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TRANSIENT ANALYSIS OF AN M/M/1 QUEUEING SYSTEM SUBJECTED TO MULTIPLE DIFFERENTIATED VACATIONS, IMPATIENT CUSTOMERS AND A WAITING SERVER WITH APPLICATION TO IEEE 802.16E POWER SAVING MECHANISM

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An $M/M/1$ queueing system subjected to multiple differentiated vacations, customer impatience and a waiting server is analyzed. The explicit transient probabilities of system size are derived using probability generating function technique, Laplace transform, continued fractions and some properties of confluent hypergeometric function. Further, the time-dependent mean and variance are obtained as the performance measures. A numerical example is presented in order to study the behavior of the system.

Key words : M/M/1 queue; customer impatience; multiple differentiated vacations; transient analysis; Laplace transform; continued fraction.

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1. INTRODUCTION

Information and communication technology (ICT) industry approximately consumes 4% of the global annual energy production [23]. A significant increase in the annual power consumption of ICT is expected in the next few years [10]. Furthermore, mobile networks contribute to a considerable share of CO_2 emissions of ICT industry [23]. This indicates that with the gradual increase of mobile device users, the power consumption of mobile networks will become a significant issue in the future [10]. Therefore, a sleep mode operation was defined by the IEEE 802.16e standard to conserve power and to increase battery life of mobile broadband wireless access (BWA) devices [8].

The IEEE 802.16e power saving mechanism has two types of sleep modes. First mode is named as Power Saving Class of Type I, which is based on binary increasing sleep window size. The second is Power Saving Class of Type II which has constant sleep window size [18]. IEEE 802.16e power saving mechanism considers the battery life of Subscriber Station (SS). This mechanism helps to reduce power consumption of SS while the battery life is sustained. In order to save energy, the sleep mode mechanism of SS is set up into repetitive sleep cycles [8]. The IEEE 802.16e sleep mode technique roughly runs as follows. In active mode (busy period), the SS communicates with Base Station (BS) to send and receive data packets. If there is no data traffic between SS and BS, SS stays in an idle state at some random time duration. When SS completes the idle state, it sends a message (mobile sleep request (MOB-SLP-REQ)) to the BS requesting permission to undergo the sleep mode. It goes to the sleeping mode after receiving the mobile sleep response (MOB-SLP-RES) by the BS [35].

Since queueing theory is a powerful mathematical tool to solve the problems of congestion in the fields of telecommunication systems and computer and communication systems, a number of researchers studied the IEEE 802.16e power saving mechanism by using the queuing theory. Seo *et al*. [24] analyzed the sleep mode operation in IEEE802.16e by modelling the base station as an $M/GI/1/N$ queue with multiple vacations. Recently, Alouf *et al.* [1] studied an $M/G/1$ queueing system (BS) that has repeated inhomogeneous vacations with application to IEEE 802.16e power saving mechanism. In their model, a queueing model was proposed to study many power saving modes proposed in the standards and in the literature. Especially, their model followed the type I and Type II power saving modes described in the IEEE 802.16e standard.

Generally, there are a number of vacation policies in queueing systems; single vacation, multiple vacation, and working vacation. In single vacation, the server goes for a vacation when there are no customers in the system. Whether customers are in the system or not, the system is ready for the service after finishing vacation time duration. In the case of multiple vacation policy, server is allowed to take another vacation even after a vacation when it sees that no customers are in the system further. In working vacation policy, the system reduces its service rate without having a complete vacation. Levy and Yechiali [19] were the first to discuss the queueing systems with server vacations. Tian and Zhang [28], Doshi [11] and Takagi [27] developed a number of models for vacation types. Servi and Finn [25] analyzed working vacations in queueing systems and derived the expressions for an $M/M/1/MV$ model. Further, this model was developed as an $M/G/1/WV$ model by by Wu and Takagi [31]. It was further extended into a $G/M/1/WV$ queueing system by Baba [7]. Arumuganathan and Jeyakumar [6] considered N-policy single server queues and focused on multiple vacations and close down times. The results on transient state of a single server queueing system were derived by Kalidass *et al*. [15] with the possibilities to occur catastrophes. Indra and Renu [14] derived transient solution for a single server queueing model with working vacation policy under two dimensional view by applying Bernoulli schedule. Sudhesh and Raj [26] obtained remarkable results for time dependent expression of a single server queueing system with multiple vacations. Time dependent results for an $M/M/1/N$ queue system with multiple vacations were derived by Yang and Wu [32]. Kalidass and Ramanath [17] derived the transient solution for a single server queue with multiple vacations. The model for N-policy single server queue with multiple exponential vacation was further considered by Vijayashree and Janani [29] to obtain the explicit transient solutions. Vijayashree and Janani [30] further extended the studies conducted by Ibe and Isijola [13] with new type of vacation model (differentiated vacation). Transient solution for the system size probabilities was derived by them using generating function technique and Laplace transform. They obtained the performance measures such as mean value, variance, and probability for a server to be busy or in vacation at a specific time. Under the perspective of differentiated vacation, two vacations types can be seen in queues, which are longer vacations and shorter vacations. These types of vacation can be found in systems such as libraries, banks, and super markets. In these the system takes longer vacations when no customers are in the system. The server allows to take a second type of vacation, if there are no customers after finishing the first type of vacation.

In this research, the BS is modelled as a single server queueing system with repeated differentiated vacations. When the BS receives a message giving permission to go to the sleep mode it takes Type-I sleep mode which is exponentially distributed. if there are no data traffic between BS and SS when it returns from Type-I sleep mode, BS is allowed to take Type-II sleep mode which is exponentially distributed. After finishing the Type-II sleep mode, it returns to the normal state if there are data traffic between BS and SS. Otherwise, BS is allowed to take Type-I sleep mode again. This process was repeated until there is data traffic between BS and SS.

Since the customers' impatience is considered as a the most important part of a queueing system, in long queues, the arrivals normally become impatient when they have to wait for their service. Therefore, waiting customers may leave the system without fulfilling their service requirements, when it is in the idle state even after the arrival of a customer to the system. This could happen due to another scenario such as the long time taken by the system to restart. In real life situations, customers may decide to leave the system when they have to wait more time for service. This affects to the stability and performance measures of the system. The applications of customer impatience can be seen in systems such as communication networks and production-inventory systems. In this model, when BS is in any type of sleep mode, a message waiting for BS may be recalled by the sender. According to queueing theory this situation is referred to as impatient customers in the queue.

In the past few decades, many authors paid attention to understanding the server vacations in queueing systems with customer impatience. Altman and Yechiali [2] studied the behaviour of impatient customers when server is in a vacation for an $M/M/1$, $M/G/1$ and $M/M/c$ queues. Further, Altman and Yechiali [3] studied the infinite capacity queue by considering server vacations and customer impatience. The explicit expression for system size probabilities was derived by making use of probability generating function methods. Perel and Yechiali [22] derived the transient result for multi-server queueing models in a 2-phase (fast and slow) Markovian random environment with impatient customers. Yue *et al.* [34] analyzed the single server queueing systems with customer impatience and working vacation. The authors paid attention to time dependent solution for different jobs in a queueing system and used the probability generating function techniques to obtain desired results. Ammar [4] obtained time-dependent solution for single server queue with customer impatience by adding multiple vacation policy and the mean and variance were derived as the performance measures.

Practically in the real world, the server may not be ready to go for a vacation just as the system is empty of customers. It is a common situation in real life and when we consider the human behaviour as a server, it waits some time duration in the system although no customers are to be served in the queue such as in the counters of libraries, banks, super markets. In our present model, the waiting server corresponds to idle time duration of BS. Boxma *et al.* [9] analyzed $M/G/1$ queue with a waiting server timer and vacations. In extending the above research, Yechiali [33] derived the expressions for $M^x/G/1$ queue with a waiting server and vacations. An $M/M/1$ queue with server vacations and a waiting server was considered by Kalidass and Ramanath [16] and they obtained the time dependent results. Steady state results for the vacation queues with impatient customers and a waiting server were derived by Padmavathy *et al.* [21]. Ammar [5] analyzed the explicit solutions for an $M/M/1$ vacation queue with impatient customers and a waiting server which is allowed to take a vacation when the system is empty after waiting for a sufficient time period.

It was found that no researchers attempted to derive transient solution for an $M/M/1$ queueing system subjected to multiple differentiated vacations with customer impatience and a waiting server. The most important part of this research is that a new concept named Multiple Differentiated Vacation was introduced. In this research, the transient solution of an $M/M/1$ queue subjected to multiple differentiated vacations with impatience customers and a waiting server were derived. Laplace transform, probability generating function technique, continued fractions and properties of confluent hyper-geometric function were used to obtain the transient solution. Accordingly, this can be taken as a further improvement of the model derived by Vijayashree and Janani [30].

This paper is organized in several major sections. Section 2 explains the model. The explicit expressions for the time-dependent system size probabilities are discussed in 3. Section 4 introduces the expressions for time-dependent expected system size and variance. Section 5 presents the numerical results of the system and conclusions are discussed under 6.

2. MODEL DESCRIPTION

A single server queueing model with multiple-differentiated vacations, waiting server and customers' impatience is considered. The assumptions of the system are built up as follows.

1. Arrivals to the system occur in accordance with Poisson process with rate λ and service takes place according to an exponential distribution with rate μ.

2. When the busy period is ended, the server waits a random duration of time before beginning a vacation. It is assumed that waiting time duration follows the exponential distribution with rate η.

3. The server begins a vacation (Type-I) after finishing the waiting time duration and returns back when at least one customer is found to be waiting for service. When the server finds an empty system upon his returns, it takes another vacation of shorter duration (Type-II). And again, the server is allowed to take the first type of vacation, if there are no customers in the system after completing the second type of vacation. If there are zero customers in the waiting queue after finishing any type of vacation the server is allowed to continuously follow this vacation process (this process is named as multiple differentiated vacations). As before, the vacation times of the server are assumed to follow exponential distribution with parameter γ_1 and γ_2 respectively.

4. Customers arriving while the system being in vacation state become impatient. That is, each customer, upon arrival, activates an individual timer, exponentially distributed with parameter ξ_1 and ξ_2 for Type-I and Type-II vacations respectively if the customer's service has not been started before the customer's timer expires, he abandons the system never to return.

5. It is assumed that inter-arrival times, service times, waiting times and vacation times are mutually independent and the service discipline is First-In, First-Out (FIFO).

Let $\{X(t), t \geq 0\}$ denotes the total number of customers in the system at time t and let $J(t)$ represents the state of the system at time t, which is defined as follows.

$$J(t) = \begin{cases} 0, & \text{if the server being in functional state at time } t \\ 1, & \text{if the server being in type-1 vacation at time } t \\ 2, & \text{if the server being in type-2 vacation at time } t. \end{cases}$$

Then $\{J(t), X(t), t \geq 0\}$ is a two-dimensional continuous time Markov process on the state space $S = \{(j,n); j = 0,1,2; n = 0,1,2,...\}$. Let $P_{j,n}(t)$ be the time dependent probabilities for the system to be in the state j with n customers at time t. Let

$$\begin{aligned} P_{0,n}(t) &= Prob\{J(t) = 0, X(t) = n\}, \quad n = 0,1,2,.... \\ P_{1,n}(t) &= Prob\{J(t) = 1, X(t) = n\}, \quad n = 0,1,2,.... \\ P_{2,n}(t) &= Prob\{J(t) = 2, X(t) = n\}, \quad n = 0,1,2,.... \end{aligned}$$

Then, the set of forward Kolmogorov differential difference equations governing the process are given by

$$P_{0,0}^{'}(t) = -(\lambda+\eta)P_{0,0}(t) + \mu P_{0,1}(t) \tag{1}$$

$$\begin{aligned} P_{0,n}^{'}(t) &= \lambda P_{0,n-1}(t) - (\lambda+\mu)P_{0,n}(t) + \mu P_{0,n+1}(t) + \gamma_1 P_{1,n}(t) \\ &\quad + \gamma_2 P_{2,n}(t); 1 \leq n \end{aligned} \tag{2}$$

$$P_{1,0}^{'}(t) = -(\lambda+\gamma_1)P_{1,0}(t) + \eta P_{0,0}(t) + \xi P_{1,1}(t) + \gamma_2 P_{2,0}(t) \tag{3}$$

$$\begin{aligned} P_{1,n}^{'}(t) &= \lambda P_{1,n-1}(t) - (\lambda + n\xi + \gamma_1)P_{1,n}(t) \\ &\quad + (n+1)\xi P_{1,n+1}(t); 1 \leq n \end{aligned} \tag{4}$$

$$P_{2,0}^{'}(t) = -(\lambda+\gamma_2)P_{2,0}(t) + \xi P_{2,1}(t) + \gamma_1 P_{1,0}(t) \tag{5}$$

$$\begin{aligned} P_{2,n}^{'}(t) &= \lambda P_{2,n-1}(t) - (\lambda + n\xi + \gamma_2)P_{2,n}(t) \\ &\quad + (n+1)\xi P_{2,n+1}(t); 1 \leq n. \end{aligned} \tag{6}$$

Initially, it is assumed that there are no customers in the queueing system and the server is in Type-I vacation, i.e., $P_{1,0}(0) = 1$, $P_{2,0}(0) = 0$ and $P_{j,n}(0) = 0$ for $n \geq 1$ and $j = 0,1,2$.

3. Transient Probabilities

In this section, the transient solution of the above described model is derived by using generating functions, Laplace transform, continued fractions and some properties of confluent hypergeometric function.

3.1 *Evaluation of* $P_{0,n}(t)$

Define the generating function as follows, for $|z| \leq 1$

$$P(z,t) = \sum_{n=0}^{\infty} P_{0,n}(t) z^n,$$

with initial condition $P(z,0) = 0$.

Multiplying the equations (1) and (2) by appropriate powers of z and summing over $n \geq 1$, we can obtain

$$\begin{aligned} P(z,t) = {} & \gamma_1 \int_0^t \left(\sum_{m=1}^{\infty} P_{1,m}(u) z^m \right) e^{-\left[\lambda(1-z)+\mu(1-z^{-1})\right](t-u)} \mathrm{d}u \\ & + \gamma_2 \int_0^t \left(\sum_{m=1}^{\infty} P_{2,m}(u) z^m \right) e^{-\left[\lambda(1-z)+\mu(1-z^{-1})\right](t-u)} \mathrm{d}u \\ & + \left[(\mu - \eta) - \frac{\mu}{z} \right] \int_0^t P_{0,0}(u) e^{-\left[\lambda(1-z)+\mu(1-z^{-1})\right](t-u)} \mathrm{d}u. \end{aligned} \tag{7}$$

It is well known that if $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$, then

$$\exp\left[\left(\lambda z + \frac{\mu}{z} \right) t \right] = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t)$$

where $I_n(\cdot)$ is the modified Bessel function of the first kind.

Comparing the coefficients of z^n in the equation (7) for $n = 1, 2, 3, ...$ leads to

$$\begin{aligned} P_{0,n}(t) = {} & \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{1,m}(u) \beta^{n-m} I_{n-m}(\alpha(t-u)) e^{-(\lambda+\mu)(t-u)} \mathrm{d}u \\ & + \gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{2,m}(u) \beta^{n-m} I_{n-m}(\alpha(t-u)) e^{-(\lambda+\mu)(t-u)} \mathrm{d}u \\ & + (\mu - \eta) \int_0^t P_{0,0}(u) \beta^n I_n(\alpha(t-u)) e^{-(\lambda+\mu)(t-u)} \mathrm{d}u \\ & - \mu \int_0^t P_{0,0}(u) \beta^{n+1} I_{n+1}(\alpha(t-u)) e^{-(\lambda+\mu)(t-u)} \mathrm{d}u. \end{aligned} \tag{8}$$

Using the fact that $I_{-n}(\cdot) = I_n(\cdot)$ and comparing the coefficients of z^{-n} in the equation (7) yields

$$\begin{aligned} 0 &= \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{1,m}(u)\beta^{n-m} I_{n+m}\left(\alpha(t-u)\right) e^{-(\lambda+\mu)(t-u)} \mathrm{d}u \\ &+ \gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{2,m}(u)\beta^{n-m} I_{n+m}\left(\alpha(t-u)\right) e^{-(\lambda+\mu)(t-u)} \mathrm{d}u \\ &+ (\mu-\eta) \int_0^t P_{0,0}(u)\beta^{n} I_{n}\left(\alpha(t-u)\right) e^{[-(\lambda+\mu)(t-u)]} \mathrm{d}u \\ &- \mu \int_0^t P_{0,0}(u)\beta^{n+1} I_{n-1}\left(\alpha(t-u)\right) e^{[-(\lambda+\mu)(t-u)]} \mathrm{d}u. \end{aligned} \tag{9}$$

Subtracting the equation (8) from the equation (9) for $n = 1, 2, 3, ...$, we have

$$\begin{aligned} P_{0,n}(t) &= \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{1,m}(u)\beta^{n-m} \left[I_{n-m}\left(\alpha(t-u)\right)\right. \\ &\left. - I_{n+m}\left(\alpha(t-u)\right)\right] e^{-(\lambda+\mu)(t-u)} \mathrm{d}u \\ &+ \gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{2,m}(u)\beta^{n-m} \left[I_{n-m}\left(\alpha(t-u)\right)\right. \\ &\left. - I_{n+m}\left(\alpha(t-u)\right)\right] e^{-(\lambda+\mu)(t-u)} \mathrm{d}u \\ &+ 2\mu n \beta^{n+1} \int_0^t P_{0,0}(u) \frac{I_n\left(\alpha(t-u)\right)}{\alpha(t-u)} e^{-(\lambda+\mu)(t-u)} \mathrm{d}u. \end{aligned} \tag{10}$$

It is clear that $P_{0,n}(t)$ are expressed in terms of $P_{1,n}(t)$, $P_{2,n}(t)$ and $P_{0,0}(t)$.

3.2 *Evaluation of* $P_{1,n}(t)$ *and* $P_{2,n}(t)$

Let $\hat{P}_{j,n}(s)$ be the Laplace transform of $P_{j,n}(t)$; $n = 1, 2, 3.....$ On taking Laplace transform of equations (4) and (6), we have

$$\begin{aligned} s\hat{P}_{1,n}(s) - P_{1,n}(0) &= \lambda \hat{P}_{1,n-1}(s) - (\lambda + n\xi + \gamma_1)\hat{P}_{1,n}(s) \\ &+ (n+1)\xi \hat{P}_{1,n+1}(s) \end{aligned} \tag{11}$$

$$\begin{aligned} s\hat{P}_{2,n}(s) - P_{2,n}(0) &= \lambda \hat{P}_{2,n-1}(s) - (\lambda + n\xi + \gamma_2)\hat{P}_{2,n}(s) \\ &+ (n+1)\xi \hat{P}_{2,n+1}(s). \end{aligned} \tag{12}$$

Applying boundary conditions to the equation (11), we have the expression

$$\frac{\hat{P}_{1,n}(s)}{\hat{P}_{1,n-1}(s)} = \frac{\lambda}{(s + \lambda + n\xi + \gamma_1) - (n+1)\,\xi \frac{\hat{P}_{1,n+1}(s)}{\hat{P}_{1,n}(s)}}.$$

This equation can be rearranged using the identity (27) of confluent hyper-geometric function.

Then it becomes the following form

$$\frac{\hat{P}_{1,n}(s)}{\hat{P}_{1,n-1}(s)} = \frac{\lambda}{\xi\left(\frac{s+\gamma_1}{\xi}+n\right)}\frac{{}_1F_1\left(n+1;\frac{s+\gamma_1}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{{}_1F_1\left(n;\frac{s+\gamma_1}{\xi}+n;-\frac{\lambda}{\xi}\right)}.$$

For $n = 1, 2, 3, ...$ invoking the above equation, we will have:

$$\hat{P}_{1,n}(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}\left(\frac{s+\gamma_1}{\xi}+i\right)}\frac{{}_1F_1\left(n+1;\frac{s+\gamma_1}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{{}_1F_1\left(1;\frac{s+\gamma_1}{\xi}+1;-\frac{\lambda}{\xi}\right)}\hat{P}_{1,0}(s)$$

let

$$\hat{\phi}_n(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}\left(\frac{s+\gamma_1}{\xi}+i\right)}\frac{{}_1F_1\left(n+1;\frac{s+\gamma_1}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{{}_1F_1\left(1;\frac{s+\gamma_1}{\xi}+1;-\frac{\lambda}{\xi}\right)}.$$

Then

$$P_{1,n}(t) = \phi_n(t) * P_{1,0}(t) \tag{13}$$

where $\phi_n(t)$ is the inverse Laplace transform of $\hat{\phi}_n(s)$ and '*' denotes convolution.

Applying boundary conditions to the equation (12) and by using the same procedure which has been used to evaluate $P_{1,n}(t)$, we can find $P_{2,n}(t)$ as follows

$$P_{2,n}(t) = \psi_n(t) * P_{2,0}(t) \tag{14}$$

where $\psi_n(t)$ is the inverse Laplace transform of $\hat{\psi}_n(s)$ and '*' denotes convolution. Where

$$\hat{\psi}_n(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}\left(\frac{s+\gamma_2}{\xi}+i\right)}\frac{{}_1F_1\left(n+1;\frac{s+\gamma_2}{\xi}+n+1;-\frac{\lambda}{\xi}\right)}{{}_1F_1\left(1;\frac{s+\gamma_2}{\xi}+1;-\frac{\lambda}{\xi}\right)}.$$

Taking the Laplace transform of the (5), we will have

$$s\hat{P}_{2,0}(s) - P_{2,0}(0) = -(\lambda+\gamma_2)\hat{P}_{2,0}(s) + \gamma_1\hat{P}_{1,0}(s) + \xi\hat{P}_{2,1}(s).$$

Substituting $n = 1$ to the equation (14) and after some algebra, we have

$$\hat{P}_{2,0}(s) = \gamma_1 C(s)\hat{P}_{1,0}(s) \tag{15}$$

where

$$C(s) = \sum_{r=0}^{\infty}\xi^r\frac{\psi_1^r(s)}{(s+\lambda+\gamma_2)^{r+1}}.$$

Taking the inverse Laplace transform of the equation (15), we can get

$$P_{2,0}(t) = \gamma_1 C(t) * P_{1,0}(t)$$

where

$$C(t) = \sum_{r=0}^{\infty} \xi^r \psi_1^r(t) * e^{-(\lambda+\gamma_2)t}\frac{t^r}{r!}.$$

It is clear that $P_{2,0}(t)$ is expressed in terms of $P_{1,0}(t)$.

Taking the Laplace transform of the (3), we will have

$$s\hat{P}_{1,0}(s) - P_{1,0}(0) = -(\lambda+\gamma_1)\hat{P}_{1,0}(s) + \eta\hat{P}_{0,0}(s) + \gamma_2\hat{P}_{2,0}(s) + \xi\hat{P}_{1,1}(s).$$

Substituting $n = 1$ to the equations (13) and (14) and after some algebra, we have

$$\hat{P}_{1,0}(s) = \left(1 + \eta\hat{P}_{0,0}(s)\right) G(s) \tag{16}$$

where

$$G(s) = \sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{m=0}^{n}(-1)^m\binom{r}{n}\binom{n}{m}\xi^{r+m-n}\gamma_1^n\gamma_2^n\hat{\phi}_1^{r-n}(s) \times\hat{\psi}_1^m(s)\frac{1}{(s+\lambda+\gamma_1)^{r+1}}\frac{1}{(s+\lambda+\gamma_{12})^{r+m}}.$$

Taking the inverse Laplace transform of the equation (16), we can get

$$P_{1,0}(t) = \left(\delta(t) + \eta\hat{P}_{0,0}(s)\right) * G(t) \tag{17}$$

where $\delta(t)$ is the Derac-Delta function and

$$G(t) = \sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{m=0}^{n}(-1)^m\binom{r}{n}\binom{n}{m}\xi^{r+m-n}\gamma_1^n\gamma_2^n\phi_1^{r-n}(t) *\psi_1^m(t) * e^{-(\lambda+\gamma_1)t}\frac{t^r}{r!} * e^{-(\lambda+\gamma_2)t}\frac{t^{r+m-1}}{(r+m-1)!}.$$

It is clear that $P_{1,0}(t)$ is expressed in terms of $P_{0,0}(t)$. Where "*" denotes convolution.

3.3 *Evaluation of* $P_{0,0}(t)$

Using the fact that $I_{-n}(\cdot) = I_n(\cdot)$ and comparing the coefficients of z^{-1} in the equation (7), we will have

$$\begin{aligned} 0 &= \gamma_1 \int_0^t \sum_{m=1}^{\infty} P_{1,m}(u)\beta^{-(m+1)} I_{m+1}\left(\alpha(t-u)\right) e^{-(\lambda+\mu)(t-u)} du \\ &\quad + \gamma_2 \int_0^t \sum_{m=1}^{\infty} P_{2,m}(u)\beta^{-(m+1)} I_{m+1}\left(\alpha(t-u)\right) e^{-(\lambda+\mu)(t-u)} du \\ &\quad + \frac{(\mu-\eta)}{\beta} \int_0^t I_1\left(\alpha(t-u)\right) e^{-(\lambda+\mu)(t-u)} P_{0,0}(u) du \\ &\quad - \mu \int_0^t I_0\left(\alpha(t-u)\right) e^{-(\lambda+\mu)(t-u)} P_{0,0}(u) du. \end{aligned}$$

Taking the inversion of the above equation and after some algebra, we will have

$$\begin{aligned} \hat{P}_{0,0}(s) &= \frac{1}{2\lambda\mu} \sum_{r=0}^{\infty} \left(\frac{\mu-\eta}{2\lambda\mu}\right)^r \left(p - \sqrt{p^2-\alpha^2}\right)^{r+1} \left\{ \sum_{m=1}^{\infty} \left[\gamma_1 \hat{P}_{1,m}(s) \right.\right. \\ &\quad \left.\left. + \gamma_2 \hat{P}_{2,m}(s)\right] \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^m \right\} \end{aligned}$$

where $p = s + \lambda + \mu$. Again, substituting the values for $\hat{P}_{1,n}(s)$ and $\hat{P}_{2,n}(s)$ and after some mathematical calculations, we can derive

$$\hat{P}_{0,0}(s) = \sum_{k=0}^{\infty} \eta^k \left[G(s)\right]^{K+1} \left[F(s)\right]^{K+1} \tag{18}$$

where

$$\begin{aligned} F(s) &= \frac{\gamma_1}{2\lambda\mu} \sum_{r=0}^{\infty} \left(\frac{\mu-\eta}{2\lambda\mu}\right)^r \left(p - \sqrt{p^2-\alpha^2}\right)^{r+1} \left\{ \sum_{m=1}^{\infty} \left[\hat{\phi}_m(s) \right.\right. \\ &\quad \left.\left. + \gamma_2 \hat{\psi}_m(s) C(s)\right] \left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^m \right\}. \end{aligned}$$

Taking the inversion of the equation (18), we have

$$P_{0,0}(t) = \sum_{k=0}^{\infty} \eta^k \left[G(t)\right]^{*(k+1)} * \left[F(t)\right]^{*(k+1)} \tag{19}$$

where

$$\begin{aligned} F(t) &= \frac{\alpha\gamma_1}{2} \sum_{r=0}^{\infty} \left[\left(\frac{\mu-\eta}{2\lambda\mu}\right)\alpha\right]^r \left[I_r(\alpha t) - I_{r+2}(\alpha t)\right] e^{-(\lambda+\mu)t} \\ &\quad * \left\{ \sum_{m=1}^{\infty} \beta^{-m} \left[\phi_m(t) + \gamma_2 \psi_m(t) * C(t)\right] \right. \\ &\quad \left. * \left[I_{m-1}(\alpha t) - I_{m+1}(\alpha t)\right] e^{-(\lambda+\mu)t} \right\} \end{aligned}$$

where "*" denotes convolution while "*$(k+1)$" represents the $(k+1)$-fold convolution.

All the time-dependent probabilities are explicitly obtained in terms of modified Bessel function of the first kind by making use of Laplace transform and probability generating function techniques along with continued fractions and some properties of the confluent hypergeometric function.

3.4 *Expression for $\phi_n(t)$ and $\psi_n(t)$*

We know that

$$\hat{\phi}_n(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{i=1}^{n}\left(\frac{s+\gamma_1}{\xi}+i\right)} \frac{{}_1F_1\left(n+1; \frac{s+\gamma_1}{\xi}+n+1; -\frac{\lambda}{\xi}\right)}{{}_1F_1\left(1; \frac{s+\gamma_1}{\xi}+1; -\frac{\lambda}{\xi}\right)}. \tag{20}$$

Using the definition of confluent hypergeometric function, we can derive

$$\frac{{}_1F_1\left(n+1; \frac{s+\gamma_1}{\xi}+n+1; -\frac{\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\gamma_1}{\xi}+i\right)} = \xi^n \sum_{k=o}^{\infty} \frac{\binom{n+k}{k}}{\prod_{i=1}^{n+k}(s+\gamma_1+i\xi)}(-\lambda)^k.$$

Taking the partial fractions, we will have

$$\frac{{}_1F_1\left(n+1; \frac{s+\gamma_1}{\xi}+n+1; -\frac{\lambda}{\xi}\right)}{\prod_{i=1}^{n}\left(\frac{s+\gamma_1}{\xi}+i\right)} = \xi^n \sum_{k=0}^{\infty} \binom{n+k}{k} \frac{(-\lambda)^k}{\xi^{n+k-1}} \sum_{i=1}^{n+k} \frac{(-1)^{i-1}}{(i-1)!(n+k-i)!} \frac{1}{(s+\gamma_1+i\xi)} \tag{21}$$

when $n=0$,

$${}_1F_1\left(1; \frac{s+\gamma_1}{\xi}+1; -\frac{\lambda}{\xi}\right) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\prod_{i=1}^{k}(s+\gamma_1+i\xi)} = \sum_{k=0}^{\infty}(-\lambda)^k a_k(s)$$

where

$$\begin{aligned} a_k(s) &= \frac{1}{\prod_{i=1}^{k}(s+\gamma_1+i\xi)} \\ &= \frac{1}{\xi^{k-1}} \sum_{r=1}^{k} \frac{(-1)^{r-1}}{(r-1)!(k-r)!} \frac{1}{(s+\gamma_1+r\xi)}, k=1,2,3... \end{aligned}$$

and $a_0(s)=1$.

Using the identity given in [12]

$$\left[{}_1F_1\left(1; \frac{s+\gamma_1}{\xi}+1; -\frac{\lambda}{\xi}\right)\right]^{-1} = \sum_{k=0}^{\infty} b_k(s)\lambda^k \tag{22}$$

where $b_0(s) = 1$ and for $k = 1, 2, 3, ...$

$$b_k(s) = \begin{vmatrix} a_1(s) & 1 & & ... & & \\ a_2(s) & a_1(s) & 1 & ... & & \\ a_3(s) & a_3(s) & a_1(s) & ... & & \\ ... & ... & ... & ... & & \\ a_{k-1}(s) & a_{k-2}(s) & a_{k-1}(s) & ... & a_1(s) & 1 \\ a_k(s) & a_{k-1}(s) & a_{k-2}(s) & ... & a_2 & a_1(s). \end{vmatrix}$$

$$b_k(s) = \sum_{i=1}^{k}(-1)^{i-1}a_i(s)b_{k-i}(s).$$

By substituting the equations (21) and (22) to the equation (20), we can obtain

$$\hat{\phi}_n(s) = \lambda^n \sum_{j=0}^{\infty}(-\lambda)^j \binom{n+j}{j} a_{n+j}(s) \sum_{k=1}^{\infty} b_k(s)\lambda^k.$$

Taking inverse Laplace transform of the above equation, we have

$$\phi_n(t) = \lambda^n \sum_{j=0}^{\infty}(-\lambda)^j \binom{n+j}{j} a_{n+j}(t) * \sum_{k=1}^{\infty} b_k(t)\lambda^k$$

where

$$a_k(t) = \frac{1}{\xi^{k-1}} \sum_{r=1}^{k} \frac{(-1)^{r-1}}{(r-1)!(k-r)!} e^{-(\gamma_1 + r\xi)t}, k = 1, 2, 3, ...$$

$$b_k(t) = \sum_{i=1}^{k}(-1)^{i-1}a_i(t) * b_{k-i}(t), k = 2, 3, 4, ...; b_1(t) = a_1(t)$$

By using the same procedure which has been used to evaluate $\phi_n(t)$, we can find $\psi_n(t)$ as follows,

$$\psi_n(t) = \lambda^n \sum_{i=0}^{\infty}(-\lambda)^i \binom{n+i}{i} d_{n+i}(t) * \sum_{m=1}^{\infty} g_m(t)\lambda^m$$

where

$$d_k(t) = \frac{1}{\xi^{k-1}} \sum_{r=1}^{k} \frac{(-1)^{r-1}}{(r-1)!(k-r)!} e^{-(\gamma_2 + r\xi)t}, k = 1, 2, 3, ...$$

$$g_k(t) = \sum_{i=1}^{k}(-1)^{i-1}d_i(t) * g_{k-i}(t), k = 2, 3, 4, ...; g_1(t) = d_1(t)$$

4. Time Dependent Mean and Variance

In this section, time-dependent expected system size and variance are derived.

4.1 *Mean*

Let $X(t)$ denotes the number of jobs in the system at time t. The average number of jobs in the system at time t is given by

$$m(t) = E(X(t)) = \sum_{n=1}^{\infty} n\left(P_{0,n}(t) + P_{1,n}(t) + P_{2,n}(t)\right)$$
$$m(0) = \sum_{n=1}^{\infty} n\left(P_{0,n}(0) + P_{1,n}(0) + P_{2,0}(t)\right) = 0$$
$$m^{'}(t) = \sum_{n=1}^{\infty} n\left(P^{'}_{0,n}(t) + P^{'}_{1,n}(t) + P^{'}_{2,n}(t)\right).$$

By equations (2), (4) and (6) and after some algebra, we have the following equation

$$m^{'}(t) = \lambda - \mu\sum_{n=1}^{\infty} P_{0,n}(t) - \xi\left[\sum_{n=1}^{\infty} n\left(P_{1,n}(t) + P_{2,n}(t)\right)\right].$$

By using the initial condition $m(0) = 0$ and integrating it by t, the solution of the above equation can be obtained as follows;

$$\begin{aligned} m(t) &= \lambda t - \mu\sum_{n=1}^{\infty}\int_0^t P_{0,n}(u)du \\ &\quad -\xi\sum_{n=1}^{\infty} n\left[\int_0^t P_{1,n}(u)du + \int_0^t P_{2,n}(u)du\right] \end{aligned} \tag{23}$$

where $P_{0,n}(t)$, $P_{1,n}(t)$ and $P_{2,n}(t)$ are given by the equations (10), (13) and (14) respectively.

4.2 *Variance*

Let $X(t)$ denotes the number of jobs in the system at time t. The variance of jobs in the system at time t is given by

$$\begin{aligned} Var(X(t)) &= E(X^2(t)) - [E(X(t))]^2, \\ Var(X(t)) &= k(t) - [m(t)]^2 \end{aligned} \tag{24}$$

where

$$k(t) = E(X^2(t)) = \sum_{n=1}^{\infty} n^2\left(P_{0,n}(t) + P_{1,n}(t) + P_{2,n}(t)\right).$$

Also,

$$k(0) = \sum_{n=1}^{\infty} n^2 \left(P_{0,n}(0) + P_{1,n}(0) + P_{2,n}(0)\right) = 0$$

and

$$k^{'}(t) \;=\; \sum_{n=1}^{\infty} n^2 \left(P^{'}_{0,n}(t) + P^{'}_{1,n}(t) + P^{'}_{2,n}(t)\right).$$

By equations (2), (4) and (6) and after some algebra, we have the following equation

$$\begin{aligned} k^{'}(t) \;&=\; 2\lambda m(t) + \lambda - \mu \sum_{n=1}^{\infty}(2n-1)P_{0,n}(t) \\ &\quad -\xi \sum_{n=1}^{\infty} n(2n-1)\left[P_{1,n}(t) + P_{2,n}(t)\right]. \end{aligned}$$

By using the initial condition $k(0) = 0$ and integrating it by t, the solution of the above equation can be obtained as follows;

$$\begin{aligned} k(t) \;&=\; 2\lambda \int_0^t m(u)du + \lambda t - \mu \sum_{n=1}^{\infty}(2n-1)\int_0^t P_{0,n}(u)du \\ &\quad -\xi \sum_{n=1}^{\infty} n(2n-1)\left[\int_0^t P_{1,n}(u)du + \int_0^t P_{2,n}(u)du\right]. \end{aligned}$$

Substituting above equation into the equation (25), we will have

$$\begin{aligned} Var(X(t)) \;&=\; 2\lambda \int_0^t m(u)du + \lambda t - \mu \sum_{n=1}^{\infty}(2n-1)\int_0^t P_{0,n}(u)du \\ &\quad -\xi \sum_{n=1}^{\infty} n(2n-1)\left[\int_0^t P_{1,n}(u)du + \int_0^t P_{2,n}(u)du\right] \\ &\quad - [m(t)]^2 \end{aligned} \tag{25}$$

where $P_{0,n}(t)$, $P_{1,n}(t)$, $P_{2,n}(t)$ and $m(t)$ are given by the equations (10), (13), (14) and (23) respectively.

5. Numerical Illustrations

In this section, the numerical example is presented to show the behaviour of the transient probabilities of the system, when the server being in busy period and vacation states against time. The variations of the time-dependent mean and variance against time t are presented. Even though, this queueing system is infinite, it is restricted to 20 considering the purpose of numerical solutions.

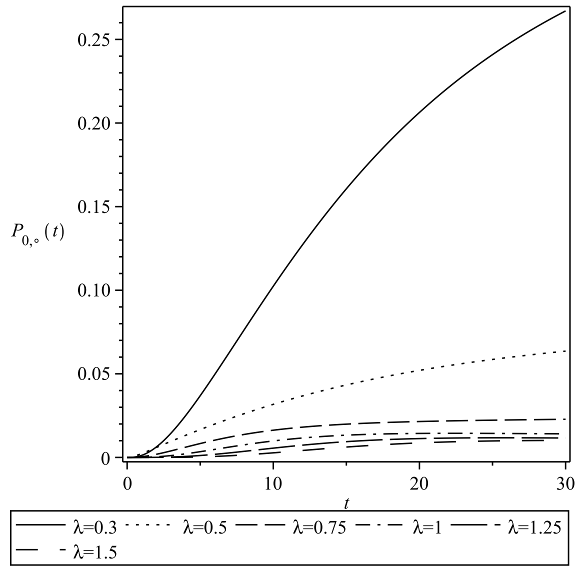

Figure 1: Behaviour of the $P_{0,n}(t)$ against t for varying values of n

Figure 1 is plotted to show the behaviour of $P_{0,n}(t)$ against time t for varying values of n with parameters $\mu = 1.5$, $\gamma_1 = 0.03$, $\gamma_2 = 0.05$, $\xi_1 = 0.02$, $\xi_2 = 0.01$ and $\lambda = 0.3$. It can be seen that all the probabilities for $P_{0,n}(t)$ start at 0 and finally, settle at steady-state.

Figure 2 also is used to plot the graph of $P_{1,n}(t)$ against time t for varying values of n with same parameter values. All the values for $P_{1,n}(t)$ start at 0 and reach to the steady-state except $P_{1,0}(t)$. Since, we assumed that there are no customers at initial state and system is in vacation Type-I, $P_{1,0}(t)$ starts at 1 and it settles at the steady state when time progresses..

Figure 3 shows the behaviour of $P_{2,n}(t)$ against time t for varying values of n with same parameter values. The values of $P_{2,n}(t)$ are equal to 0 at the initial state. They become the steady-state when time progresses.

Figure 4 shows the possibility of having zero customers in the queue when server is in the functional state against time t for $\mu = 1.5$, $\gamma_1 = 0.03$, $\gamma_2 = 0.05$, $\xi_1 = 0.02$, $\xi_2 = 0.01$ and varying values of λ (0.3, 0.5, 0.75, 1, 1.25, 1.5). It can be seen that the possibility of having zero customers in the system increases with the increase of the arrival rate.

Figures 5 illustrates the behaviour of $P_{1,0}(t)$ against time t for same parameter values and varying values of λ. It is observed that the possibility of having zero customers in the system when server is in Type-I vacation decreases when arrival rate increases.

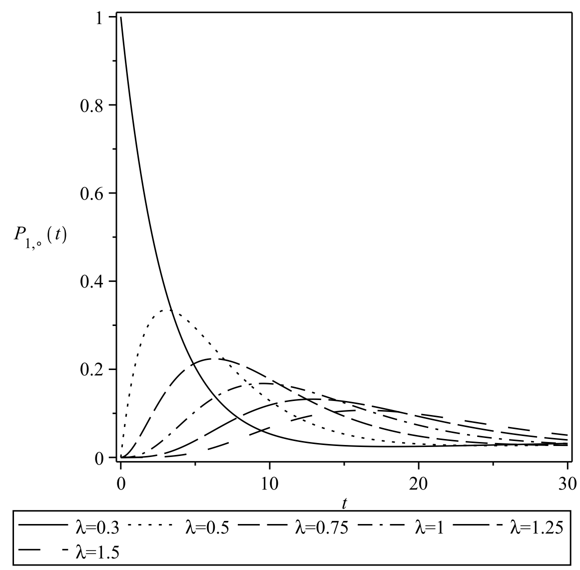

Figure 2: Behaviour of the $P_{1,n}(t)$ against t for varying values of n

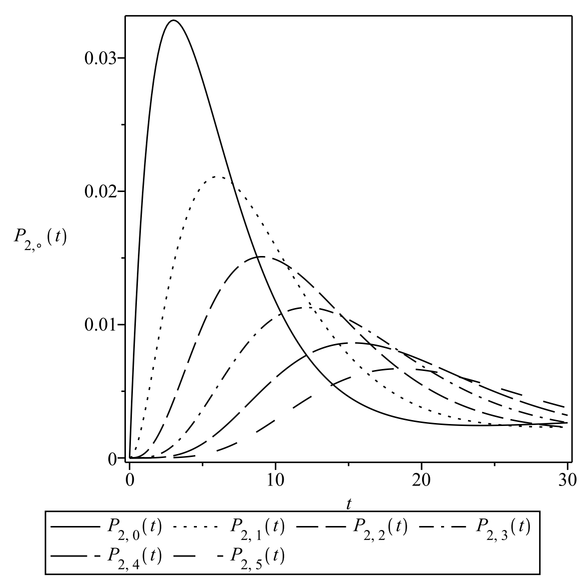

Figure 3: Behaviour of the $P_{2,n}(t)$ against t for varying values of n

Figure 6 also depicts the behaviour of $P_{1,0}(t)$ against time t for same parameter values and varying values of λ. $P_{2,0}(t)$ also behave same as $P_{1,0}(t)$ when arrival rate increases.

Figure 7 shows the variation of the mean number of the customers in the system against time t

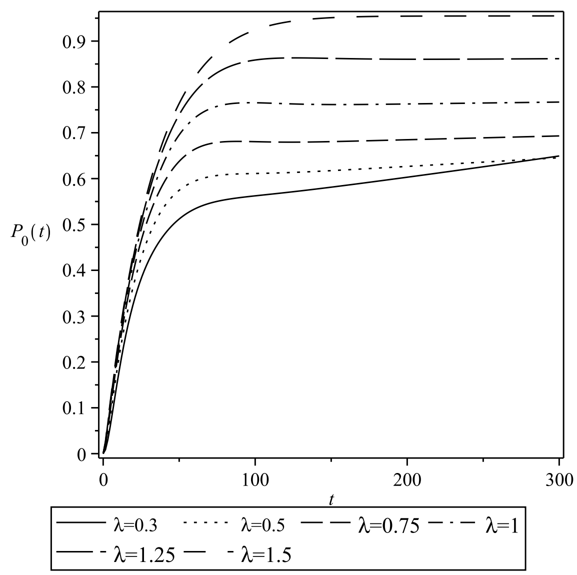

Figure 4: Behaviour of $P_{0,0}(t)$ against t for varying values of λ

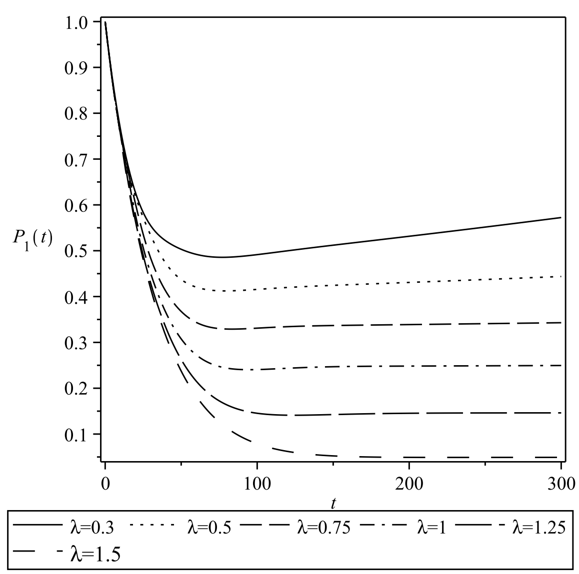

Figure 5: Behaviour of $P_{1,0}(t)$ against t for varying values of λ

for all values of λ. The mean number of the customers in the system rapidly increases when time progresses and finally, it arrives at steady-state. Figure 8 also depicts the behaviour of the variance of the system size against time t where when λ increases, the variance of the number of customers in the system also increases.

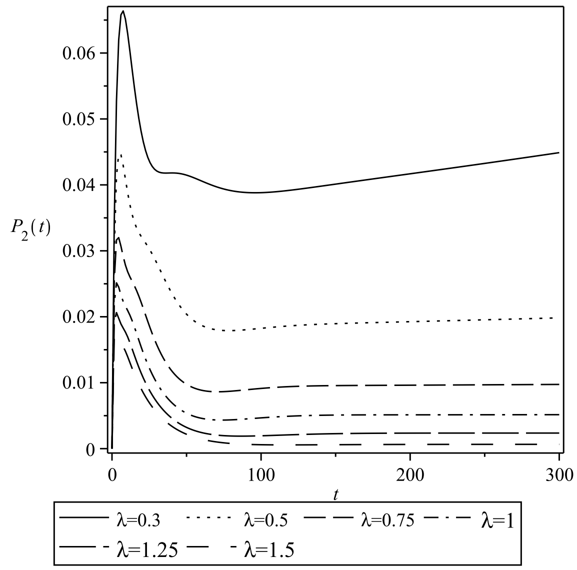

Figure 6: Behaviour of $P_{2,0}(t)$ against t for varying values of λ

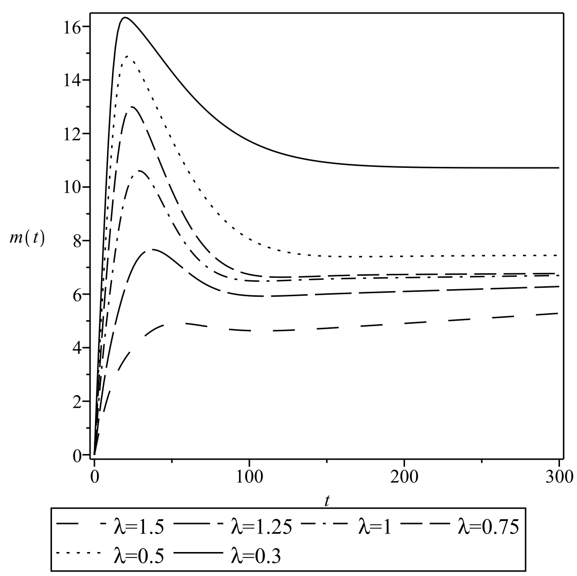

Figure 7: Behaviour of the mean against t for varying values of λ

6. Conclusions

A single server queueing system subjected to multiple differentiated vacations with customers' impatience and waiting server is studied in transient regime. The time-dependent expressions for system

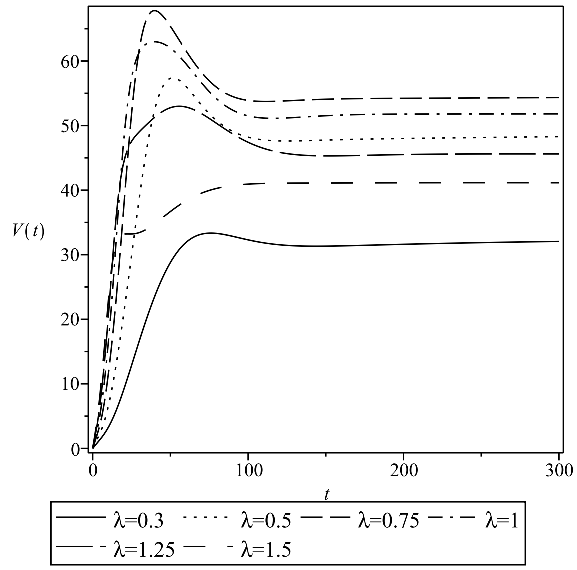

Figure 8: Behaviour of the variance against t for varying values of λ

size probabilities are derived in terms of the modified Bessel function of first kind. Additionally, mean and variance of the system size are obtained as the performance measures. Finally, their numerical result is illustrated. It can be seen that when time progresses the time-dependent probabilities converge to the corresponding steady-state probabilities.

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A CONFLUENT HYPERGEOMETRIC FUNCTION

In this section, the definition of confluent hypergeometric function and some properties of this function is expressed.

The confluent hypergeometric function is denoted by ${}_1F_1(a;c;z)$ and is defined by the power series

$$
\begin{aligned}
{}_1F_1(a;c;z) &= 1 + \frac{a}{c}\frac{z}{1\,!} + \frac{a}{c}\frac{(a+1)}{(c+1)}\frac{z^2}{2\,!} + ... \\
&= \sum_{k=0}^{\infty} \frac{(a)_{(k)}}{(c)_{(k)}}\frac{z^k}{k\,!} \qquad (26)
\end{aligned}
$$

provided that ${}_1F_1(a;c;z)$ does not exist when c is a negative integer. Here $(\alpha)_{(k)}$ is the rising factorial function (the Pochhammer symbol) which is defined as

$$
\begin{aligned}
(\alpha)_{(0)} &= 1 \\
(\alpha)_{(n)} &= \alpha(\alpha+1)(\alpha+2)........(\alpha+n-1)
\end{aligned}
$$

the Pochhammer symbol has following characteristic

$$
\frac{(\alpha)_{(k)}}{k!} = \binom{\alpha+k}{k}
$$

It is observed that

$$
{}_1F_1(0;c;z) = 1
$$

The recurrence relation for the confluent hypergeometric function is given by

$$
c(c-1){}_1F_1(a-1;c-1;z) - az{}_1F_1(a+1;c+1;z) = c(c-1-z){}_1F_1(a;c;z)
$$

The quotient of two hypergeometric functions may be expressed as continued fractions. The following identity has been developed by Lorentzen and Waadeland [20].

$$
\frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)} = \frac{c}{c-z}\frac{(a+1)z}{+c-z+1}\frac{(a+2)z}{+c-z+2}.......,
$$

which can be rewritten as

$$
c\frac{{}_1F_1(a;c;z)}{{}_1F_1(a+1;c+1;z)} - (c-z) = \frac{(a+1)z}{c-z+1}\frac{(a+2)z}{+c-z+2}....... \qquad (27)
$$

and

$$
\sum_{k=0}^{\infty} \frac{(a)_{(k)}}{(c)_{(k)}}\frac{y^k}{k!} \times_1 F_1(a+k;c+k;x) = {}_1F_1(a;c;x+y) \qquad (28)
$$

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