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TARGET SET SELECTION ON GENERALIZED PANCAKE GRAPHS

Haining Jiang[1]

School of Mathematics and Statistics, Heze University, Heze, Shandong, 274000,
People's Republic of China
and
School of Mathematical Sciences, Xiamen University, Xiamen, Fujian, 361005,
People's Republic of China
e-mail: hnjiangsd@163.com

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Target Set Selection (TSS) was initially proposed to study the problem of the spread of information, ideas or influence through a social network and had formulated many problems arising in various practical applications. We consider a particular type of graphs, namely n-dimensional m-sided pancake graph mP_n, which is one class of Cayley graphs and is widely used in the symmetric interconnection networks. We establish a bound of TSS on mP_n by the minimum feedback vertex set.

Key words : Target set selection; feedback vertex set; n-dimensional m-sided pancake graph.

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1. INTRODUCTION

A group-theoretical model for designing, analyzing, and improving symmetric interconnection networks was proposed recently which is based on Cayley graphs. Pancake graph [19, 23], one of a specific Cayley graphs, which offer more attractive perspectives than n-cubes for design of parallel architectures, have received much attention lately. An attractive feature of pancake graph is that they have sublogarithmic diameter and have a great deal of symmetry akin to the binary hypercube. Justan [22] generated the pancake graph, i.e., state the m-sided pancake flipping problem, and describe its graph as the m-sided pancake graph. The n-dimensional m-sided pancake graph mP_n is also isomorphic to one class of Cayley graph.

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We want to research on Target Set Selection of mP_n.

Target Set Selection (TSS) was initially proposed by Kempe, Kleinberg and Tardos [24] to study the social network analysis involving the problem of the spread of information, ideas or influence through a social network [5, 8]. This model can also formulate many problems arising in economy, sociology, medicine and computer science and, therefore, receives much attention from both theoretical and practical interests [1, 7, 13, 14, 17, 26, 28].

In Target Set Selection problem, a network is mathematically modeled as an undirected simple graph with its vertices equipped with a threshold function $\theta(v) : V(G) \rightarrow \mathbb{N}$, where $V(G)$ is the vertex set of G and $\mathbb{N}$, the set of positive integers. In this paper, we only discuss that $\theta(v)$ is a constant number k.

In a k-threshold process, we fix a number k and assume that a vertex changes its state at time $t+1$ if at least k of its neighbors are in the opposite state at time t. In the irreversible k-threshold process, once a vertex becomes active, it remains active for the entire process. The process ends when no more vertices can get activated. If S can activates all the vertices of G in irreversible k-threshold process then we call S a *k-target set* of G, k is a positive integer.

Target Set Selection (TSS)

Select a k-target set S with minimum number of vertices.

A k-target set with minimum number of vertices is also called an *optimal k-target set* of G and its size, denoted by $\min_k(G)$ [2, 3], is called the *k-target number*.

It is not surprising that the Target Set Selection problem is in general NP-hard. In fact, Dreyer and Roberts [14] showed that it is NP-hard to compute $\min_k(G)$ for any constant $k \geq 3$. Therefore, much research interests on this subject focus on particular graph structures, e.g., block-cactus graph, chordal graph and Hamming graph [10], tori [18], hexagonal grid [9], sparse graph and ‘cliquish’ graph [11]; tree, multipartite graph and grid [14]. In [14], Dreyer and Roberts gave the following theorem.

***Theorem* 1** — *Let $G = (V,\ E)$ be an $(r+1)$-regular graph, and let $S \subseteq V$. Then S is a r-target set if and only if S is a minimum feedback vertex set.*

In a graph $G = (V,\ E)$, a subset $F \subset V(G)$ is a feedback vertex set of G if the subgraph induced by $V(G)\backslash F$ is acyclic. If the cardinality $|F|$ is minimum among all possible feedback vertex sets, then F is called a minimum feedback vertex set, denoted by $F_{\min}(G)$.

It gives us another method to determine Target Set Selection on mP_n. In this paper, we get the bound of $\min_k(mP_n)$ by the minimum feedback vertex set.

2. Lower Bound of n-Dimensional m-Sided Pancake Graph

Definition 1 — [22]. Consider a permutation $[(k_1, p_1), (k_2, p_2), \cdots, (k_i, p_i), (k_{i+1}, p_{i+1}), \cdots, (k_n, p_n)]$ where $[p_1, p_2, \cdots, p_i, \cdots, p_n] \in S_n$ and $k_j = 0, 1, \cdots, m-1$, for $1 \leq j \leq n$.

An ***i-flip*** action is defined by $[(k_1, p_1), (k_2, p_2), \cdots, (k_i, p_i), (k_{i+1}, p_{i+1}), \cdots, (k_n, p_n)] \vdash [(k_i + 1_{(mod\ m)}, p_i), \cdots, (k_2 + 1_{(mod\ m)}, p_2), (k_1 + 1_{(mod\ m)}, p_1), (k_{i+1} + 1_{(mod\ m)}, p_{i+1}), \cdots, (k_n + 1_{(mod\ m)}, p_n)]$.

An ***i-flop*** action is defined by $[(k_1, p_1), (k_2, p_2), \cdots, (k_i, p_i), (k_{i+1}, p_{i+1}), \cdots, (k_n, p_n)] \vdash [(k_i - 1_{(mod\ m)}, p_i), \cdots, (k_2 - 1_{(mod\ m)}, p_2), (k_1 - 1_{(mod\ m)}, p_1), (k_{i+1} - 1_{(mod\ m)}, p_{i+1}), \cdots, (k_n - 1_{(mod\ m)}, p_n)]$.

Definition 2 — [22]. The n-dimensional m-sided pancake graph mP_n has vertices labeled with $[(k_1, p_1), (k_2, p_2), \cdots, (k_i, p_i), (k_{i+1}, p_{i+1}), \cdots, (k_n, p_n)]$ for each $[p_1, p_2, \cdots, p_i, \cdots, p_n)] \in S_n$ and k_j=0, 1, $\cdots$, m-1 for $1 \leq j \leq n$, and there is an edge between two vertices x and y where y can be obtained from x by either a flip action or a flop action.

Definition 3 — Let $G < Sym(X)$ be permutation group of degree m and $H < Sym(Y)$ be a permutation group of degree n. The wreath product $G \times H$, of G by H is a permutation group of degree mn acting on $X \times Y$.

***Theorem* 2** — *The wreath product $G \times H$ is a group.*

$Z'_m = \{[1,\ 2,\ \cdots,\ m],\ [2,\ 3,\ \cdots,\ m,\ 1],\ \cdots,\ [m,\ m-1,\ \cdots,\ 1]\}$.

***Theorem* 3** — *$< Z'_m,\ * >$ is isomorphic to $< Z_m,\ + >$.*

***Theorem* 4** — *Let* T_1=$\{g_1,\ g_2,\ ...,\ g_n;\ [i,\ i-1,\ ...,\ 3,\ 2,\ 1,\ i+1,\ i+2,\ ...,n]\},\ 1 \leq i \leq n$, *where*

$$g_i = \begin{cases} [<2,\ 3,\ ...,\ m-1,\ m>,\ 1] & \textit{if}\ 1 \leq j \leq i, \\ [1,\ 2,\ 3,\ ...,\ m-1,\ m] & \textit{if}\ i+1 \leq j \leq n. \end{cases}$$

Let T_2=$\{(g_1,\ g_2,\ ...,\ g_n;\ [i,\ i-1,\ ...,\ 3,\ 2,\ 1,\ i+1,\ i+2,\ ...,\ n])\},\ 1 \leq i \leq n$, *where*

$$g_i = \begin{cases} [m, <2, 3, ..., m-1>] & \textit{if}\ 1 \leq j \leq i, \\ [1, 2, 3, ..., m-1, m] & \textit{if}\ i+1 \leq j \leq n. \end{cases}$$

$T_1 \bigcup T_2$ is a set of generators for the wreath products $Z_m^{'} \times S_n$.

***Theorem* 5** — *The n-dimensional m-sided pancake graph, mP_n, $m \geq 2$, is isomorphic to the Cayley graph of $Z_m' \times S_n$, with $T_1 \bigcup T_2$ as the set of generators.*

The identity of $Z_m' \times S_n$ is denoted by I_n. S_n is the symmetric group of order n.

Let T_n be the set $\{1,\ 2,\ \cdots,\ n\}$, and let $P(m,\ n,\ t)$ be the set of $(m, n,\ t)$-permutations, $1 \leq t \leq n$.

$P(m,\ n,\ t)$=$\{$[(k_{n-t+1}, p_{n-t+1}), (k_{n-t+2}, p_{n-t+2}), $\cdots$, (k_n, p_n)]—[$p_i \in T_n$, $p_i \neq p_{i'}$ and k_j = 0, 1, $\cdots$, m-1, for $n - t + 1 \leq i,\ i^{'},\ j \leq n.\}$

Let $p(m,\ n,\ t)$=$|P(m,\ n,\ t)|$. Then $p(m,\ n,\ t)$=$m^n n!/(n-t)!$.

When $t = n$, we denote $P(m,\ n,\ t)$ by $P(m,\ n)$.

Definition 4 — Define a relation R on $P(m,\ n)$:

For any two elements x=$[(k_1,\ p_1),\ (k_2,\ p_2),\ \cdots,\ (k_i,\ p_i),\ (k_{i+1},\ p_{i+1}),\ \cdots,\ (k_n,\ p_n)]$ and y=$[(j_1,\ q_1),\ (j_2,\ q_2),\ \cdots,\ (j_i,\ q_i),\ (j_{i+1},\ q_{i+1}),\ \cdots,\ (j_n,\ q_n)]$ in $P(m, n)$, we have

$$xRy \Leftrightarrow \begin{cases} (k_i,\ p_i) = (j_i,\ q_i)\ for\ each\ i = 4,\ 5,\ \cdots,\ n & \text{if } m = 1,\ n \geq 4, \\ (k_i,\ p_i) = (j_i,\ q_i)\ for\ each\ i = 3,\ 4,\ \cdots,\ n & \text{if } m = 2,\ n \geq 3, \\ (k_i,\ p_i) = (j_i,\ q_i)\ for\ each\ i = 2,\ 3,\ 4, \cdots,\ n & \text{if } m \geq 3,\ n \geq 2. \end{cases}$$

We can easily prove that the relation R on $P(m,\ n)$ is an equivalence relation. For $u \in P(m,\ n)$, the equivalence class of u is the set

$$[u] = \{xRu, x \in P(m,\ n)\}.$$

All equivalence classes form a partition of $P(m,\ n)$, denoted by

$$\mathcal{P}(m,\ n) = \{[u]|u \in P(m,\ n)\}.$$

In Figure, we replace [(k_1, p_1), $\cdots$, (k_n, p_n)] with $k_1p_1, \cdots, k_np_n$ for short.

For example, [(0, 1), (0, 2), (0, 3)] is a vertex in $2P_3$,

$[u]$=$\{$[(0, 1), (0, 2), (0, 3)], [(1, 1), (0, 2), (0, 3)], [(1, 2), (0, 1), (0, 3)], [(0, 2), (0, 1), (0, 3)], [(1, 1), (1, 2), (0, 3)], [(0, 1), (1, 2), (0, 3)], [(0, 2), (1, 1), (0, 3)], [(1, 2), (1, 1), (0, 3)]$\}$.

$\mathcal{P}(2, 3)$ is

$\{\{$[(0, 1), (0, 2), (0, 3)], [(1, 1), (0, 2), (0, 3)], [(1, 2), (0, 1), (0, 3)], [(0, 2), (0, 1), (0, 3)],

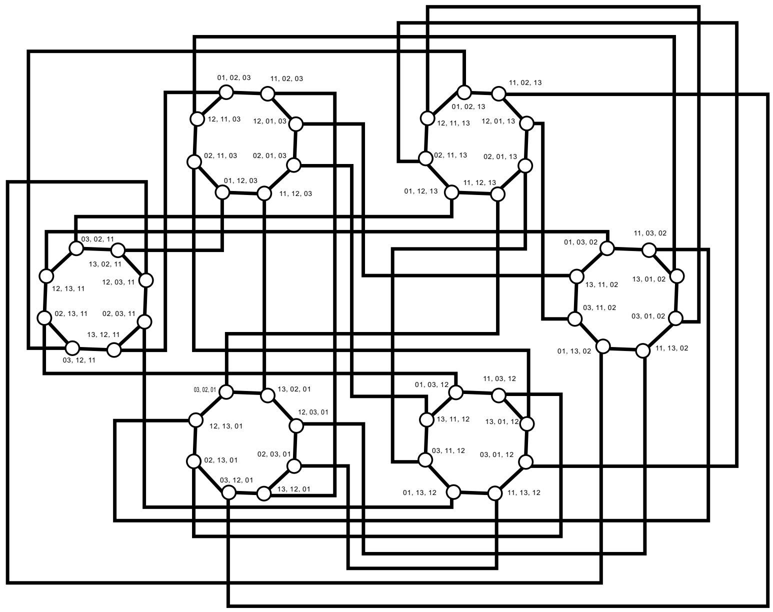

Figure 1. $2P_3$.

[(1, 1), (1, 2), (0, 3)], [(0, 1), (1, 2), (0, 3)], [(0, 2), (1, 1), (0, 3)], [(1, 2), (1, 1), (0, 3)]},
{[(0, 1), (0, 2), (1, 3)], [(1, 1), (0, 2), (1, 3)], [(1, 2), (0, 1), (1, 3)], [(0, 2), (0, 1), (1, 3)],
[(1, 1), (1, 2), (1, 3)], [(0, 1), (1, 2), (1, 3)], [(0, 2), (1, 1), (1, 3)], [(1, 2), (1, 1), (1, 3)]},
{[(0, 1), (0, 3), (0, 2)], [(1, 1), (0, 3), (0, 2)], [(1, 3), (0, 1), (0, 2)], [(0, 3), (0, 1), (0, 2)],
[(1, 1), (1, 3), (0, 2)], [(0, 1), (1, 3), (0, 2)], [(0, 3), (1, 1), (0, 2)], [(1, 3), (1, 1), (0, 2)]},
{[(0, 1), (0, 3), (1, 2)], [(1, 1), (0, 3), (1, 2)], [(1, 3), (0, 1), (1, 2)], [(0, 3), (0, 1), (1, 2)],
[(1, 1), (1, 3), (1, 2)], [(0, 1), (1, 3), (1, 2)], [(0, 3), (1, 1), (1, 2)], [(1, 3), (1, 1), (1, 2)]},
{[(0, 3), (0, 2), (0, 1)], [(1, 3), (0, 2), (0, 1)], [(1, 2), (0, 3), (0, 1)], [(0, 2), (0, 3), (0, 1)],
[(1, 3), (1, 2), (0, 1)], [(0, 3), (1, 2), (0, 1)], [(0, 2), (1, 3), (0, 1)], [(1, 2), (1, 3), (0, 1)]},
{[(0, 3), (0, 2), (1, 1)], [(1, 3), (0, 2), (1, 1)], [(1, 2), (0, 3), (1, 1)], [(0, 2), (0, 3), (1, 1)],
[(1, 3), (1, 2), (1, 1)], [(0, 3), (1, 2), (1, 1)], [(0, 2), (1, 3), (1, 1)], [(1, 2), (1, 3), (1, 1)]}}.

The cycle of length k in n-dimensional m-sided pancake graph mP_n as a reduced finite sequence of reversals (ρ_1, ρ_2, $\cdots$, ρ_k), all of which lie in the appropriate generating set, and such that $\rho_k \cdots \rho_1 = I_n$. By a reduced sequence, it means that $\rho_1 \neq \rho_k$ and $\rho_{i+1} \neq \rho_i$ for all $1 \leq i \leq k-1$. The girth of a pancake graph will be the minimal length of a cycle of reversals taken from that graphs's generating set [12].

Observation 1 : Every mP_{n_1} graph embeds in all mP_{n_2} graphs, $n_2 \geq n_1$.

The observation will be very useful in the following section.

Lemma 2.1 — The subgraph induced by $[u]$ is a cycle.

PROOF : $m = 1$, $n \geq 4$. As P_3 is a cycle and P_3 embeds in all higher order pancake graphs, by the definition of $[u]$, the subgraph induced by $[u]$ is a cycle.

$m = 2$, $n \geq 3$ ($m \geq 3$, $n \geq 2$). $2P_2$ (mP_1) is a cycle. By Observation 1, we can get the conclusion. □

The subgraph induced by $[u]$ is a cycle, denoted by $C([u])$.

Now, we define a mapping

$$\theta = \begin{cases} P(m,\ n,\ n-3) \longrightarrow \mathcal{P}(m,\ n) & \text{if } m = 1,\ n \geq 4, \\ P(m, n,\ n-2) \longrightarrow \mathcal{P}(m,\ n) & \text{if } m = 2,\ n \geq 3, \\ P(m,\ n,\ n-1) \longrightarrow \mathcal{P}(m,\ n) & \text{if } m \geq 3,\ n \geq 2. \end{cases}$$

In short,

$$\theta = P(m,\ n,\ t) \longrightarrow \mathcal{P}(m,\ n),$$

the value of t depends on m and n by the above definition.

For any α=$[(k_{n-t+1}, p_{n-t+1}), (k_{n-t+2}, p_{n-t+2}), \cdots, (k_n, p_n)] \in P(m,\ n,\ t)$,

$\theta(\alpha)$=$\{[(k_1,\ p_1),\ (k_2,\ p_2),\ \cdots,\ (k_{n-t},\ p_{n-t}),\ (k_{n-t+1},\ p_{n-t+1}),\ (k_{n-t+1},\ p_{n-t+1}),\ (k_{n-t+2}, p_{n-t+2}), \cdots, (k_n, p_n)]$—$k_1, \cdots, k_{n-t} \in \{0,\ 1, \cdots, m{-}1\}, p_i \in T_n\backslash\{p_{n-t+1}, p_{n-t+2}, \cdots, p_n\},\ 1 \leq i \leq n-t\}$.

Lemma 2.2 — The map θ is a bijection from $P(m,\ n,\ t)$ to $\mathcal{P}(m,\ n)$.

PROOF : For any two distinct elements,

α=$[(k_{n-t+1}, p_{n-t+1}), (k_{n-t+2}, p_{n-t+2}), \cdots, (k_n, p_n)]$ and β=$[(j_{n-t+1}, q_{n-t+1}), (j_{n-t+2}, q_{n-t+2}), \cdots, (j_n, q_n)]$ in $P(m,\ n,\ t)$,

we have

$\theta(\alpha)$=$\{[(k_1,\ p_1),\ (k_2,\ p_2),\ \cdots,\ (k_{n-t},\ p_{n-t}),\ (k_{n-t+1},\ p_{n-t+1}),\ (k_{n-t+1},\ p_{n-t+1}),\ (k_{n-t+2}, p_{n-t+2}), \cdots, (k_n, p_n)]$—$k_1, \cdots, k_{n-t} \in \{0,\ 1,\ \cdots,\ m{-}1\}, p_i \in T_n\backslash\{p_{n-t+1}, p_{n-t+2}, \cdots, p_n\},\ 1 \leq i \leq n-t\}$.

$\theta(\beta)$=$\{[(j_1, q_1), (j_2, q_2), \cdots, (j_{n-t}, q_{n-t}), (j_{n-t+1}, q_{n-t+1}), (j_{n-t+1}, q_{n-t+1}), (j_{n-t+2}, q_{n-t+2}), \cdots, (j_n, q_n)]$—$j_1, \cdots, j_{n-t} \in \{0,\ 1,\ \cdots,\ m-1\},\ q_i \in T_n\backslash\{q_{n-t+1},\ q_{n-t+2},\ \cdots, q_n\},\ 1 \leq i \leq n-t\}$.

Obviously, $\theta(\alpha) \bigcap \theta(\beta) = \emptyset$, which means θ is an injection.

For any $[u] \in \mathcal{P}(m,\ n)$, say

u=[(k_1, p_1), (k_2, p_2), $\cdots$, (k_{n-t}, p_{n-t}), (k_{n-t+1}, p_{n-t+1}), (k_{n-t+1}, p_{n-t+1}), (k_{n-t+2}, p_{n-t+2}), $\cdots$, (k_n, p_n)],

we set α=[(k_{n-t+1}, p_{n-t+1}), (k_{n-t+1}, p_{n-t+1}), (k_{n-t+2}, p_{n-t+2}), $\cdots$, (k_n, p_n)].

Then $\alpha \in P(m,\ n,\ t)$ and

$\theta(\alpha)$={[(k_1, p_1), (k_2, p_2), $\cdots$, (k_{n-t}, p_{n-t}), (k_{n-t+1}, p_{n-t+1}), (k_{n-t+1}, p_{n-t+1}), (k_{n-t+2}, p_{n-t+2}), $\cdots$, (k_n, p_n)]}=$[u]$.

Then θ is a surjection.

It follows that θ is a bijection from $P(m,\ n,\ t)$ to $\mathcal{P}(m,\ n)$. □

By Lemma 2.2, $\mathcal{P}(m,\ n,\ t) = \{\theta(\alpha)|\alpha \in P(m,\ n,\ t)\}$, which is a partition of vertex set $V(mP_n)$. It means that, $V(mP_n)$=$\bigcup_{\alpha\in P(m,\ n,\ t)} \theta(\alpha)$, and $\theta(\alpha_i) \bigcap \theta(\alpha_j) = \emptyset$ for any distinct α_i, α_j $\in P(n,\ m,\ t)$.

By Lemma 2.1 and Lemma 2.2, we can get the following lemma.

Lemma 2.3 — If $[u_1] \neq [u_2]$, then $C([u_1])$ is disjoint with $C([u_2])$.

By the definition of minimum feedback vertex set and Lemma 2.3, we immediately have the following conclusion.

Lemma 2.4 — $F_{\min}(mP_n) \geq p(m,\ n,\ t)$

Lemma 2.5 — [6]. IF G is a connected graph with p vertices $(p > 2)$, q edges, and maximum degree d, then,

$F_{\min}(G) \geq \frac{q-p+1}{d-1}$.

Graph	Vertices	Degree
P_n	$n!$	$n-1$
$2P_n$	$n!2^n$	n
$3P_n$	$n!3^n$	$2n$
mP_n	$n!m^n$	$2n$

Table 1.

By the simple calculations, we find that Lemma 2.4 is better than Lemma 2.5.

If m=1, n=3, we let t=n-2=1. Note that mP_n is a regular graph ,The next result directly follows from Theorem 1 and Lemma 2.5.

***Theorem* 6** — $\min_\theta(mP_n) \geq p(m,\ n,\ t),\ \theta = \begin{cases} n-2 & \textit{if}\ m=1, n\geq 3, \\ n-1 & \textit{if}\ m=2, n\geq 3, \\ 2n-1 & \textit{if}\ m\geq 3, n\geq 2. \end{cases}$

3. UPPER BOUND OF n-DIMENSIONAL m-SIDED PANCAKE GRAPH

Definition 5 — [22]. The n-dimensional 2-sided pancake graph $2P_n$ has vertices labeled with $[(k_1, p_1), (k_2, p_2), \cdots, (k_i, p_i), (k_{i+1}, p_{i+1}), \cdots, (k_n, p_n)]$ for each $[p_1, p_2, \cdots, p_i, \cdots, p_n)] \in S_n$ and k_j=0 or 1, for $1 \leq j \leq n$, and there is an edge between two vertices x and y where y can be obtained from x by a flip action.

Definition 6 — [15]. A sequence of n integers ω=$(u_1, u_2, \cdots, u_n)$ that satisfies the condition $\{|u_1|, |u_2|, \cdots, |u_n|\}$ = $< n >$ is called a signed permutation of $< n >$. For a signed permutation ω=$(u_1, u_2, \cdots, u_n)$ and an integer i $(1 \leq i \leq n)$, the operation $\omega^i = (-u_i, -u_{i-1}, \cdots, -u_1, u_{i+1}, \cdots, u_n)$ is called signed prefix reversal.

Definition 7 — [15]. An n-burnt pancake graph is an undirected graph $G(V, E)$ defined by V=$\{\omega|\omega$ is a signed permutation of $< n >\}$, E=$\{(\omega, \omega^i)|\omega \in V, 1 \leq i \leq n\}$.

By Definition 7 and Definition 5, we easily know the following lemma.

Lemma 3.1 — $2P_n$ is isomorphic to burnt pancake graphs.

In [21], they give the upper bound of the number of the vertices in the minimum feedback vertex set of pancake graph and burnt pancake graph. Then, we only consider $mP_n(m \geq 3)$.

In different computer languages, it is not the same in the modulo operation processing of negative number. For different two integers, the principle of the C++/Java language is to make quotient large as possible, many new language and web calculator principle is to make quotient as small as possible. For example, in C++/Java language, $(-1)mod(3) = -1$, but in Python language, $(-1)mod(3) = 2$. By Definition 2, we can think Justan [22] use the later principle. In order to avoid controversy, As $(-1)mod(m) = (m-1)mod(m)$, we replace -1 with m-1. We give a similar definition of n-dimensioned m-sided pancake graph and easily know they are equivalent.

For example, in $3P_2$, we replace [(2, 1), (0, 2)] with [(-1, 1), (0, 2)].

Now, we give the new following definition.

Consider a permutation $[(k_1, p_1), (k_2, p_2), \cdots, (k_i, p_i), (k_{i+1}, p_{i+1}), \cdots, (k_n, p_n)]$ where $[p_1, p_2, \cdots, p_i, \cdots, p_n)] \in S_n$ and k_j=0, 1, $\cdots$, m-1 for $1 \leq j \leq n$.

An ***(i,h)-flip*** action is defined by $[(k_1, p_1), (k_2, p_2), \cdots, (k_i, p_i), (k_{i+1}, p_{i+1}), \cdots, (k_n, p_n)] \vdash [(k_i + h_{(mod\ m)}, p_i), \cdots, (k_2 + h_{(mod\ m)}, p_2), (k_1 + h_{(mod\ m)}, p_1), (k_{i+1} + h_{(mod\ m)}, p_{i+1}), \cdots, (k_n + h_{(mod\ m)}, p_n)]$ for $h = 1\ or\ m-1$.

Definition 8 — The n-dimensional m-sided pancake graph mP_n has vertices labeled with $[(k_1,$

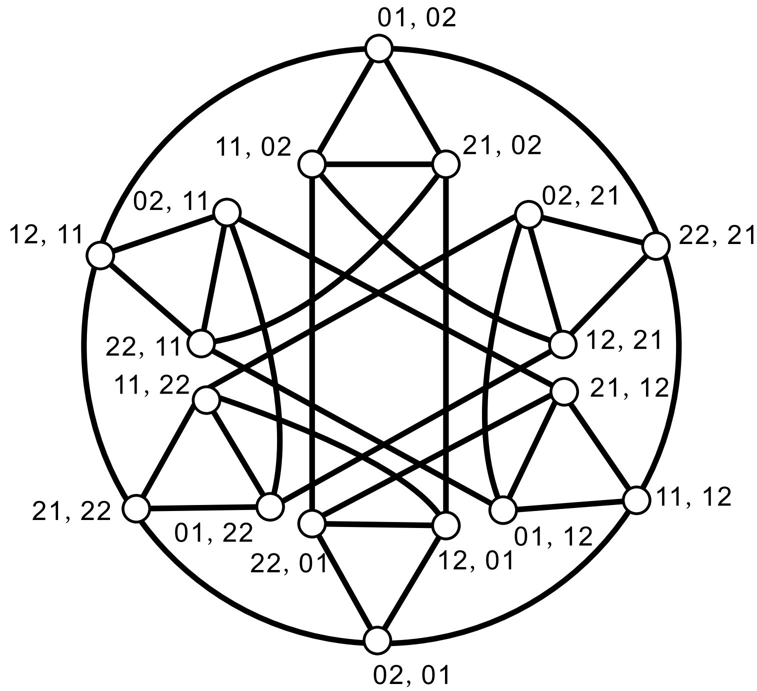

Figure 2. $3P_2$.

p_1), (k_2, p_2), $\cdots$, (k_i, p_i), (k_{i+1}, p_{i+1}), $\cdots$, (k_n, p_n)] for each $[p_1,\, p_2,\, \cdots,\, p_i,\, \cdots,\, p_n)] \in S_n$ and k_j=0, 1, $\cdots$, m-1 for $1 \leq j \leq n$, and there is an edge between two vertices x and y where y can be obtained from x by ***(i,h)-flip*** action.

In an $mP_n(m \geq 3)$, the subgraph $mP_n\{(0,k)\}$ induced by the subset of vertices that have a common element $(0,k)$ at the leftmost positions of the permutations does not have any edge with $k = 0,\, \cdots,\, m-1$. Let us consider the m-partite subgraph induced by $mP_n\{(0,k)\}$, $mP_n\{(1,k)\}$, $\cdots$, $mP_n\{(m-1,k)\}$ with edges among them. By Definition 8, the degree of any vertex in m-partite subgraph is 2. In the other word, m-partite subgraph is union of disjoint cycle and the length of these cycle in m-partite subgraph is at least m.

There are $p(m,\, n,\, n-1)$ vertices in the m-partite subgraph induced by $mP_n\{(0,\, k)\}$, $mP_n\{(1,\, k)\}$, $\cdots$, $mP_n\{(m-1,\, k)\}$. Then, if we delete $p(m,\, n,\, n-1)/m$ in the m-partite subgraph, there would be no cycle. Then, if $p(m,\, n) - p(m,\, n,\, n-1) + p(m,\, n,\, n-1)/m$ vertices are deleted from mP_n, there would be no cycle in the residual graph. If m=3, n=1, let t=n-1=0.

Lemma 3.2 — $F_{\min}(mP_n) \leq p(m,\, n) - p(m,\, n,\, n-1) + p(m,\, n,\, n-1)/m,\; m \geq 3,\; n \geq 1.$

***Theorem* 7** — $\min_{2n-1}(mP_n) \leq p(m,\, n) - p(m,\, n,\, n-1) + p(m,\, n,\, n-1)/m,\; m \geq 3,\; n \geq 1.$

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REFERENCES

1. E. Ackerman, O. Ben-Zwi, and G. Wolfovitz, Combinatorial model and bounds for target set selection, *Theoret. Comput. Sci.*, **411** (2010), 4017-4022.
2. S. S. Adams, Z. Brass, C. Stokes, and D. S. Troxell, Irreversible k-threshold and majority conversion processes on complete multipartite graphs and graph products, *Australas. J. Combin.*, **56** (2013), 47-60.
3. S. S. Adams, D. S. Troxell, and S. L. Zinnen, Dynamic monopolies and feedback vertex sets in hexagonal grids, *Comput. Math. Appl.*, **62** (2011), 4049-4057.
4. S. Bao and L. W. Beineke, The decycling number of graphs, *Australas. J. Combin.*, **25** (2002), 285-298.
5. O. Ben-Zwi, D. Hermelin, and D. Lokshtanov, An exact almost optimal algorithm for target set selection in social networks, *Proceedings of the tenth ACM conference on Electronic commerce*, 2009, Stanford, CA, USA.
6. L. Beineke and R. Vandell, Decycling graphs, *J. Graph Theory*, **25** (1997), 59-77.
7. E. Berger, Dynamic monopolies of constant size, *J. Combin. Theory Ser. B*, **83** (2001), 191-200.
8. N. Chen, On the approximability of influence in social networks, *SIAM J. Discrete Math.*, **23** (2009), 1400-1415.
9. C. Y. Chiang, L. H. Huang, and H. G. Yeh. Target set selection problem for honeycomb networks, *SIAM J. Discrete Math.*, **27** (2013), 310-328.
10. C. Y. Chiang, L. H. Huang, B. J. Li, J. Wu, and H. G. Yeh, Some results on the target set selection problem, *J. Comb. Optim.*, **25** (2011), 702-715.
11. M. Chopin, A. Nichterlein, R. Niedermeier, and M. Weller, Constant thresholds can make target set selection tractable, *Theory. Comput. Systems*, **55** (2014), 61-83.
12. P. Compeau, Girth of pancake graphs, *Discrete Appl. Math.*, **159** (2011), 1641-1645.
13. P. Domingos and M. Richardson, Mining the network value of customers, *Seventh International Conference on Knowledge Discovery and Data Mining*, 2001, San Francisco, CA, USA.
14. P. A. Dreyer and F. S. Roberts, Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion, *Discrete Appl. Math.*, **157** (2009), 1615-1627.
15. T. Iwasaki, Fault-tolerant routing in burnt pancake graphs, *Inform. Process. Lett.*, **110** (2010), 535-538.
16. P. Festa, P. Pardalos, and Mauricio G. C. Resende, Feedback set problems, Springer, US, (1999).

17. P. Flocchini, R. Královič, P. Ruzicka, and N. Santoro, On time versus size for monotone dynamic monopolies in regular topologies, *J. Discrete Algorithms*, **2** (2002), 135-156.

18. P. Flocchini, E. Lodi, F. Luccio, L. Pagli, and N. Santoro, Dynamic monopolies in tori, *Discrete Appl. Math.*, **137** (2004), 197-212.

19. M. Heydari and I. Sudborough, On sorting by prefix reversals and the dimaeter of pancake networks, Parallel Archirectures and Their Efficient Use, *Lecture Notes in Computer Science*, **678** (1993), 218-227.

20. C. Hoffman, *Group theoretic algorithms and graph isomorphism*, Springer, New York (1982).

21. S. Jung and K. Kaneko, *A feedback vertex set on pancake graphs*, International Conference on Parallel and distributed Processing Techniques and Applications, 2010, San Francisco, CA, USA.

22. M. Justan, F. Muga II, and I. Sudborough, *On the generalization of the pancake network*, Proceedings of the International Symposium on Parallel Archiectures, Algorithms and Networks, 2002, Sydney, Australia.

23. K. Kaneko and Y. Suzuki, Node-to-set disjoint paths problem in pancake graphs, *IEICE Trans. Inf. Syst.*, **E86-D** (2003), 1628-1633.

24. D. Kempe, J. Kleinberg, and E. Tardos, *Maximizing the spread of influence through a social network*, In Proceedings of the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, 2003, London, UK.

25. F. Luccio, Almost exact minimum feedback vertex set in meshes and butterflies, *Inform. Process. Lett.*, **66** (1998), 59-64.

26. D. Peleg, Local majorities, coalitions and monopolies in graphs: A review, *Theoret. Comput. Sci.*, **282** (2002), 231-257.

27. D. A. Pike and Y. Zou, Decycling Cartesian products of two cycles, *SIAM J. Discrete Math.*, **19** (2005), 651-663.

28. M. Zaker, On dynamic monopolies of graphs with general thresholds, *Discrete Math.*, **312** (2012), 1136-1143.