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THE PRICING OF TOTAL RETURN SWAP UNDER DEFAULT CONTAGION MODELS WITH JUMP-DIFFUSION INTEREST RATE RISK

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In this paper, we consider a two-firm default contagion model with counterparty risk and jump-diffusion interest rate risk. Under this model, we study the pricing of total return swap(TRS). We assume that the interest rate follows the Vasicek jump-diffusion model, and obtain the Libor market interest rate. The case that default is related to the interest rate is considered. Using the method of change of measure and the properties of conditional expectations, the joint conditional survival probability and joint conditional density function are derived. Applying the arbitrage-free pricing principle of TRS in the complete market, we price the swap rate of TRS and obtain the closed-form solution. And we analyze the effect of various factors on the price of TRS.

Key words : Credit risk; default contagion; interest rate risk; jump-diffusion risk; Total return swap.

2010 Mathematics Subject Classification : 60A10, 60K35, 60J60, 60J75, 91B25

1. Introduction

As a type of credit derivatives and a financing and leverage tool, TRS is an important off-balance sheet tool, particularly for hedge funds and for banks seeking additional fee income. The corporate bonds and their credit derivatives are typically financial tools in the markets which undertake and avoid the credit risk of the companies. Therefore, the key of Management of credit risk is pricing credit derivatives fairly. Especially after 2007 financing crisis, the contagion effect of credit risk has attracted a huge attention of financial market regulators and financial institutions. For pricing credit derivatives, default contagion models are developed rapidly in order to make them more realistic after the subprime mortgage crisis.

There are mainly two approaches of modeling the pricing of credit derivatives: the value-of-the-firm (or structural) approach and the intensity-based approach (or reduced form approach). The structural model is based on the work of Merton [1], Black and Cox [2] and Geske [3]: the default occurs when the firm assets are insufficient to meet payments on debt or the value of the firm asset falls below a prespecified level.

Reduced-form models are developed by Artzner and Delbaen [4], Duffie *et al*. [5], Jarrow and Turnbull [7], and Madan and Unal [7]. This approach shows that default is an exogenous variable and the default intensity is modeled as a random process. The common feature of the reduced-form models is that default cannot be predicted and can occur at any time. Therefore, reduced-form models have been used to price a wide variety of instruments. Duffie and Lando [8] show that a reduced-form model can be obtained from a structural model with incompelete accounting information. This type of reduced-form model is that the default time is the first jump of an exogenously given jump process with an intensity. Jarrow and Yu [9] set up a reduced-form model in which estimation can be based on bond prices as well as credit default swap prices. Bai *et al*. [10] show that the default contagion of counterparty risk has the attenuation effect and give a hyperbolic attenuation contagion model. A systematic development of mathematical tools for reduced-form models has been given by Elliott *et al*. [11]. Jamshidian [12] develops the change of numeraire methodology for reduced-form models. Leung and Kwok [13] consider a three-firm default contagion model and discuss the pricing of CDS. Wang and Ye [14] explore a three-firm contagion model with an interaction term, namely, the case where the default intensity of one firm is affected by the simultaneous default of other two firms.

Interest rate risk is not be ignored in the pricing of credit derivatives. Wang and Ye [15] study the pricing of CDS with Vasicek interest rate risk, and they [16] set up a default contagion model with HJM interest rate risk. Thus, in the previous research, interest rate processes are mainly continuous, such as CIR interest rate process, Vasicek interest rate process and HJM interest rate process. This is clearly not in line with the actual financial market. Because there are financial crises, sudden changes in monetary policy, and other sudden events such as terrorist attacks in financial markets, all these will mutate financial markets. Chacko and Das [17] show that there is a series of jumping behaviors in the interest rate market. Das suggests that the interest rate process can be decomposed into pure random jumps and Brownian motion. He also assumes that the jump part and the continuous part are independent of each other, namely the compound jump-diffusion process.

A TRS is a bilateral financial contract between a total return payer, and a total return receiver. One party (the total return payer) pays the total return of a reference security (or reference securities) and receives a form of payment from the other party (the receiver of the total rate of return).

Often payment is a floating rate payment, a spread to LIBOR. The reference assets can be indices, bonds (emerging market, sovereign, bank debt, mortgage-backed securities, corporate), loans (term or revolver), equities, real estate receivables, lease receivables, or commodities. Ye and Zhuang [18] consider the pricing of TRS only when reference asset defaults, they obtain TRS pricing formula when default is independent of the interest rate. When the default is related to the interest rate, using the hybrid model given in Das and Sundaram [19], they model default time and the interest rate, give the Monte Carlo simulation result. Wang and Ye [16] study the pricing of TRS under the default contagion model with counterparty risk and the interest rate risk.

In this paper, we consider the counterparty default contagion risk between the total return receiver (firm B) and the reference asset (firm C), and set up a two-firm contagion mode. The cash flow of a TRS in this model is provided by figure 1 below.

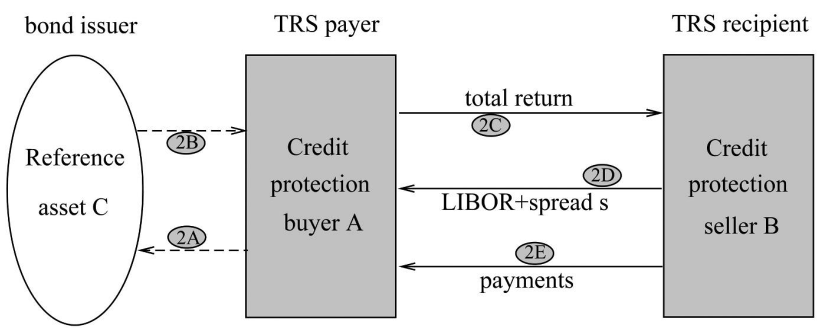

Figure 1: The structure of TRS

Suppose that firm A (a corporate bond investing firm, credit protection buyer) holds a corporate bond(reference asset) issued by firm C (a corporate bond issuer) (refer to 2A in Fig. 1), and firm C is subject to default. At bond maturity, if firm C doesn't default, it will pay the bond principle and interest to firm A (refer to 2B). On the other hand, to hedge the default risk of firm C, firm A and firm B (credit protection seller, subject to default also) enter into a TRS contract. If firm C has no default, firm A will make its total return to firm B (refer to 2C), and in exchange, firm B gives the Libor plus a spread s to A (refer to 2D). Firm B promises to compensate A for its loss in the event of default of firm C (refer to 2E).

The structure of this paper is organized as follows: In Section 2, we give the basic setup and the affine jump-diffusion interest rate model. We discuss the two-firm contagion model. Moreover the default is related to the stochastic interest rate. The Libor rate is obtained. In Section 3, using the change of measure, we derive the joint conditional survival probabilities and the joint conditional density. In section 4, with the arbitrage-free pricing principle and the properties of conditional expectation, the analytic formula is obtained under this two-firm contagion model and the interest rate risk. Section 5 is the conclusion.

2. Default Contagion Model and Affine Jump-Diffusion Interest Rate Model

We consider a filtered probability space $(\Omega, \mathscr{F}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P)$ which is an uncertain economy with a time horizon of T^*, satisfying the usual conditions of right-continuity and completeness with respect to P-null sets, where $\mathscr{F} = \mathscr{F}_{T^*}$ and P is an equivalent martingale measure under which discounted bond prices are martingales. We assume the existence and uniqueness of P, so that bond markets are complete and there is no arbitrage, as shown in discrete time case by Harrison and Kreps [20] and in continuous time case by Harrison and Pliska [21]. In this paper, subsequent specifications of the model are all under the equivalent martingale measure (or risk neutral measure) P.

On this probability space there is an $\mathbb{R}^d$-valued process X_t, which presents d dimensional economy-wide state variables. In this paper, we consider the only state variable which is the interest rate denoted by r_t. There are also two point processes, $N^i (i = B, C)$, initialized at 0, representing the default processes of the firm B and firm C respectively such that the default of the firm i occurs when N^i jumps from 0 to 1.

According to the above information mentioned, define the enlarged filtration:

$$\mathscr{F}_t = \mathscr{F}_t^r \vee \mathscr{F}_t^B \vee \mathscr{F}_t^C \tag{1}$$

where

$$\mathscr{F}_t^r = (r_s, 0 \leqslant s \leqslant t) \tag{2}$$

$$\mathscr{F}_t^i = (N_t^i, 0 \leqslant s \leqslant t), i = B, C \tag{3}$$

$\mathscr{F}_t$ is the filtration generated by r_t and N_t^i. Denote

$$\mathscr{G}_t^B = \mathscr{F}_t^B \vee \mathscr{F}_{T^*}^r \vee \mathscr{F}_{T^*}^C = \mathscr{F}_t^B \vee \mathscr{G}_{T^*}^{-B} \tag{4}$$

$$\mathscr{G}_t^C = \mathscr{F}_t^C \vee \mathscr{F}_{T^*}^r \vee \mathscr{F}_{T^*}^B = \mathscr{F}_t^C \vee \mathscr{G}_{T^*}^{-C} \tag{5}$$

where $\mathscr{G}_{T^*}^{-B} = \mathscr{F}_{T^*}^r \vee \mathscr{F}_{T^*}^C, \mathscr{G}_{T^*}^{-C} = \mathscr{F}_{T^*}^r \vee \mathscr{F}_{T^*}^B$, $\mathscr{G}_{T^*}^{-i} (i = B, C)$ contains all the information about state variable r and the default of firms up to time T^* other than firm i.

Let τ^i be the default time of firm i, namely, τ^i be the first jump time of N^i, denote

$$\tau^i = inf\{t : \int_0^t \lambda_s^i ds \geq E^i\} \tag{6}$$

where λ_t^i is $\mathscr{F}_t^r$-measurable, which is the default intensity of company i. E^i is the random variable which subjects to the unit exponential distribution, independent of $\mathscr{F}_t^r$.

Suppose that τ^i is a $\mathscr{F}_t$-predictable process with a right continuous sample path, with the default intensity λ_t^i. According to the Doob-Meyer decomposition theorem,

$$M_t^i := N_t - \int_0^{t\wedge\tau^i} \lambda_s^i ds \tag{7}$$

is a $(\mathscr{F}_t, P)$ martingale.

Firm B and firm C have the default correlation and contagion due to the impact of counterparty risk, the default intensities λ_t^B, λ_t^C of firms B and C are no longer independent under the condition $\mathscr{F}_t^r$. Moreover, the default is related to the interest rate risk. Therefore, we consider the following intensity model about default intensity λ_t^B and λ_t^C,

$$\lambda_t^B = b_0 + br_t + b_1\mathbb{I}_{\{\tau^C \leq t\}} \tag{8}$$

$$\lambda_t^C = c_0 + cr_t + c_1\mathbb{I}_{\{\tau^B \leq t\}} \tag{9}$$

where $b_0 > 0, c_0 > 0, b > 0, c > 0, b_0 + b + b_1 > 0, c_0 + c + c_1 > 0$. b_0 and c_0 reflect the influence of the system factor on the company's default intensity. b and c reflect the impact of the interest rate on the default intensity. b_1 and c_1 reflect the correlation and contagion between firm B and firm C.

The conditional survival probability and unconditional survival probability of the default time $\tau^i(i = B, C)$ are

$$P\{\tau^i > t \mid \mathscr{G}_{T^*}^{-i}\} = \exp\left(-\int_0^{t\wedge\tau^i} \lambda_s^i ds\right), t \in [0, T] \tag{10}$$

$$P\{\tau^i > t\} = E\left[\exp(-\int_0^{t\wedge\tau^i} \lambda_s^i ds)\right], t \in [0, T] \tag{11}$$

respectively.

In this paper, we consider the default-free interest rate follows the Vasicek affine jump-diffusion process

$$dr_t = \alpha(K - r_t)dt + \sigma dW_t + q_t dN_t \tag{12}$$

where α, σ and K are constants, q_t is a deterministic function, W_t is a standard Brownian motion on probability space $(\Omega, \mathscr{F}, \mathscr{F}_{t=0}^{T^*}, P)$, N_t is a Poissiion process with the intensity μ, independent of W_t.

The interest rate r_t has the following solution

$$r_t = r_0 e^{-\alpha t} + \alpha K \int_0^t e^{-\alpha(t-s)} ds + \sigma \int_0^t e^{-\alpha(t-s)} dW_s + \int_0^t q_s e^{-\alpha(t-s)} dN_s \tag{13}$$

According to the result in Jarrow and Yu [9], the forward rate at time 0 can be substituted for the forward rate of time t. Denote $f(0,t) = r_0 e^{-\alpha u}$, for any $u \geq t$, equation (13) becomes

$$r_u = f(t,u) + \alpha K \int_t^u e^{-\alpha(u-s)} ds + \sigma \int_t^u e^{-\alpha(u-s)} dW_s + \int_t^u q_s e^{-\alpha(u-s)} dN_s \tag{14}$$

where

$$f(t,u) = f(0,u) + \alpha K \int_0^t e^{-\alpha(t-s)} ds + \sigma \int_0^t e^{-\alpha(t-s)} dW_s + \int_0^t q_s e^{-\alpha(t-s)} dN_s \tag{15}$$

Moreover, by the properties of W_t and N_t, r_t is a $\mathscr{F}_t^r$ Markov process.

The instantaneous rate at time t is $r_t = f(t,t)$. $D(t) = \exp\left(-\int_0^t r_u du\right)$ is the discount process. Denote $B(t,T)$ is the zero coupon bond with face value 1 at maturity date T. According to (14),

$$B(t,T) = \exp\left(-\int_t^T r_u du\right), 0 \leq t \leq T \leq \bar{T} \tag{16}$$

Suppose the Libor interest rate $L(t,T)$ is the yield from T to $T+\delta$ which is locked at time t, we have

$$L(t,T) = \frac{B(t,T) - B(t,T+\delta)}{\delta B(t,T+\delta)} \tag{17}$$

For $0 \leq t < T$, $L(t,T)$ is called the forward Libor interest rate, $L(T,T)$ is called the spot Libor rate, δ is the duration of Libor, normally 0.25 year or a half year.

3. The Joint Conditional Distribution and The Joint Conditional Probability Density Function

To derive the joint conditional default probability and the joint conditional density of τ^B and τ^C under the default intensities (8)-(9), we define a new probability measure $P^i(i = A, B, C)$: it depends on the probability measure of company i, which is absolutely continuous with respect to measure P on

$[0,\tau^i)$ and 0 almost everywhere on $[\tau^i,+\infty)$. Under measure P^i, the default probability of company i is 0 before time t. The Radon-Nikodym derivative of P^i with respect to P is

$$Z_t^i := \frac{dP^i}{dP} = \mathbb{I}_{\{\tau^i>t\}} \exp\left(\int_0^t \lambda_s^i ds\right) \tag{18}$$

With the result in Wang and Ye [15] and the change of measure mentioned above, the joint conditional survival probability and the joint conditional density of default time (τ^B,τ^C) are given by the following theorem.

***Theorem* 3.1** — *Suppose that the default intensities of firm B and firm C are given by $(8)-(9)$. The joint conditional survival probability of default time τ^B,τ^C can be derived. The result is as follows*

$$\begin{aligned}P(\tau^B>t_1,\tau^C>t_2|\mathscr{F}_{T^*}^r) &= \exp(-c_0(t_2-t)-cR_{t,t_2})\\ &\cdot[\exp(-c_1(t_2-t_1))-\exp(-b_0(t_2-t_1)-bR_{t_1,t_2})+\exp(-b_0(t_2-t)-bR_{t,t_2})\\ &+c_1\int_{t_1}^{t_2}\exp(-b_0(u-t_1)-c_1(t_2-u)-bR_{t_1,u})du], t<t_1<t_2<T\end{aligned} \tag{19}$$

$$\begin{aligned}P(\tau^B>t_1,\tau^C>t_2|\mathscr{F}_{T^*}^r) &= \exp(-b_0(t_1-t)-bR_{t,t_1})\\ &\cdot[\exp(-b_1(t_1-t_2))-\exp(-c_0(t_1-t_2)-cR_{t_2,t_1})+\exp(-c_0(t_1-t)-cR_{t,t_1})\\ &+b_1\int_{t_2}^{t_1}\exp(-c_0(u-t_2)-b_1(t_1-u)-cR_{t_2,u})du], t<t_2\le t_1<T\end{aligned} \tag{20}$$

where $R_{s,t}=\int_s^t r_u du$, $\mathscr{F}_{T^*}^r=(r_t, 0\le t\le T^*)$.

PROOF : With the default intensities (8)-(9), for $t<t_1<t_2<T$,

$$\begin{aligned}&P(\tau^B>t_1,\tau^C>t_2|\mathscr{F}_{T^*}^r)\\ =\ & E[\mathbb{I}_{\{\tau^B>t_1,\tau^C>t_2\}}|\mathscr{F}_{T^*}^r]\\ =\ & E^C[\mathbb{I}_{\{\tau^B>t_1\}}\exp(-c_0(t_2-t)-cR_{t,t_2}-\mathbb{I}_{\{\tau^B\le t_2\}}c_1(t_2-\tau^B))|\mathscr{F}_t^B\vee\mathscr{F}_{T^*}^r]\\ =\ & E^C[[\mathbb{I}_{\{t_1<\tau^A\le t_2\}}\cdot\exp(-b_0(t_2-t)-bR_{t,t_2}-b_1(t_2-\tau^A))\\ &+\mathbb{I}_{\{\tau^A>t_2\}}\cdot\exp(-b_0(t_2-t)-bR_{t,t_2}))]|\mathscr{F}_t^B\vee\mathscr{F}_{T^*}^r]\\ \triangleq\ & I_1+I_2,\end{aligned} \tag{21}$$

where E^C denotes the expectation under probability measure P^C and $F^C(\cdot|\tilde{\mathcal{F}}_t^B\vee\mathcal{F}_{T^*}^r)$ the conditional probability distribution of τ^C .

τ^B has the following conditional distribution on $\tau^B > t$

$$P(\tau^B > t_2|\tilde{\mathcal{F}_t^B} \vee \mathcal{F}^r_{T^*}) = \exp(-\int_t^{t_2} \lambda_s^B ds)$$

Moreover, under probability measure P^C,

$$P^C(\tau^B > t_2|\tilde{\mathcal{F}_t^B} \vee \mathcal{F}^r_{T^*}) = \exp(-b_0(t_2 - t) - bR_{t,t_2}).$$

Thus

$$\begin{aligned} I_1 &= \int_{t_1}^{t_2} \exp(-c_0(t_2 - t) - cR_{t,t_2} - c_1(t_2 - u))dF^C_{\tau^B}(u) \\ &= \exp(-c_0(t_2 - t) - cR_{t,t_2})\Big[\exp(-c_1(t_2 - t_1)) - \exp(-(b_0(t_2 - t_1) - bR_{t_1,t_2})) \\ &\quad + c_1 \int_{t_1}^{t_2} \exp(-b_0(u - t_1) - c_1(t_2 - u) - bR_{u,t_1})du\Big], \\ I_2 &= \exp(-c_0(t_2 - t) - cR_{t,t_2}) \cdot P^C(\tau^B > t_2|\tilde{\mathcal{F}_t^B} \vee \mathcal{F}^r_{T^*}) \\ &= \exp(-(b_0 + c_0)(t_2 - t) - (b + c)R_{t,t_2}), \end{aligned}$$

Substituting I_1 and I_2 to (21), we have

$$\begin{aligned} & P(\tau^B > t_1, \tau^C > t_2|\mathcal{F}_t \vee \mathcal{F}^r_{T^*}) \\ = \; & \exp(-c_0(t_2 - t) - cR_{t,t_2})\Big[\exp(-c_1(t_2 - t_1)) - \exp(-(b_0(t_2 - t_1) - bR_{t_1,t_2})) \\ & + \exp(-b_0(t_2 - t) - bR_{t,t_2}) + c_1 \int_{t_1}^{t_2} \exp(-b_0(u - t_1) - c_1(t_2 - u)) - bR_{t_1,u})du\Big], \\ & \qquad \text{for } \; t < t_1 < t_2 < T, \end{aligned}$$

Similar to the above derivation, we can obtain formula (20). This completes the proof. □

Take the partial derivative of the joint conditional probability, we have the following joint conditional density.

***Theorem* 3.2** — *Assume that the default intensities of firm B and firm C are given by $(8) - (9)$, then the joint conditional density function of the default time τ^B and τ^C is as follows*

$$f_1(t_1, t_2|\mathscr{F}^r_{T^*}) = (b_0 + br_{t_1})(c_0 + c_1 + cr_{t_2}) \cdot e^{-(b_0 - c_1)t_1 - (c_0 + c_1)t_2 - bR_{0,t_1} - cR_{0,t_2}}, t_1 < t_2 \quad (22)$$

$$f_2(t_1, t_2|\mathscr{F}^r_{T^*}) = (c_0 + cr_{t_2})(b_0 + b_1 + br_{t_1}) \cdot e^{-(b_0 + b_1)t_1 - (c_0 - b_1)t_2 - bR_{0,t_1} - cR_{0,t_2}}, t_2 \le t_1 \quad (23)$$

4. TRS VALUATION

In this section, we will derive the analytic price of TRS under the models (8)-(9) and (12).

Assume the reference asset C is a defaultable coupon bond with face value 1 and the same maturity T with TRS contract. $T_0, T_1, T_2, \cdots, T_n$ are bond interest payment dates, where $0 = T_0 < T_1 < \cdots < T_n = T$, for $0 \le i \le n-1$, $T_{i+1} - T_i = \triangle T$, $n\triangle T = T$.

Denote C_i be the time-T_i cash flow of TRS payer $(i = 1, \cdots, n)$, where C_i are the interest payments at time $T_i (i = 1, \cdots, n-1)$, C_n is the sum of interest payment and value-added of the bond at time $T_n = T$, which are all determined at $T_0 = 0$. Let M be the notional principal, δ is the maturity of Libor interest rate, the same with the bond's payment cycle $\triangle T$, namely $\delta = \triangle T$. For simplification, the default recovery rate of reference asset C is 0.

At time 0, for TRS payment leg, the discounted expectation of the cash flow F_B is

$$E[F_B] = E\left[\sum_{i=1}^{n} D(T_i) C_i \mathbb{I}_{\{\tau^B \wedge \tau^C > T_i\}}\right] \tag{24}$$

At time 0, for TRS recipient leg, the discounted expectation of the cash flow F_C is

$$E[F_C] = E\left[\sum_{i=1}^{n} D(T_i)\delta M(L(T_{i-1}, T_i) + s)\mathbb{I}_{\{\tau^B \wedge \tau^C > T_i\}} + D(\tau^C + \theta)\mathbb{I}_{\{\tau^C \le T\}}\mathbb{I}_{\{\tau^B > \tau^C + \theta\}}\right] \tag{25}$$

where E is the expectation under the risk neutral measure P, θ is the length of settlement period, $\tau^C + \theta$ is the settlement date. $L(T_{i-1}, T_{i-1})$ is time-T_{i-1} locked T_i-Libor interest rate. s is the TRS spread, $D(.)$ is the discounted factor. Suppose that the market is complete, namely there is no arbitrage. Therefore, according to the arbitrage-free pricing principle, we have

$$E[F_B] = E[F_C] \tag{26}$$

Substitute $(24) - (25)$ into (26), we have the price formula of TRS

$$\begin{aligned} s &= \frac{E\left[\sum_{i=1}^{n} D(T_i) C_i \mathbb{I}_{\{\tau^B \wedge \tau^C > T_i\}}\right] - E\left[\sum_{i=1}^{n} D(T_i)\delta M L(T_{i-1}, T_{i-1})\mathbb{I}_{\{\tau^B \wedge \tau^C > T_i\}}\right]}{E\left[\sum_{i=1}^{n} D(T_i)\delta M \mathbb{I}_{\{\tau^B \wedge \tau^C > T_i\}}\right]} \\ &- \frac{E\left[D(\tau^C + \theta)\mathbb{I}_{\{\tau^C \le T\}}\mathbb{I}_{\{\tau^B > \tau^C + \theta\}}\right]}{E\left[\sum_{i=1}^{n} D(T_i)\delta M \mathbb{I}_{\{\tau^B \wedge \tau^C > T_i\}}\right]} \end{aligned} \tag{27}$$

To obtain the analytic formula of TRS, we need to derive several conditional expectations above. The result is given by the following lemma.

Lemma 4.1 — Let the interest rate process is given by (12), denote

$$E_t\left[e^{-mR_{t,T}}\right] := L_1(m;t,T)$$

$$E_{t_0}\left[e^{-m_1R_{t_0,t_1}-m_2R_{t_1,t_2}}\right] := L_2(m_1,m_2;t_0,t_1,t_2)$$

Then

$$\begin{aligned} L_1(m;t,T) &= \exp\left(\int_t^T(-mf(t,u)+\frac{1}{2}m^2\sigma^2h_T^2(u)+\mu(e^{-mq_uc_T(u)}-1))du\right) \\ &\cdot \exp\left(mK(T-t)+Kh_T(t)\right) \end{aligned} \quad (28)$$

$$\begin{aligned} &L_2(m_1,m_2;t_0,t_1,t_2) = \exp\left(-m_1\int_{t_0}^{t_1}f(t_0,u)du - F(m_2;t_0,t_1,t_2)\right) \\ &\cdot \exp\left(-\sum_{i=1}^2 m_iK(t_i-t_{i-1}) + m_1Kh_{t_1}(t_0) + m_2(K-r_0)d(t_1,t_2,0)\right) \\ &\cdot \exp\left(\mu\sum_{i=1}^2\int_{t_{i-1}}^{t_i}\left[\exp(-q_um_ih_{t_i}(u)+m_{i+1}d(t_i,t_{i+1},u)\mathbb{I}_{\{i+1\leq 2\}})-1\right]du\right) \\ &\cdot \exp\left(\frac{1}{2}\sigma^2\sum_{i=1}^2\int_{t_{i-1}}^{t_i}(m_ih_{t_i}(u)+m_{i+1}d(t_i,t_{i+1},u)\mathbb{I}_{\{i+1\leq 2\}})^2du\right) \end{aligned} \quad (29)$$

where

$$h_T(t) = \frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$d(t_i,t_{i+1},u) = \frac{1}{\alpha}\left(e^{-\alpha(t_i-u)} - e^{-\alpha(t_{i+1}-u)}\right)$$

$$F(m_i;t_{i-2},t_{i-1},t_i) = m_i\left(\int_0^{t_{i-2}}\sigma d(t_{i-1},t_i,u)dW_u + \int_0^{t_{i-2}}q_ud(t_{i-1},t_i,u)dN_u\right)$$

PROOF : See [25]. □

Using Lemma 4.1 and formula (27), we can obtain the following theorem.

Theorem 4.1 — *Suppose that the default intensities λ^B,λ^C of the TRS seller B and the reference asset C are given by $(8)-(9)$ respectively, and the interest rate process is given by (12). Assume the settlement length $\theta=0$, the price s of TRS is given by*

$$\begin{aligned} s &= \frac{\sum_{i=1}^n e^{-(b_0+c_0)T_i}[(C_i+M)L_1(1+b+c;0,T_i) - ML_2(1+b+c,b+c;0,T_{i-1},T_i)]}{\delta M\sum_{i=1}^n e^{-(b_0+c_0)T_i}L_1(1+b+c;0,T_i)} \\ &- \frac{\int_0^T c_0e^{-(b_0+c_0)v}dv + \frac{c}{1+b+c}[1-e^{-(b_0+c_0)T}L_1(1+b+c;0,T)]}{\delta M\sum_{i=1}^n e^{-(b_0+c_0)T_i}L_1(1+b+c;0,T_i)} \end{aligned} \quad (30)$$

PROOF : By (27),

$$
\begin{aligned}
s &= \frac{E\left[\sum_{i=1}^{n} D(T_i)C_i\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}\right] - E\left[\sum_{i=1}^{n} D(T_i)\delta M L(T_{i-1},T_i)\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}\right]}{E\left[\sum_{i=1}^{n} D(T_i)\delta M\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}\right]} \\
&- \frac{E\left[D(\tau^C)\mathbb{I}_{\{\tau^C\le T\}}\mathbb{I}_{\{\tau^B>\tau^C\}}\right]}{E\left[\sum_{i=1}^{n} D(T_i)\delta M\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}\right]} := \frac{H_1-H_2-H_3}{H} \qquad (31)
\end{aligned}
$$

We need only derive H, H_1, H_2, H_3. With the property of the conditional expectation and Fubini theorem,

$$
\begin{aligned}
H &= \sum_{i=1}^{n} E\left[D(T_i)\delta M\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}\right] \\
&= \delta M\sum_{i=1}^{n} E\left[E\left[D(T_i)\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}|\mathscr{F}_{T^*}^r\right]\right] = \delta M\sum_{i=1}^{n} E\left[e^{-R_{0,T_i}}E\left[D(T_i)\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}|\mathscr{F}_{T^*}^r\right]\right] \\
&= \delta M\sum_{i=1}^{n} E\left[e^{-R_{0,T_i}}e^{-(b_0+c_0)T_i-(b+c)R_{0,T_i}}\right] = \delta M\sum_{i=1}^{n} e^{-(b_0+c_0)T_i}L_1(1+b+c;0,T_i) \qquad (32)
\end{aligned}
$$

$$
H_1 = E\left[\sum_{i=1}^{n} D(T_i)C_i\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}\right] = \sum_{i=1}^{n} C_i e^{-(b_0+c_0)T_i}L_1(1+b+c;0,T_i) \qquad (33)
$$

$$
\begin{aligned}
H_2 &= E\left[\sum_{i=1}^{n} D(T_i)\delta M L(T_{i-1},T_{i-1})\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}\right] \\
&= \delta M\sum_{i=1}^{n} E\left[D(T_i)E\left[L(T_{i-1},T_{i-1})\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}|\mathscr{F}_{T^*}^r\right]\right] \\
&= M\sum_{i=1}^{n} E\left[e^{-R_{0,T_i}}\frac{e^{-R_{T_{i-1},T_{i-1}}} - e^{-R_{T_{i-1},T_i}}}{e^{-R_{T_{i-1},T_i}}}E\left[\mathbb{I}_{\{\tau^B\wedge\tau^C>T_i\}}|\mathscr{F}_{T^*}^r\right]\right] \\
&= M\sum_{i=1}^{n} e^{-(b_0+c_0)T_i}[L_2(1+b+c,b+c;0,T_{i-1},T_i) - L_1(1+b+c;0,T_i)] \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
H_3 &= E\left[D(\tau^C)\mathbb{I}_{\{\tau^C\le T\}}\mathbb{I}_{\{\tau^B>\tau^C\}}\right] = E\left[E\left[D(\tau^C)\mathbb{I}_{\{\tau^C\le T\}}\mathbb{I}_{\{\tau^B>\tau^C\}}|\mathscr{F}_{T^*}^r\right]\right] \\
&= E\left[\int_0^T dv\int_v^{+\infty} e^{R_{0,v}}f(u,v|\mathscr{F}_{T^*}^r)du\right] \\
&= E\left[\int_0^T (c_0+cr_v)e^{(-b_0+c_0)v}e^{-(1+b+c)R_{0,v}}dv\right] \\
&= E\left[\int_0^T c_0e^{(-b_0+c_0)v}e^{-(1+b+c)R_{0,v}}dv - \frac{c}{1+b+c}\int_0^T e^{(-b_0+c_0)v}e^{-(1+b+c)R_{0,v}}dv\right]
\end{aligned}
$$

$$= \int_0^T c_0 e^{(-b_0+c_0)v} L_1(1+b+c;0,v)dv + \frac{c}{1+b+c}\left[1 - e^{(-b_0+c_0)T} L_1(1+b+c;0,T)\right] \tag{35}$$

Substituting (32)-(35) into (31), we have (30). The proof is completed. □

Remark 4.1 : Compare to our previous research [16], the price of TRS is effected either by the default contagion risk, the interest rate risk, or effected by the jump-diffusion risk. Therefore, the jump-diffusion risk is ignorable. However, from formula (30), we conclude that if the length θ of the reference asset's settlement period is 0, the price s of TRS is only related to the interest rate risk and the systematic risk.

5. Conclusion

In this paper, we mainly study the default contagion model with counterparty risk and Vasicek jump-diffusion interest rate risk. Under this framework, we obtain the analytic price expression of TRS. From the expression, we show that the default risk and the jump-difussion interest rate risk both have effects on the pricing of TRS. Therefore, the model in our paper has the certain practical significance. Moreover, in the future research, we will claim that the contagion effect between the reference asset and the protection seller is not ignorable when the settlement length is not 0. We will further discuss other contagion models and interest rate models, and study credit derivatives valuation.

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