Indian J. Pure Appl. Math., **51**(1): 345-360, March 2020

DOI: 10.1007/s13226-020-0404-x

n-DIMENSIONAL LAPLACE TRANSFORMS OF OCCUPATION TIMES FOR PRE-EXIT DIFFUSION PROCESSES[1]

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(*Received* 26 *March* 2018, *accepted* 7 *February* 2019)

In this paper, we adopt a Poisson approach to find Laplace transforms of joint occupation times over n disjoint intervals for pre-exit diffusion processes. Then we generalize previous result for the 2-dimensional case and the 3-dimensional case.

Key words : Diffusion processes; occupation time; Laplace transform; exit time.

2010 Mathematics Subject Classification : 60J60, 60G17

1. Introduction

The occupation time is the amount of time a stochastic process stays within a certain range. During the past several years there have been a series of papers concerning occupation time related problems for Levy processes. These problems arise from both theoretical interests and the applications in risk theory and finance, see Cai *et al.* [3], Landriault and Shi [12], Guerin *et al.* [6] and Jin *et al.* [7] etc.

[1]This work is partially supported by the National Natural Science Foundation of China (Nos. 71701068, 11701175, 11671132), the Natural Science Foundation of Hunan Province, China (No. 2018JJ3360), the Scientific Research Fund of Hunan Provincial Education Department, China (Nos. 19B343, 19A342, 17C1001, 17K057), and the Applied Economics of Hunan Province.

A spectrally negative Levy process (SNLP for short) is a levy process with no upward jumps that finds many applications in risk theory, mathematical finance and branching process. We can see Bertoin [1] and Kyprianou [8] for nice introductions of SNLPs. For a general SNLP, Laplace transform of occupation times (of the negative half-line) are computed in Landriault *et al*. [11]. They used fluctuation identities for SNLPs and, as a consequence, the results are expressed in terms of the so-called scale functions of the SNLP and its Laplace exponent. And then, the SNLP in Loeffen *et al*. [14] is approximated by a SNLP with sample paths of bounded variation whose Laplace transform for the occupation time can be found directly. They identified Laplace transforms of occupation times of intervals until first passage times for SNLPs. New analytical identities for scale function are derived and therefore the results are explicitly stated in terms of the scale functions of the process. Kyprianou *et al*. [9] used a method similar to Loeffen *et al*. [14] and excursion theory to consider a more general process, namely a refracted spectrally negative Levy process. They extended the results of Landriault *et al*. [11] and bore relevance to Parisian-type financial instruments and insurance scenarios. Applications of such model can be found in Perez and Yamazaki [20] and Renaud [21] etc.

In order to get around the approximation arguments in the aforementioned work, Li and Zhou [15] first adopted a Poisson approach to find joint Laplace transform for occupation times over two disjoint intervals for general SNLPs. This approach uses a property of Poisson random measure and can be effectively implemented. With this method, Kuang and Zhou [10] futher found joint Laplace transforms of occupation times over n-disjoint subintervals resulting from a partition of a infinite interval for SNLPs. Equivalently, they found Laplace transforms for weighted occupation times with step weight functions. And they have also recovered the main results of Loeffen *et al*. [14] and Li and Zhou [15]. For more details on occupation times of SNLPs, the reader is referred to Li *et al*. [17] and Guerin *et al*. [6].

Not only for spectrally negative levy processes, occupation times of other stochastic processes were also studied, just like diffusion processes. Some results to occupation times for general diffusion processes can be found in Pitman and Yor [18, 19]. Cai *et al*. [3] provided Laplace transform-based analytical solutions to pricing problems of various occupation-time-related derivatives under double exponential jumps diffusion model. Wu and Zhou [22] considered the hyper-exponential jump diffusion process, and used the strong Markov property to derives analytical formulas for the Laplace transform of the joint distribution of a hyper-exponential jump diffusion process and its occupation times. Zhou *et al*. [23] extended their model to the Levy-driven Ornstein-Uhlenbeck processes with two-sided exponential jumps.

Li and Zhou [13] adopted the perturbation approach of Landriault *et al*. [11] to find the joint

Laplace transforms of occupation times for time-homogeneous diffusion processes. The results were expressed in terms of solutions to the differential equation associated to the diffusion generator. Li *et al.* [16] used the same way to find expressions of double Laplace transforms for diffusion processes. More recently, the Poisson approach also works well for diffusion processes. We can see Chen *et al.* [4, 5]. The direct approach of Li and Zhou [15] allows us to handle more complex quantities involving Laplace transforms of occupation times. It thus has some advantages over the previous approaches.

In this paper, for a diffusion process, we adopt the Poisson approach of Li and Zhou [15] to find joint Laplace transforms of occupation times over n-disjoint subintervals resulting from a partition of a finite interval. The expressions are in terms of solutions to the associated differential equations. To our best knowledge, such results have not been known before. In addition, our results can also be applied to find more explicit Laplace transforms of the occupation times.

The rest of the paper is arranged as follows. In Section 2, we review the basic facts we need for the time-homogeneous diffusion processes. In Section 3, the desires results on n-dimensional Laplace transforms of occupation times for diffusion processes are obtained. In Section 4, we find explicit expressions on the 2-dimensional Laplace transforms which has the same results as Chen *et al.* [4] and 3-dimensional Laplace transforms of diffusion processes.

2. PRELIMINARIES

We now introduce the one-dimensional diffusion process X considered in this paper. For $-\infty \leq l_1 < l_2 \leq +\infty$, write I for the interval with end points l_1 and l_2. The I-values regular time-homogeneous diffusion process $X = \{X_t, t \geq 0\}$, defined on a filtered probability space $\{\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\}$, is specified by the following stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{2.1}$$

where $X_0 = x_0$ is the initial value and $\{W_t, t \geq 0\}$ is a one-dimensional standard Brownian motion. The law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$, and we write $\mathbb{P}$ and $\mathbb{E}$ when $X = 0$. Throughout the paper we assume that equation (2.1) allows a unique weak solution, which is guaranteed if there exists a constant $K > 0$ such that, for all $x, y \in I$,

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K|x - y|, \qquad \mu^2(x) + \sigma^2(x) \leq K^2(1 + x^2). \tag{2.2}$$

Two basic characteristics of the diffusion process X, the speed measure m and the scale function s, are given by

$$m(dx) = m(x)dx := 2e^{B(x)}/\sigma^2(x)dx, \qquad s(x) := \int^x e^{-B(y)}dy,$$

when $l_1 < x < l_2$. Respectively, where

$$B(x) := \int^x 2\mu(y)/\sigma^2(y)dy.$$

Let $p(\cdot;\cdot,\cdot)$ be the transition density of X with respect to the speed measure for diffusion processes,we have

$$\mathbb{P}_x\{X_t \in dy\} = p(t;x,y)m(dy).$$

For $\lambda > 0$, let $g_{-,\lambda}(\cdot)$ and $g_{+,\lambda}(\cdot)$ be two independent positive solutions to the (generalized) differential equation associated to the generator of X

$$\frac{1}{2}\sigma^2(x)g''(x) + \mu(x)g'(x) = \lambda g(x), \tag{2.3}$$

with $g_{-,\lambda}(\cdot)$ decreasing and $g_{+,\lambda}(\cdot)$ increasing, respectively. For some examples of diffusion processes, equation (2.3) yields explicit expressions for $g_{-,\lambda}(\cdot)$ and $g_{+,\lambda}(\cdot)$, see Borodin and Salminen [2]. Here, a solution $g(x)$ to (2.3) satisfies

$$\lambda \int_{[a,b)} g(x)m(dx) = g^-(b) - g^-(a), \tag{2.4}$$

where

$$g^-(x) := \lim_{h\to 0^+} \frac{g(x) - g(x-h)}{s(x) - s(x-h)}.$$

The green function for X is

$$G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t;x,y)dt.$$

Then

$$G_\lambda(x,y) = \begin{cases} \omega_\lambda^{-1} g_{+,\lambda}(x)g_{-,\lambda}(y), & x \le y, \\ \omega_\lambda^{-1} g_{+,\lambda}(y)g_{-,\lambda}(x), & x \ge y, \end{cases}$$

where

$$\omega_\lambda := g^+_{+,\lambda}(x)g_{-,\lambda}(x) - g^+_{-,\lambda}(x)g_{+,\lambda}(x) = g^-_{+,\lambda}(x)g_{-,\lambda}(x) - g^-_{-,\lambda}(x)g_{+,\lambda}(x),$$

is the so-called $Wronskian$ with

$$g^+(x) := \lim_{h\to 0^+} \frac{g(x+h) - g(x)}{s(x+h) - s(x)}.$$

It is known that ω_λ is independent of x.

We refer to Chapter II of Borodin and Salminen [2] for the above facts and more details about diffusion processes.

Furthermore, for $\lambda > 0$, define

$$f_\lambda(y,z) := g_{-,\lambda}(y)g_{+,\lambda}(z) - g_{-,\lambda}(z)g_{+,\lambda}(y),$$

we have the following well-known solutions to the exit problems.

Let
$$\tau_x := \inf\{t \geq 0 : X_t = x\},$$
be the first passage time of X at level x with the convention $\inf\emptyset = \infty$. For any $a < x < b$ and $\lambda \geq 0$,

$$\mathbb{E}_x[e^{-\lambda\tau_a}; \tau_a < \tau_b] = \frac{f_\lambda(x,b)}{f_\lambda(a,b)},$$

and

$$\mathbb{E}_x[e^{-\lambda\tau_b}; \tau_b < \tau_a] = \frac{f_\lambda(a,x)}{f_\lambda(a,b)},$$

see e.g., Borodin and Salminen [2] for more discussion.

The potential measure for diffusion processes is needed for our main result in Section 3. Throughout the paper, we always assume $a, b \in (l_1, l_2)$. For $a < x, y < b$ and $\lambda \geq 0$, we have its expression

$$\int_0^\infty \mathbb{P}_x\{X_t \in dy, t < \tau_a \wedge \tau_b\}e^{-\lambda t}dt = [G_\lambda(x,y) - \frac{f_\lambda(x,b)}{f_\lambda(a,b)}G_\lambda(a,y) - \frac{f_\lambda(a,x)}{f_\lambda(a,b)}G_\lambda(b,y)]m(dy),$$

which is mentioned in Chen *et al*. [4].

3. Main Results

In this section, we proceed to find the n-dimensional laplace transform of occupation times for diffusion processes. In the following sections, we always denote $\lambda_i, i = 0, 1, 2, 3, 4...n-1$ for n nonnegative constants.

***Theorem* 3.1** — *Let* $a_0 < a_1 < ... < a_{n-1} < a_n$, *for* $a_i < x < a_{i+1}, 0 \leq i < n$, *we have*

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{n-1}\lambda_j\int_0^{\tau_{a_0}\wedge\tau_{a_n}} 1_{(a_j,a_{j+1})}(X_s)ds\}, \tau_{a_0} < \tau_{a_n}]$$
$$= \frac{f_{\lambda_i}(x,a_{i+1})}{f_{\lambda_i}(a_i,a_{i+1})}f_-(a_i) + \frac{f_{\lambda_i}(a_i,x)}{f_{\lambda_i}(a_i,a_{i+1})}f_-(a_{i+1}),$$

where $\{f_-(a_i), 0 < i < n\}$ *is the solution of the following equation set:*

$$\begin{cases} A^-_{11}f_-(a_1) + A^-_{12}f_-(a_2) + ... + A^-_{1n-1}f_-(a_{n-1}) = B^-_1, \\ A^-_{21}f_-(a_1) + A^-_{22}f_-(a_2) + ... + A^-_{2n-1}f_-(a_{n-1}) = B^-_2, \\ ... \\ A^-_{n-11}f_-(a_1) + A^-_{n-12}f_-(a_2) + ... + A^-_{n-1n-1}f_-(a_{n-1}) = B^-_{n-1}, \end{cases}$$

where

$$A_{ij}^- = \begin{cases} 1-(\lambda-\lambda_{j-1})\int_{a_{j-1}}^{a_j} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_{j-1}}(a_{j-1},x)}{f_{\lambda_{j-1}}(a_{j-1},a_j)}m(dx) \\ -(\lambda-\lambda_j)\int_{a_j}^{a_{j+1}} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_j}(x,a_{j+1})}{f_{\lambda_j}(a_j,a_{j+1})}m(dx), & i=j, \\ \\ -(\lambda-\lambda_{j-1})\int_{a_{j-1}}^{a_j} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_{j-1}}(a_{j-1},x)}{f_{\lambda_{j-1}}(a_{j-1},a_j)}m(dx) \\ -(\lambda-\lambda_j)\int_{a_j}^{a_{j+1}} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_j}(x,a_{j+1})}{f_{\lambda_j}(a_j,a_{j+1})}m(dx),, & i\neq j, \end{cases}$$

$$B_i^- = \frac{f_\lambda(a_i,a_n)}{f_\lambda(a_0,a_n)} + (\lambda-\lambda_0)\int_{a_0}^{a_1} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx),$$

and

$$I_\lambda(a_0,a_n,a_i,x) := G_\lambda(a_i,x) - \frac{f_\lambda(a_i,a_n)}{f_\lambda(a_0,a_n)}G_\lambda(a_0,x) - \frac{f_\lambda(a_0,a_i)}{f_\lambda(a_0,a_n)}G_\lambda(a_n,x),$$

$$\lambda = \sum_{k=0}^{n-1}\lambda_k = \lambda_0+\lambda_1+...+\lambda_{n-1},$$

particularly, we have $f_-(a_0)=1$ *and* $f_-(a_n)=0$*.*

PROOF : Define

$$f_-(x) := \begin{cases} \mathbb{E}_x[\exp\{-\sum_{j=0}^{n-1}\lambda_j\int_0^{\tau_{a_0}\wedge\tau_{a_n}} 1_{(a_j,a_{j+1})}(X_s)ds\},\tau_{a_0}<\tau_{a_n}], & a_0<x<a_n, \\ 1, & x\le a_0. \end{cases}$$

It is obvious that $f_-(a_0)=1$ and $f_-(a_n)=0$. Write $0<T_1^k<T_2^k<...$ for the arrival times of independent Poisson processes with rates $\lambda_k,\ k=0,1,...,n-1$. We also assume that these Poisson processes are independent of process X. By a property of Poisson process, we observe that $f_-(x)=\mathbb{P}_x\{D_-\}$ for event

$$D_- := \bigcup_{k=0}^{n-1}\{\{T_i^k\}\bigcap\{s<\tau_{a_0}<\tau_{a_n}: a_k<X_s<a_{k+1}\}\}=\emptyset.$$

Then for independent exponential random variables T_k with rate $\lambda_k,\ k=0,1,...,n-1$, respectively, we have

$$
\begin{aligned}
f_-(a_i) &= \mathbb{P}_{a_i}\{\tau_{a_0}<\tau_{a_n}\wedge T_0\wedge T_1\wedge ...\wedge T_{n-1},D_-\}+\mathbb{P}_{a_i}\{T_0<\tau_{a_0}\wedge\tau_{a_n}\wedge T_1\wedge T_2\wedge ...\wedge\\
&\quad T_{n-1},D_-\}+\sum_{i=1}^{n-2}[\mathbb{P}_{a_i}\{T_i<\tau_{a_0}\wedge\tau_{a_n}\wedge T_0\wedge T_1\wedge ..\wedge T_{i-1}\wedge T_{i+1}\wedge ..\wedge T_{n-1},D_-\}]\\
&\quad +\mathbb{P}_{a_i}\{T_{n-1}<\tau_{a_0}\wedge\tau_{a_n}\wedge T_0\wedge T_1\wedge ...\wedge T_{n-2},D_-\}\\
&= \sum_{i=1}^{n-2}[(\int_{a_0}^{a_n}-\int_{a_i}^{a_{i+1}})\mathbb{P}_{a_i}\{T_i<\tau_{a_0}\wedge\tau_{a_n}\wedge T_0\wedge T_1\wedge ..\wedge T_{i-1}\wedge T_{i+1}\wedge ..\wedge T_{n-1},X(t)\\
&\quad \in dx\}f_-(x)]+\int_{a_0}^{a_{n-1}}\mathbb{P}_{a_i}\{T_{n-1}<\tau_{a_0}\wedge\tau_{a_n}\wedge T_0\wedge T_1\wedge ...\wedge T_{n-2},X(t)\in dx\}f_-(x)\\
&\quad +\int_{a_1}^{a_n}\mathbb{P}_{a_i}\{T_0<\tau_{a_0}\wedge\tau_{a_n}\wedge T_1\wedge T_2\wedge ...\wedge T_{n-1},X(t)\in dx\}f_-(x)\\
&\quad +\mathbb{P}_{a_i}\{\tau_{a_0}<\tau_{a_n}\wedge T_0\wedge T_1\wedge ...\wedge T_{n-1}\}\\
&= \sum_{i=0}^{n-1}[\lambda_i(\int_{a_0}^{a_n}-\int_{a_i}^{a_{i+1}})\int_0^\infty\mathbb{P}_{a_i}\{t<\tau_{a_0}\wedge\tau_{a_n},X(t)\in dx\}e^{-\lambda t}dtf_-(x)\\
&\quad +\mathbb{E}_{a_i}[e^{-\lambda\tau_{a_i}};\tau_{a_0}<\tau_{a_n}]\\
&= \sum_{i=0}^{n-1}\lambda_i(\int_{a_0}^{a_n}-\int_{a_i}^{a_{i+1}})[G_\lambda(a_i,x)-\frac{f_\lambda(a_i,a_n)}{f_\lambda(a_0,a_n)}G_\lambda(a_0,x)\\
&\quad -\frac{f_\lambda(a_0,x)}{f_\lambda(a_0,a_n)}G_\lambda(a_n,x)]m(dx)f_-(x)+\frac{f_\lambda(a_i,a_n)}{f_\lambda(a_0,a_n)}, \qquad (3.1)
\end{aligned}
$$

for $a_i<x<a_{i+1}$,

$$
\begin{aligned}
f_-(x) &= \mathbb{E}_x[e^{-\lambda_i\tau_{a_i}};\tau_{a_i}<\tau_{a_{i+1}}]f_-(a_i)+\mathbb{E}_x[e^{-\lambda_i\tau_{a_{i+1}}};\tau_{a_{i+1}}<\tau_{a_i}]f_-(a_{i+1})\\
&= \frac{f_\lambda(x,a_{i+1})}{f_\lambda(a_i,a_{i+1})}f_-(a_i)+\frac{f_\lambda(a_i,x)}{f_\lambda(a_i,a_{i+1})}f_-(a_{i+1}). \qquad (3.2)
\end{aligned}
$$

Combining (3.1) and (3.2), after some algebras,we can obtain

$$
\begin{cases}
A^-_{11}f_-(a_1)+A^-_{12}f_-(a_2)+...+A^-_{1n-1}f_-(a_{n-1})=B^-_1,\\
A^-_{21}f_-(a_1)+A^-_{22}f_-(a_2)+...+A^-_{2n-1}f_-(a_{n-1})=B^-_2,\\
...\\
A^-_{n-11}f_-(a_1)+A^-_{n-12}f_-(a_2)+...+A^-_{n-1n-1}f_-(a_{n-1})=B^-_{n-1},
\end{cases}
$$

with $A^-_{ij}, i = 1,2,..,n-1, j = 1,2,...,n-1$ and $B^-_i, i = 1,2,...,n-1$ given by

$$A^-_{ij} = \begin{cases} 1-(\lambda-\lambda_{j-1})\int_{a_{j-1}}^{a_j} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_{j-1}}(a_{j-1},x)}{f_{\lambda_{j-1}}(a_{j-1},a_j)}m(dx) \\ -(\lambda-\lambda_j)\int_{a_j}^{a_{j+1}} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_j}(x,a_{j+1})}{f_{\lambda_j}(a_j,a_{j+1})}m(dx), & i=j, \\ \\ -(\lambda-\lambda_{j-1})\int_{a_{j-1}}^{a_j} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_{j-1}}(a_{j-1},x)}{f_{\lambda_{j-1}}(a_{j-1},a_j)}m(dx) \\ -(\lambda-\lambda_j)\int_{a_j}^{a_{j+1}} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_j}(x,a_{j+1})}{f_{\lambda_j}(a_j,a_{j+1})}m(dx), & i\neq j, \end{cases} \tag{3.3}$$

$$B^-_i = \frac{f_\lambda(a_i,a_n)}{f_\lambda(a_0,a_n)} + (\lambda-\lambda_0)\int_{a_0}^{a_1} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx). \tag{3.4}$$

These are the results of Theorem 3.1.

Throughout the paper, we denote $I_\lambda(a_0,a_n,a_i,x)$ as in Theorem 3.1.

***Theorem* 3.2** — *Let $a_0 < a_1 < ... < a_{n-1} < a_n$, for $a_i < x < a_{i+1}, 0 \le i < n$, we have*

$$\begin{aligned} &\mathbb{E}_x[\exp\{-\sum_{j=0}^{n-1}\lambda_j\int_0^{\tau_{a_0}\wedge\tau_{a_n}} 1_{(a_j,a_{j+1})}(X_s)ds\}, \tau_{a_n} < \tau_{a_0}] \\ &= \frac{f_{\lambda_i}(x,a_{i+1})}{f_{\lambda_i}(a_i,a_{i+1})}f_+(a_i) + \frac{f_{\lambda_i}(a_i,x)}{f_{\lambda_i}(a_i,a_{i+1})}f_+(a_{i+1}), \end{aligned}$$

where $\{f_+(a_i), 0 < i < n\}$ is the solution of the following equation set:

$$\begin{cases} A^+_{11}f_-(a_1) + A^+_{12}f_-(a_2) + ... + A^+_{1n-1}f_-(a_{n-1}) = B^+_1, \\ A^+_{21}f_-(a_1) + A^+_{22}f_-(a_2) + ... + A^+_{2n-1}f_-(a_{n-1}) = B^+_2, \\ ... \\ A^+_{n-11}f_-(a_1) + A^+_{n-12}f_-(a_2) + ... + A^+_{n-1n-1}f_-(a_{n-1}) = B^+_{n-1}, \end{cases}$$

where

$$A^+_{ij} = \begin{cases} 1-(\lambda-\lambda_{j-1})\int_{a_{j-1}}^{a_j} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_{j-1}}(a_{j-1},x)}{f_{\lambda_{j-1}}(a_{j-1},a_j)}m(dx) \\ -(\lambda-\lambda_j)\int_{a_j}^{a_{j+1}} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_j}(x,a_{j+1})}{f_{\lambda_j}(a_j,a_{j+1})}m(dx), & i=j, \\ \\ -(\lambda-\lambda_{j-1})\int_{a_{j-1}}^{a_j} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_{j-1}}(a_{j-1},x)}{f_{\lambda_{j-1}}(a_{j-1},a_j)}m(dx) \\ -(\lambda-\lambda_j)\int_{a_j}^{a_{j+1}} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_j}(x,a_{j+1})}{f_{\lambda_j}(a_j,a_{j+1})}m(dx), & i\neq j, \end{cases}$$

$$B^+_i = \frac{f_\lambda(a_i,a_n)}{f_\lambda(a_0,a_n)} + (\lambda-\lambda_0)\int_{a_0}^{a_1} I_\lambda(a_0,a_n,a_i,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx),$$

particularly, $f_+(a_0) = 0$ and $f_+(a_n) = 1$.

PROOF : Define

$$f_+(x) := \begin{cases} \mathbb{E}_x[\exp\{-\sum_{j=0}^{n-1}\lambda_j \int_0^{\tau_{a_0}\wedge\tau_{a_n}} 1_{(a_j,a_{j+1})}(X_s)ds\}, \tau_{a_n} < \tau_{a_0}], & a_0 < x < a_n, \\ 1, & x \le a_0. \end{cases}$$

Observed that $f_+(x) = \mathbb{P}_x\{D_+\}$ for event

$$D_+ := \bigcup_{k=0}^{n-1}\{\{T_i^k\}\bigcap\{s < \tau_{a_n} < \tau_{a_0} : a_k < X_s < a_{k+1}\}\} = \emptyset.$$

Then for independent exponential random variables T_k with rate $\lambda_k,\ k = 0, 1, ..., n-1$, respectively. Following similar arguments in the proof of Theorem 3.1, we have the results of Theorem 3.2.

Combing Theorem 3.1 and Theorem 3.2, we have the following results.

***Theorem* 3.3 —** *Let $a_0 < a_1 < ... < a_{n-1} < a_n$, for $a_i < x < a_{i+1}, 0 \le i < n$, we have*

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{n-1}\lambda_j \int_0^{\tau_{a_0}\wedge\tau_{a_n}} 1_{(a_j,a_{j+1})}(X_s)ds\}]$$

$$= \frac{f_{\lambda_i}(x, a_{i+1})}{f_{\lambda_i}(a_i, a_{i+1})}[f_-(a_i) + f_+(a_i)] + \frac{f_{\lambda_i}(a_i, x)}{f_{\lambda_i}(a_i, a_{i+1})}[f_-(a_{i+1}) + f_+(a_{i+1})],$$

where $f_-(a_i)$ and $f_+(a_i), 0 < i < n$ have been defined in Theorem 3.1 and Theorem 3.2, respectively.

4. EXAMPLES

In this section, we consider two special cases. One case is that n=2, and the other case is that n=3.

Example 4.1 : Let $X = \{X_t, t \ge 0\}$ be a time-homogeneous diffusion process, and $a_0 < a_1 < a_2$, we consider

$$\mathbb{E}_x[\exp\{-\lambda_0 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_0,a_1)}(X_s)ds - \lambda_1 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_1,a_2)}(X_s)ds\}, \tau_{a_0} < \tau_{a_2}], \tag{4.1.1}$$

$$\mathbb{E}_x[\exp\{-\lambda_0 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_0,a_1)}(X_s)ds - \lambda_1 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_1,a_2)}(X_s)ds\}, \tau_{a_2} < \tau_{a_0}], \tag{4.1.2}$$

$$\mathbb{E}_x[\exp\{-\lambda_0 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_0,a_1)}(X_s)ds - \lambda_1 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_1,a_2)}(X_s)ds\}]. \tag{4.1.3}$$

For (4.1.1), by Theorem 3.1, when $a_0 < x < a_1$, we have

$$\mathbb{E}_x[\exp\{-\lambda_0 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_0,a_1)}(X_s)ds - \lambda_1 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_1,a_2)}(X_s)ds\}, \tau_{a_0} < \tau_{a_2}] = \frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)} f_-(a_0) + \frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)} f_-(a_1),$$

when $a_1 < x < a_2$, we have

$$\mathbb{E}_x[\exp\{-\lambda_0 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_0,a_1)}(X_s)ds - \lambda_1 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_1,a_2)}(X_s)ds\}, \tau_{a_0} < \tau_{a_2}] = \frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)} f_-(a_1) + \frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)} f_-(a_2),$$

where $f_-(a_0) = 1, f_-(a_2) = 0$, and $f_-(a_1)$ satisfy

$$A_1^- f_-(a_1) = B_1^-.$$

By (3.3) and (3.4), set $\lambda = \lambda_0 + \lambda_1$, we have

$$A_1^- = 1 - \lambda_1 \int_{a_0}^{a_1} I_\lambda(a_0,a_2,a_1,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)} m(dx) - \lambda_0 \int_{a_1}^{a_2} I_\lambda(a_0,a_2,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)} m(dx),$$

$$B_1^- = \frac{f_\lambda(a_1,a_2)}{f_\lambda(a_0,a_2)} + \lambda_1 \int_{a_0}^{a_1} I_\lambda(a_0,a_2,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)} m(dx).$$

After some algebras, we can obtain that for any $a_0 < x < a_1$,

$$\mathbb{E}_x[\exp\{-\lambda_0 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_0,a_1)}(X_s)ds - \lambda_1 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_1,a_2)}(X_s)ds\}, \tau_{a_0} < \tau_{a_2}] = \frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)} + \frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)} f_-(a_1),$$

for any $a_1 < x < a_2$,

$$\mathbb{E}_x[\exp\{-\lambda_0 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_0,a_1)}(X_s)ds - \lambda_1 \int_0^{\tau_{a_0}\wedge\tau_{a_2}} 1_{(a_1,a_2)}(X_s)ds\}, \tau_{a_0} < \tau_{a_2}] = \frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)} f_-(a_1),$$

where

$$f_-(a_1) = \frac{\frac{f_\lambda(a_1,a_2)}{f_\lambda(a_0,a_2)} + \lambda_1 \int_{a_0}^{a_1} I_\lambda(a_0,a_2,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)} m(dx)}{1 - \lambda_1 \int_{a_0}^{a_1} I_\lambda(a_0,a_2,a_1,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)} m(dx) - \lambda_0 \int_{a_1}^{a_2} I_\lambda(a_0,a_2,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)} m(dx)}.$$

Remark 4.1 : This result is the same as the Theorem 3.1 in Chen *et al*. [4]. With similar computations, we can get the (4.1.2) and (4.1.3). So we don't show it in this article.

Example 4.2 : Let $X = \{X_t, t \geq 0\}$ be a time-homogeneous diffusion process, and $a_0 < a_1 < a_2 < a_3$, we first consider

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{2}\lambda_j\int_0^{\tau_{a_0}\wedge\tau_{a_3}}1_{(a_j,a_{j+1})}(X_s)ds\},\tau_{a_0}<\tau_{a_3}]. \tag{4.2.1}$$

For (4.2.1), by Theorem 3.1, for any $a_0 < x < a_1$, we have

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{2}\lambda_j\int_0^{\tau_{a_0}\wedge\tau_{a_3}}1_{(a_j,a_{j+1})}(X_s)ds\},\tau_{a_0}<\tau_{a_3}]$$

$$=\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}f_-(a_0)+\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}f_-(a_1),$$

for any $a_1 < x < a_2$, we have

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{2}\lambda_j\int_0^{\tau_{a_0}\wedge\tau_{a_3}}1_{(a_j,a_{j+1})}(X_s)ds\},\tau_{a_0}<\tau_{a_3}]$$

$$=\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}f_-(a_1)+\frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}f_-(a_2),$$

for any $a_2 < x < a_3$, we have

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{2}\lambda_j\int_0^{\tau_{a_0}\wedge\tau_{a_3}}1_{(a_j,a_{j+1})}(X_s)ds\},\tau_{a_0}<\tau_{a_3}]$$

$$=\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}f_-(a_2)+\frac{f_{\lambda_2}(a_2,x)}{f_{\lambda_2}(a_2,a_3)}f_-(a_3),$$

where $f_-(a_0) = 1$, $f_-(a_3) = 0$, and $f_-(a_1), f_-(a_{12})$ satisfy

$$\begin{cases} A_{11}^- f_-(a_1) + A_{12}^- f_-(a_2) = B_1^-, \\ A_{21}^- f_-(a_1) + A_{22}^- f_-(a_2) = B_2^-. \end{cases}$$

By (3.3) and (3.4), set $\lambda = \lambda_0 + \lambda_1 + \lambda_2$, we obtain

$$A_{11}^- = 1-(\lambda_1+\lambda_2)\int_{a_0}^{a_1}I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$

$$-(\lambda_0+\lambda_2)\int_{a_1}^{a_2}I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx),$$

$$A_{12}^{-} = -(\lambda_0+\lambda_2)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}m(dx)$$

$$-(\lambda_0+\lambda_1)\int_{a_2}^{a_3} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}m(dx),$$

$$A_{21}^{-} = -(\lambda_1+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$

$$-(\lambda_0+\lambda_2)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx),$$

$$A_{22}^{-} = 1-(\lambda_0+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(a_0,x)}{f_{\lambda_1}(a_0,a_1)}m(dx)$$

$$-(\lambda_0+\lambda_1)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_2}(x,a_2)}{f_{\lambda_2}(a_1,a_2)}m(dx),$$

$$B_1^{-} = \frac{f_\lambda(a_1,a_3)}{f_\lambda(a_0,a_3)} + (\lambda_1+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx),$$

$$B_2^{-} = \frac{f_\lambda(a_2,a_3)}{f_\lambda(a_0,a_3)} + (\lambda_1+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx),$$

further,

$$\begin{aligned}
\begin{vmatrix} A_{11}^{-} & A_{12}^{-} \\ A_{21}^{-} & A_{22}^{-} \end{vmatrix} = {} & 1+(\lambda_1+\lambda_2)(\lambda_0+\lambda_2)[\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx) \\
& \int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(a_0,x)}{f_{\lambda_1}(a_0,a_1)}m(dx) \\
& -\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x) \\
& \frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}m(dx)] + (\lambda_1+\lambda_2)(\lambda_0+\lambda_1)[\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x) \\
& \frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_2}(x,a_2)}{f_{\lambda_2}(a_1,a_2)}m(dx) \\
& -\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)\int_{a_2}^{a_3} I_\lambda(a_0,a_3,a_1,x)
\end{aligned}$$

$$\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}m(dx)] + (\lambda_0+\lambda_2)(\lambda_0+\lambda_2)[\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)$$
$$\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(a_0,x)}{f_{\lambda_1}(a_0,a_1)}m(dx)$$
$$-\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}m(dx)]$$
$$+(\lambda_0+\lambda_2)(\lambda_0+\lambda_1)[\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)$$
$$\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_2}(x,a_2)}{f_{\lambda_2}(a_1,a_2)}m(dx)$$
$$-\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)\int_{a_2}^{a_3} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}m(dx)]$$
$$-(\lambda_1+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$
$$-(\lambda_0+\lambda_2)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)$$
$$-(\lambda_0+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(a_0,x)}{f_{\lambda_1}(a_0,a_1)}m(dx)$$
$$-(\lambda_0+\lambda_1)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_2}(x,a_2)}{f_{\lambda_2}(a_1,a_2)}m(dx),$$

$$\begin{vmatrix} A_{11}^- & B_1^- \\ A_{21}^- & B_2^- \end{vmatrix} = \frac{f_\lambda(a_2,a_3)}{f_\lambda(a_0,a_3)} + (\lambda_1+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$
$$-(\lambda_1+\lambda_2)(\lambda_1+\lambda_2)[\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$
$$\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$
$$-\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$
$$\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)]$$
$$-(\lambda_0+\lambda_2)(\lambda_1+\lambda_2)[\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)$$
$$\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)$$
$$-\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)$$
$$\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)]$$

$$
\begin{aligned}
&-(\lambda_0+\lambda_2)[\frac{f_\lambda(a_2,a_3)}{f_\lambda(a_0,a_3)}\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)\\
&-\frac{f_\lambda(a_1,a_3)}{f_\lambda(a_0,a_3)}\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(x,a_2)}{f_{\lambda_1}(a_1,a_2)}m(dx)]\\
&-(\lambda_1+\lambda_2)[\frac{f_\lambda(a_2,a_3)}{f_\lambda(a_0,a_3)}\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)\\
&-\frac{f_\lambda(a_1,a_3)}{f_\lambda(a_0,a_3)}\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(a_0,x)}{f_{\lambda_0}(a_0,a_1)}m(dx)],
\end{aligned}
$$

$$
\begin{aligned}
\begin{vmatrix} B_1^- & A_{12}^- \\ B_2^- & A_{22}^- \end{vmatrix} \;=\; &\frac{f_\lambda(a_1,a_3)}{f_\lambda(a_0,a_3)}+(\lambda_1+\lambda_2)\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)\\
&-(\lambda_0+\lambda_2)(\lambda_1+\lambda_2)[\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)\\
&\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}m(dx)\\
&-\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\\
&\frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}m(dx)]\\
&-(\lambda_0+\lambda_1)(\lambda_1+\lambda_2)[\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{a_2}^{a_3} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}m(dx)\\
&-\int_{a_0}^{a_1} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_0}(x,a_1)}{f_{\lambda_0}(a_0,a_1)}m(dx)\int_{a_2}^{a_3} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}m(dx)]\\
&-(\lambda_0+\lambda_2)[\frac{f_\lambda(a_1,a_3)}{f_\lambda(a_0,a_3)}\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}m(dx)\\
&-\frac{f_\lambda(a_2,a_3)}{f_\lambda(a_0,a_3)}\int_{a_1}^{a_2} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_1}(a_1,x)}{f_{\lambda_1}(a_1,a_2)}m(dx)]\\
&-(\lambda_0+\lambda_1)[\frac{f_\lambda(a_1,a_3)}{f_\lambda(a_0,a_3)}\int_{a_2}^{a_3} I_\lambda(a_0,a_3,a_1,x)\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}m(dx)\\
&-\frac{f_\lambda(a_2,a_3)}{f_\lambda(a_0,a_3)}\int_{a_2}^{a_3} I_\lambda(a_0,a_3,a_2,x)\frac{f_{\lambda_2}(x,a_3)}{f_{\lambda_2}(a_2,a_3)}m(dx)].
\end{aligned}
$$

According to the Cramer's Rule, we can obtain

$$
f_-(a_1)=\frac{\begin{vmatrix} B_1^- & A_{12}^- \\ B_1^- & A_{22}^- \end{vmatrix}}{\begin{vmatrix} A_{11}^- & A_{12}^- \\ A_{21}^- & A_{22}^- \end{vmatrix}},
$$

$$f_-(a_2) = \frac{\begin{vmatrix} A_{11}^- & B_1^- \\ A_{21}^- & B_2^- \end{vmatrix}}{\begin{vmatrix} A_{11}^- & A_{12}^- \\ A_{21}^- & A_{22}^- \end{vmatrix}}.$$

With similar computations, we also can get the

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{2} \lambda_j \int_0^{\tau_{a_0} \wedge \tau_{a_3}} 1_{(a_j, a_{j+1})}(X_s) ds\}, \tau_{a_3} < \tau_{a_0}]$$

and

$$\mathbb{E}_x[\exp\{-\sum_{j=0}^{2} \lambda_j \int_0^{\tau_{a_0} \wedge \tau_{a_3}} 1_{(a_j, a_{j+1})}(X_s) ds\}].$$

We will let this for readers.

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