Indian J. Pure Appl. Math., **51**(1): 289-296, March 2020

DOI: 10.1007/s13226-020-0401-0

ON THE STRONG AND THE SEMI-STRONG PATH PARTITION CONJECTURE

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Dedicated to the memory of Professor Claude Berge

(*Received* 3 *September* 2017*, after final revision* 18 *August* 2018;
accepted 28 *January* 2019)

We define two families of graphs and prove the strong and the weak path partition conjectures for subfamilies of these graphs.

Key words : Coloring; strong coloring; stable set; elementary path.

2010 Mathematics Subject Classification : 05Cxx: 3: 68R10

1. Introduction

We follow the notation and terminology of Berge [1]. Let G be a directed graph with vertex set X and arc set U. The number of vertices is denoted by n and the number of arcs (directed edges) by m. Unless stated otherwise, our graphs are directed graphs. A directed *elementary path* will be simply referred to as *path*. A *stable set* of G is a set of vertices of G no two of which are joined by an arc.

A *partial k-coloring* of G is a family of k pairwise disjoint (possibly empty) stable sets. A partial $k-$coloring is sometimes denoted by $(S_1, S_2, \ldots, S_k)$ where S_i is a stable set for $i = 1, 2, \ldots, k$. The vertices belonging to S_i are said to be colored with the color i. The vertices not belonging to $\cup_{i=1}^{k} S_i$ are not colored at all by the partial $k-$coloring $(S_1, S_2, \ldots, S_k)$.

A partial $k-$coloring $(S_1, S_2, \ldots, S_k)$ is *optimal* if $| \cup_{i=1}^{k} S_i| = \alpha_k$, the maximum number of vertices of an induced subgraph of G which can be colored in k colors. In particular, α_1 denoted simply by α, is the *stability number* or the *independence number* of G.

If P is a path, we also denote by P the set of vertices of the path P. The following is the classic theorem of Dilworth [3] concerning the partially ordered set (poset).

***Theorem* 1.1** — *In the graph of a partially ordered set, the minimum number of paths partitioning the vertex set is equal to the cardinality of a maximum stable set.*

The graph of a partially ordered set is transitive. If the condition of transitivity is dropped, we have the following theorem due to Gallai and Milgram [5].

***Theorem* 1.2** — *In a directed graph, the minimum number of paths partitioning the vertex set is less than or equal to the stability number.*

This theorem is the best possible in the sense that, equality is always attained if we *reorient* all the arcs of G incident to a maximum stable set S either towards exterior of S or towards interior of S [1].

This observation gives the following formula:

In any *undirected* graph G we have:

The stability number $\alpha = \max(\min(|\mathcal{P}| \mid \mathcal{P}$ is a path partition of an orientation of $G))$ where $\mathcal{P}$ is a path partition of G and $|\mathcal{P}|$ is the number of paths in the path partition $\mathcal{P}$. The maximum is taken over all *possible* orientations of G.

If the graph is a transitive graph, every path induces a clique and inversely a complete graph contains a hamilton path (see Berge [1]).

The following is the dual of the Dilworth's theorem (see Berge [1]).

***Theorem* 1.3** — *In the graph of a partially ordered set, the minimum number of stable sets partitioning the vertex set is equal to the number of vertices in a longest path.*

If the condition of transitivity is dropped from the graph, we have the following theorem proved independently by Gallai [4] and Roy [11].

***Theorem* 1.4** — *In any graph, the chromatic number γ is equal to or less than the number of vertices in a longest path.*

This theorem is the best possible in the following sense.

We can always find an orientation of undirected graph such that the chromatic number of the graph is equal to the number of vertices in a longest path (see Berge [1]).

In order to unify the theorems of Gallai-Milgram and Gallai-Roy, Berge [2] posed the following two conjectures. To state the two conjectures, we need a few definitions.

Definition 1.4.1 — Let P be a path in a directed graph and k an integer such that $1 \leq k \leq \max_P |P|$ where $|P|$ is the number of vertices of the path P. Consider now a partial $k-$coloring of G. The path P is said to be strongly colored or well colored or the partial $k-$coloring is strong for P, if the number of different colors that the path P meets is exactly $\min\{k, |P|\}$. That is, if $|P| \leq k$, then, all the vertices of the path P are distinctly colored. On the other hand, if, $|P| > k$, then P meets all the k colors.

Definition 1.4.2 — Consider a directed graph and a path partition $\mathcal{P} = (P_1, P_2, \ldots, P_s)$ of the vertex set X. Let k be any integer satisfying $1 \leq k \leq \max_P |P|$. Then consider the sum

$$B_k(\mathcal{P}) = \sum_{i=1}^{s} (k, |P_i|).$$

A path partition $\mathcal{P}'$ is k-*f*iner than $\mathcal{P}$ if $B_k(\mathcal{P}') < B_k(\mathcal{P})$. A path partition $\mathcal{P}''$ is k-*f*inest or k-optimal if there is no other k-finer partition of the vertex set. In other words, the partition $\mathcal{P}''$ minimizes the sum $B_k(\mathcal{P}'')$.

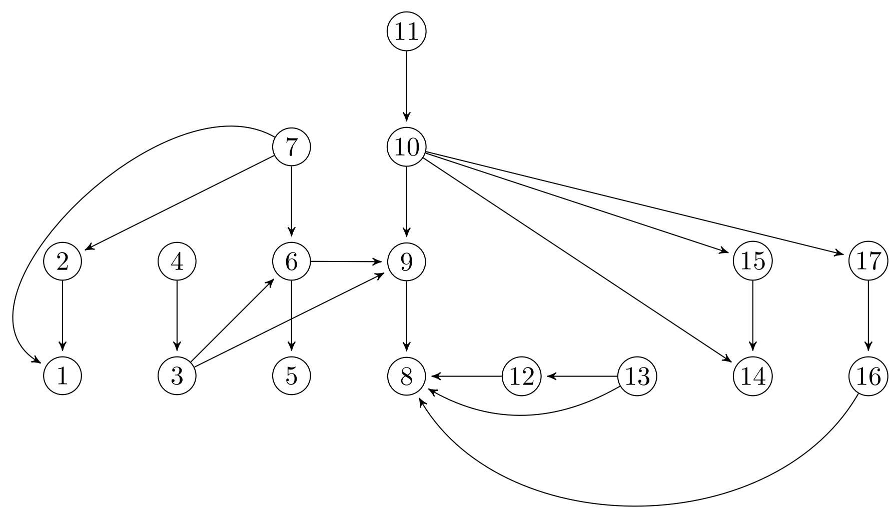

Figure 1: Graph G: Illustration of a 2-optimal path partition

Example 1.4.1 : Let us refer to the graph G of figure 1. Consider the path partition $\mathcal{P} = (P_1, P_2, \ldots, P_8)$ where $P_1 = (2, 1), P_2 = (4, 3), P_3 = (7, 6, 5), P_4 = (11, 10, 9, 8), P_5 = (12), P_6 = (13), P_7 = (15, 14), P_8 = (17, 16)$.

$B_2(\mathcal{P}) = 14$ and $\mathcal{P}$ is not a 2-optimal partition of the vertex set of G as the path partition $\mathcal{P}' = (P'_1, P'_2, \ldots, P'_8)$ where $P'_1 = (7, 2, 1), P'_2 = (3, 6, 9), P'_3 = (13, 12, 8), P'_4 = (10, 15, 14), P'_5 =$

$(17,16), P'_6 = (4), P'_7 = (5), P'_8 = (11)$ satisfies $B_2(\mathcal{P}') = 13$ and hence $\mathcal{P}'$ is 2-*f*iner than the partition $\mathcal{P}$.

The partition $\mathcal{P}'' = (P''_1, P''_2, \ldots, P''_6)$ where $P''_1 = (4,3,6,9), P''_2 = (11,10,17,16), P''_3 = (13,12,8), P''_4 = (7,2,1), P''_5 = (15,14), P''_6 = (5)$ can be seen to be a 2-*o*ptimal partition of the graph G of figure 1 and $B_2(\mathcal{P}'') = 11$.

We are now ready to state the conjectures of Berge [2].

Conjecture 1.4.1 : (Strong Path Partition Conjecture). For every k-optimal path partition of a directed graph, there is a partial k-coloring which colors all paths of the partition strongly.

For $k = 1$, this conjecture is equivalent to a slightly stronger result of the Gallai-Milgram [5] theorem, proved by Linial [9].

If k is equal to the number of vertices of a longest path, then we get a stronger result of the Gallai-Roy Theorem proved by Berge [2].

In fact, the above conjecture can be considered as an extension of the following celebrated generalization of the Dilworth's theorem due to Greene and Kleitman [6].

***Theorem* 1.5** — *Let G be the graph of a partially ordered set and let $\mathcal{P}$ be a k-optimal path partition of the vertex set of G. Then*

$$B_k(\mathcal{P}) = \alpha_k.$$

We now state the weak path partition conjecture.

Conjecture 1.5.1 — (The Weak Path Partition Conjecture). Every graph G admits a path partition $\mathcal{P}$ such that $B_k(\mathcal{P}) \leq \alpha_k$ where α_k is the maximum number of vertices of an induced subgraph which can be colored in k colors.

It can be seen that the Strong Path Partition Conjecture (SPPC) implies the Weak Path Partition Conjecture (WPPC). The SPPC gives a structural approach to the WPPC.

Definition 1.5.1 — (Level Sets Associated with a Path Partition). Consider a path partition $\mathcal{P} = (P_1, P_2, \ldots, P_s)$ of the vertex set X. A vertex $x \in P_i$, is said to be at level k, if the section of the path P_i starting from the vertex x to the terminal vertex of P_i contains exactly k vertices. The set of vertices at level k of all the paths of the partition $\mathcal{P}$ is denoted by $L_i(\mathcal{P})$.

It is not difficult to see that

$$\sum_{i=1}^{s} \min\{k, |P_i|\} = |L_1(\mathcal{P})| + |L_2(\mathcal{P})| + \cdots + |L_k(\mathcal{P})|$$

Set $|P_i| = r_i$ for $i = 1, 2, \ldots, s$. Let r_k^* be the number of integers of the sequence (r_i) which are greater than or equal to the integer $k(\geq 1)$.

The sequence (r_i^*) is called the *d*ual of the sequence (r_i). Let us note that $r_k^* = 0$ if $k > \max_i r_i$ and that $|L_i(\mathcal{P})| = r_i^*$.

The number of paths in the partition $\mathcal{P}$ is denoted by $|\mathcal{P}|$.

2. THE MAIN RESULTS

We have posed the following conjecture in [13].

Conjecture 2.0.1 — (Semi-Strong Path Partition Conjecture (SSPPC)). For any graph, there is a path partition whose paths can be strongly colored with k colors.

The following theorem is proved in [13].

***Theorem* 2.1** — *Let G be a graph with a $(k+1)$-optimal path partition $\mathcal{P} = (P_1, P_2, \ldots, P_s)$. Let this path partition satisfy the following property: The vertices of the paths of the partition $\mathcal{P}$ from the level 1 to the the level k (including the level k) can be colored strongly using k colors. Then the paths of the partition $\mathcal{P}$ can also be colored strongly with $k+1$ colors, that is, (SSPPC) is true for $k+1$. In particular, the WPPC is true for $k+1$.*

For an excellent survey paper on the Berge's path partition conjecture see [7]. The conjecture is proved for $k = 2$ by Berger and Hartman [8].

We now define the following family of graphs $(\mathcal{F}_k)$ satisfying property P_k.

The family $(\mathcal{F}_k)$: The family $(\mathcal{F}_k)$ consists of all graphs G with the following characterizing property P_k : There is a $k-$coloring of G with $k \geq \gamma$ such that all paths in G of length k or less meet different colors.

Example 2.1.1 : Clearly, a bipartite graph G satisfies the the above property P_2.

***Theorem* 2.2** — *Let G be a graph satisfying property P_k. Then (SPPC) is true for kp and $kp+1$ for $p \geq 1$.*

PROOF : Consider a $k-$coloring $(X_1, X_2, \ldots, X_k)$ of G with the property that all paths in G of cardinality k or less meet different colors. We distinguish two cases:

Case 1 : The case kp.

Let $\mathcal{P} = (P_1, P_2, \ldots, P_s)$ be any (not necessarily optimal) path partition of the vertex set X of

G. Consider a matrix $C = (c_{ij})_{k\times p}$ of kp colors where C is a matrix of k rows and p columns. We now process each path $P_1, P_2, \ldots, P_s$ separately. Color the vertices $(x_{j1}, x_{j2}, \ldots, x_{jq_j})$ of the path P_j where $x_{ji} \in L_i(P_j)$ (x_{ji} is in the i-th level of the path P_j) in this order as follows (Note that $|P_j| = q_j$): If $x_{j1} \in X_r$, then assign the color c_{r1} to x_{j1}, if $x_{j2} \in X_p$ then assign the color c_{p1} to x_{j2} and so on. In this procedure, the color assigned to $x_{jt} \in X_l$ is c_{lw} where the colors $c_{l1}, c_{l2}, \ldots, c_{l(w-1)}$ have already all been assigned to some $(w-1)$ vertices of P_j, that is, c_{lw} is the first available color not yet used from the $l-$th row of the matrix C.

Since the graph G satisfies property P_k, one can verify that this procedure colors the different paths of $\mathcal{P}$ strongly. Note that if $q_j > kp$ then the vertices of P_j from the level $kp+1$ onwards will not be colored at all by our assignment of colors and if $q_j \leq kp$ then all the vertices of P_j are assigned distinct colors.

To illustrate the above coloring procedure, let us refer to the graph G of figure 2 whose chromatic number is 3. Set $k = 3$, $p = 2$ and hence $kp = 6$. Take the 3×2 matrix of colors $C = (c_{ij})_{3\times 2}$. A 3-coloring of G is (S_1, S_2, S_3) where $S_1 = \{1, 2, 3\}$, $S_2 = \{4, 5, 6, 11\}$ and $S_3 = \{7, 8, 9, 10\}$. A partition of the vertex set of G into paths is $P_1 = (11, 10, 1, 4, 9, 3, 5)$, $P_2 = (7, 2, 6)$ and $P_3 = (8)$. A strong coloring of the paths P_1, P_2, P_3 of the graph G by the colors of the matrix C is indicated in the diagram of Figure 2.

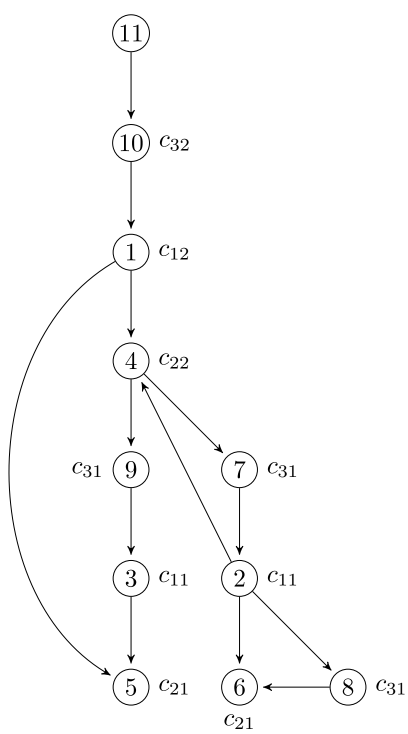

Figure 2: Graph G: Illustration of case 1

Case 2: The case $kp + 1$.

Case 2 uses the the coloring procedure employed in the proof of case 1. Let $\mathcal{P} = (P_1, P_2, \ldots, P_s)$ be a $(kp + 1)$-optimal path partition of the vertex set X of G. First color the vertices of the induced subgraph under the kp-level (including the kp-th level) of the paths of the partition $\mathcal{P}$, that is, color the vertices of the graph induced by

$$H =< \{L_1(\mathcal{P}) \cup L_2(\mathcal{P}) \cup \cdots \cup L_{kp}(\mathcal{P})\} >$$

by using the procedure as in case 1. The path partition $\mathcal{P}$ induces a path partition of H and these induced paths are strongly colored by the procedure in case 1. We have to assign one more color, say, c (since kp colors have already been assigned), so that the coloring becomes a $kp + 1$ strong coloring of the paths of $\mathcal{P}$. Now all the conditions of the Theorem 2.1 are satisfied for the paths of $\mathcal{P}$ with kp playing the role of k. Hence by Theorem 2.1 the paths can also be colored strongly by $kp + 1$ colors. This completes the proof of the theorem. □

Corollary 2.2.1 — (Theorem of Berge [2]) (SPPC) is true for bipartite graphs.

PROOF : Clearly, bipartite graphs satisfy property P_2. Hence, by Theorem 2.2, and Linial's extension of Gallai-Milgram Theorem [9], the corollary follows. □

We now consider the following family of graphs satisfying the following property:

Property Q_k : Consider the family of all directed graphs G such that any two paths of length k (paths of cardinality $k + 1$) do not hit at a vertex.

We prove the Weak Path Partition Conjecture for k for graphs satisfying property Q_k.

***Theorem* 2.3** — *Let G be a graph satisfying property Q_k. Then the (WPPC) is true for k.*

PROOF : Consider an induced subgraph H with a maximum number of vertices which can be colored in k colors. By the definition of H, the chromatic number of the induced subgraph $< H \cup v >$ is $k + 1$ for any vertex v belonging to the graph $G \setminus H$. By the Gallai-Roy theorem, the induced subgraph $< H \cup v >$ contains a path of length k (cardinality $k + 1$).

Let $\{ v_1, v_2, \ldots, v_s \}$ be the set of all vertices belonging to the graph $G \setminus H$. Then each of the induced subgraph $H_i =< H \cup v_i >$ contains a path P_i of length k for $i = 1, 2 \ldots, s$.

Then the path partition

$$\mathcal{P} = (P_1, P_2, \ldots, P_s, P_{s+1}, P_{s+2}, \ldots, P_r)$$

is a path partition of the vertex set of the whole graph G, where $P_{s+1}, P_{s+2}, \ldots, P_r$ are the singleton (trivial) paths of the vertex set of the graph $G \setminus (P_1 \cup P_2 \cup \cdots \cup P_s)$.

It can be easily verified that

$$B_k(\mathcal{P}) \leq \alpha_k.$$

Hence the theorem is proved. □

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