_j}$. Then $\langle r\rangle = \langle r^{m'_2}\rangle \times \langle r^{\frac{m_2}{m'_2}}\rangle$, and then $H = \langle r, s\rangle = \langle r^{m'_2}\rangle \times \langle r^{\frac{m_2}{m'_2}}, s\rangle$.

Let $t \equiv 1 \pmod{m}$. Since $t^2 \equiv 1 \pmod{m_2}$, one has $t^2 \equiv 1 \pmod{m'_2}$. If $t \not\equiv 1 \pmod{m'_2}$, then since m_2 is odd, one has $t \not\equiv 1 \pmod{p_i}$ for some $1 \le i \le j$. This is impossible because $p_i \mid m$ and $t \equiv 1 \pmod{m}$. Thus, $t \equiv 1 \pmod{m'_2}$. Then there exist $\beta_t \in \mathrm{Aut}(H)$ such that

$$(r^{m'_2})^{\beta_t} = r^{-tm'_2}, (r^{\frac{m_2}{m'_2}})^{\beta_t} = r^{-\frac{m_2}{m'_2}}, s^{\beta_t} = s^{-1}.$$

An easy computation shows that $R^{\beta_t} = L, L^{\beta_t} = R$ and $S^{\beta_t} = S^{-1}$. Clearly, β_t has order 2. Again by Proposition 2.2 (2), X would be a Cayley graph, a contradiction.

Let $t \equiv -1 \pmod{m}$. With a similar argument as in the above paragraph, we obtain that $t \equiv -1 \pmod{m'_2}$. Then there exist $\beta_t \in \mathrm{Aut}(H)$ such that

$$(r^{m'_2})^{\beta_t} = r^{tm'_2}, (r^{\frac{m_2}{m'_2}})^{\beta_t} = r^{-\frac{m_2}{m'_2}}, s^{\beta_t} = s^{-1}.$$

An easy computation shows that $R^{\beta_t} = L, L^{\beta_t} = R$ and $S^{\beta_t} = S^{-1}$. Clearly, β_t has order 2. Again by Proposition 2.2 (2), X would be a Cayley graph, a contradiction.

Thus, $t^2 \equiv -1 \pmod{m_2}$, and hence $t^2 \equiv -1 \pmod{m}$. On the other hand, by Eq. (5), we have $t^2 \equiv 1 \pmod{m}$. This implies that $m \mid 2$, and so $m = 1$ because m is odd. So $\langle r\rangle \cap \langle s\rangle = 1$, and hence $H = \langle r\rangle \times \langle s\rangle$. Consequently, we have $X \cong X_{m_1,m_2,t}$, completing the proof. □

ACKNOWLEDGEMENT

This work was partially supported by the National Natural Science Foundation of China (11671030, 11271012) and the Fundamental Research Funds for the Central Universities (2015JBM110).

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Appendix I — The Adjacency Matrix of $X_{3,5,2}$

$$
\begin{pmatrix}
0&0&1&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0\\
0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0\\
0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1\\
0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0
\end{pmatrix}
$$